CHARACTERIZATIONS OF COMMUTATIVITY FOR C^*-ALGEBRAS

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Let $\mathscr{A}$ be a C^*-algebra acting on the Hilbert space H and let $\mathscr{S}$ be the self-adjoint elements of $\mathscr{A}$. The following characterization of commutativity is due to I. Kaplansky (see Dixmier [**3**, p. 58]).

THEOREM 1. *$\mathscr{A}$ is commutative if and only if* 0 *is the only nilpotent element of $\mathscr{A}$.*

In this note we use the above result of Kaplansky to give two numerical characterizations of commutativity. Ogasawara [**5**], Sherman [**6**], and Fukamiya, Misonou and Takeda [**4**] characterize commutativity for $\mathscr{A}$ in terms of the usual order structure on $\mathscr{S}$. We show that Kaplansky's theorem reduces the proofs of these order characterizations to simple computations.

1. Numerical characterizations. Taylor [**7**, Lemma 3.3] proves that, if A and B are self-adjoint elements of $\mathscr{A}$ with $0 \neq \|A\| \geqq \|B\|$, then

$$\|A+B\| \leqq \|A\| + k\frac{\|AB\|}{\|A\|}, \tag{1}$$

where k may be taken as 2. If $\mathscr{A}$ is commutative, the inequality holds with $k=1$. Taylor asks if the converse is true; in Theorem 2 we prove this.

Note that an inequality of the form (1) can hold for all elements of a Banach algebra $\mathscr{B}$ only if $\mathscr{B}$ is commutative. For, setting $B = A$ in (1), we obtain $\|A\|^2 \leqq k\|A^2\|$ and thence $\|A\| \leqq kr(A)$, where $r(A)$ is the spectral radius of A. Thus $\mathscr{B}$ is commutative (see, for example, [**1**, p. 33]).

A simple argument shows that inequality (1) holds if and only if it holds for self-adjoint A, B of norm 1.

REMARK. We assume that $\mathscr{A}$ has a unit element when there is no loss of generality in so doing.

THEOREM 2. *$\mathscr{A}$ is commutative if and only if*

$$\|A+B\| \leqq 1 + \|AB\|$$

for all self-adjoint elements $A, B \in \mathscr{A}$ with $\|A\| = \|B\| = 1$.

Proof. If $\mathscr{A}$ is commutative, the result follows from the inequality

$$(I-A)(I-B) \geqq 0.$$

Assume that $\mathscr{A}$ is not commutative. By Theorem 1, there exists nonzero $T \in \mathscr{A}$ such that $T^2 = 0$. Let H_1 be the subspace $(TH)^-$ and let H_2 be the orthogonal complement of H_1 in

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H. If we represent H as $H_1 \oplus H_2$, T, T^* are represented by the 2×2 matrices of operators

$$T = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \quad T^* = \begin{bmatrix} 0 & 0 \\ S^* & 0 \end{bmatrix}.$$

We may suppose that $\| S \| = 1$. Let

$$A = TT^*, \quad B = \alpha TT^* + \alpha T^*T + \beta T + \beta T^*,$$

where $\alpha, \beta > 0$, $\alpha + \beta = 1$, so that $A, B \in \mathscr{A}$. Then

$$A = \begin{bmatrix} SS^* & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \alpha SS^* & \beta S \\ \beta S^* & \alpha S^*S \end{bmatrix}.$$

Clearly $\| A \| = 1$. Since $\| S \| = 1$, there exist $x_n \in H_1$ such that $\| x_n \| = 1$ and $SS^*x_n - x_n \to 0$. To see this, note that $\| SS^*x_n - x_n \|^2 = \| SS^*x_n \|^2 - 2\| S^*x_n \|^2 + \| x_n \|^2 \leqq 2(\| x_n \|^2 - \| S^*x_n \|^2)$, and choose x_n such that $\| S^*x_n \| \to 1$. Hence

$$B(x_n + S^*x_n) - (x_n + S^*x_n) \to 0,$$

and so $\| B \| \geqq 1$. But

$$\| B \| \leqq \alpha \| TT^* + T^*T \| + \beta \| T + T^* \| \leqq 1,$$

and so $\| B \| = 1$. Next,

$$\begin{aligned} \| AB \| &= \sup\{\| \alpha SS^*SS^*x + \beta SS^*Sy \| : \| x \|^2 + \| y \|^2 = 1\} \\ &\leqq \sup\{\alpha \| x \| + \beta \| y \| : \| x \|^2 + \| y \|^2 = 1\} \\ &= (\alpha^2 + \beta^2)^{\frac{1}{2}}. \end{aligned}$$

Let $\lambda = \alpha + \frac{1}{2} + (\frac{1}{4} + \beta^2)^{\frac{1}{2}}$, so that λ satisfies the equation

$$(\lambda - \alpha)(\lambda - \alpha - 1) = \beta^2.$$

Let x_n be as above and let $y_n = \beta(\lambda - \alpha)^{-1}S^*x_n$. Then

$$(A + B)(x_n + y_n) - \lambda(x_n + y_n) \to 0,$$

so that $\| A + B \| \geqq \lambda$. If we choose α, β so that

$$\alpha + \tfrac{1}{2} + (\tfrac{1}{4} + \beta^2)^{\frac{1}{2}} > 1 + (\alpha^2 + \beta^2)^{\frac{1}{2}},$$

then we have $\| A + B \| > 1 + \| AB \|$. It is enough to take

$$\alpha = \tfrac{2}{3}, \quad \beta = \tfrac{1}{3}.$$

REMARK. If $\mathscr{A}$ is commutative, we even have $\| A + B \| \leqq 1 + \| AB \|$ for all elements $A, B \in \mathscr{A}$ with $\| A \| = \| B \| = 1$.

We recall that the numerical index $n(\mathscr{A})$ of $\mathscr{A}$ is defined by

$$n(\mathscr{A}) = \inf\{w(A) : A \in \mathscr{A}, \| A \| = 1\},$$

where

$$w(A) = \sup\{| \langle Ax, x \rangle | : x \in H, \| x \| = 1\},$$

and that $\frac{1}{2} \leqq n(\mathscr{A}) \leqq 1$ (see [1, pp. 43, 44]).

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THEOREM 3. *$\mathcal{A}$ is commutative or not commutative according as $n(\mathcal{A})$ is 1 or $\frac{1}{2}$.*

Proof. If $\mathcal{A}$ is commutative, each $A\in\mathcal{A}$ is normal and so has $w(A)=\|A\|$. If $\mathcal{A}$ is not commutative, then, by Theorem 1, there exists $T\in\mathcal{A}$, with $T\neq 0$, $T^2=0$. A result of Bouldin [2, Corollary 2, p. 214] shows that $w(T)=\frac{1}{2}\|T\|$, so that $n(\mathcal{A})=\frac{1}{2}$. (The condition T^*H orthogonal to TH in [2] is equivalent to $T^2=0$.)

2. Order characterizations. We recall that the usual order on $\mathcal{S}$ is defined by

$$A\geqq B\Leftrightarrow\langle (A-B)x, x\rangle\geqq 0\quad (x\in H).$$

Let T, S be as in the proof of Theorem 2. Let

$$P=\begin{pmatrix} SS^* & 0\\ 0 & 0\end{pmatrix},\quad Q=\begin{pmatrix} 0 & (SS^*)^{\frac{1}{2}}S\\ S^*(SS^*)^{\frac{1}{2}} & 0\end{pmatrix},\quad R=\begin{pmatrix} 0 & 0\\ 0 & S^*S\end{pmatrix},$$

so that $P, Q, R\in\mathcal{A}$. We make frequent use of the following lemma.

LEMMA 4. *Let $\alpha,\beta,\gamma\in\mathbb{R}$ with $\gamma>0$. Then $\alpha P+\beta Q+\gamma R\geqq 0$ if and only if $\alpha\gamma-\beta^2\geqq 0$.*

Proof. For $x\in H_1, y\in H_2$ we have

$$\langle(\alpha P+\beta Q+\gamma R)(x+y), x+y\rangle = \|\beta\gamma^{-\frac{1}{2}}(SS^*)^{\frac{1}{2}}x+\gamma^{\frac{1}{2}}Sy\|^2+\gamma^{-1}(\alpha\gamma-\beta^2)\|S^*x\|^2.$$

Since $(TH)^-=H_1$, for any $x\in H_1$ there exist $y_n\in H_2$ such that $\gamma^{\frac{1}{2}}Sy_n\to-\beta\gamma^{-\frac{1}{2}}(SS^*)^{\frac{1}{2}}x$. The result follows.

$\mathcal{S}$ is said to be *lattice ordered* if, for each $U\in\mathcal{S}$, there exists $U^+\geqq 0$ such that $U^+\geqq U$ and $U^+\leqq V$ for any V such that $V\geqq 0$ and $V\geqq U$. $\mathcal{S}$ is said to have the *decomposition property* if, given $A, B, C\in\mathcal{S}$ with $0\leqq A\leqq B+C$, $B\geqq 0$, $C\geqq 0$, there exist $A_1, A_2\in\mathcal{S}$ with $A=A_1+A_2$, $0\leqq A_1\leqq B$, $0\leqq A_2\leqq C$.

THEOREM 5. ([4], [5], [6].) *The following statements are equivalent.*

(i) *$\mathcal{A}$ is commutative.*
(ii) $A, B\in\mathcal{A}$, $A\geqq B\geqq 0\Rightarrow A^2\geqq B^2$.
(iii) *$\mathcal{S}$ is lattice ordered.*
(iv) *The dual space of $\mathcal{S}$ is lattice ordered.*
(v) *$\mathcal{S}$ has the decomposition property.*

Proof. If $\mathcal{A}$ is commutative, the Gelfand–Naimark theorem readily shows that conditions (ii)–(v) hold. Assume that $\mathcal{A}$ is not commutative and let T be as in the proof of Theorem 2.

(ii)$\Rightarrow$(i). With the above notation, let $A=8P+2R$, $B=4P+2Q+R$. Then $A, B\in\mathcal{A}$ and $A\geqq B\geqq 0$, by Lemma 4. For $y\in H_2$, we have $\langle(A^2-B^2)y, y\rangle=-\langle(S^*S)^2y, y\rangle$, so that $A^2\ngeqq B^2$.

(iii)$\Rightarrow$(i). Let $\mathcal{S}$ be lattice ordered and let $U=P-R$. Then $U\in S$ and it is elementary that $U^+=P$. Let $V=2P+2^{\frac{1}{2}}Q+R$, and we have $V\in A$, $V\geqq 0$, $V\geqq U$, but $V\ngeqq U^+$, by Lemma 4.

(iv)$\Rightarrow$(i). Let $\mathscr{S}'$ be the (real) dual space of $\mathscr{S}$ with the induced dual order and let $\mathscr{S}'$ be lattice ordered. Given $x \in H_1$ and $y \in H_2$, let $f, g \in \mathscr{S}'$ be defined by

$$f(V) = \langle V_1 x, x\rangle - \langle V_3 y, y\rangle, \quad g(V) = \langle V_1 x, x\rangle,$$

where

$$V = \begin{bmatrix} V_1 & V_2 \\ V_2^* & V_3 \end{bmatrix}.$$

If $V \geqq 0$, then $V_1 \geqq 0$ and $V_3 \geqq 0$. Hence $f \leqq g$ and so $f^+ \leqq g$, since $g \geqq 0$. Then $f(P) \leqq f^+(P) \leqq g(P)$ gives $f^+(P) = \langle Px, x\rangle$ and $0 \leqq f^+(R) \leqq g(R)$ gives $f^+(R) = 0$. Also $(g-f^+)(P \pm Q + R) = \mp f^+(Q) \geqq 0$, so that $f^+(Q) = 0$. Define $h \in \mathscr{S}'$ by

$$h(V) = \langle V(2^{\frac{1}{2}}x+y), 2^{\frac{1}{2}}x+y\rangle = 2\langle V_1 x, x\rangle + 22^{\frac{1}{2}}\mathrm{Re}\langle V_2 y, x\rangle + \langle V_3 y, y\rangle.$$

Then

$$(h-f)(V) = \langle V_1 x, x\rangle + 22^{\frac{1}{2}}\mathrm{Re}\langle V_2 y, x\rangle + 2\langle V_3 y, y\rangle = \langle V(x+2^{\frac{1}{2}}y), x+2^{\frac{1}{2}}y\rangle,$$

which gives $h - f \geqq 0$. But

$$(h-f^+)(P+Q+R) = \langle Px, x\rangle + 22^{\frac{1}{2}}\mathrm{Re}\langle Q_2 y, x\rangle + \langle Ry, y\rangle = \langle (P+2^{\frac{1}{2}}Q+R)(x+y), x+y\rangle,$$

and, by Lemma 4, we can choose x, y so that $h \ngeqq f^+$.

(v)$\Rightarrow$(i). Let $A = \frac{1}{2}P$, $B = P+Q+R$, $C = 4P+2Q+R$. Then $0 \leqq A \leqq B + C$, by Lemma 4. Suppose that $A = A_1 + A_2$, with $0 \leqq A_1 \leqq B$, $0 \leqq A_2 \leqq C$. Since $A_1 \leqq A$, it is easy to show that A_1 is of the form

$$A_1 = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, since $A_1 \leqq B$, for $x \in H_1$ and $y \in H_2$ we have $\langle Xx, x\rangle \leqq \langle (P+Q+R)(x+y), x+y\rangle = \| (SS^*)^{\frac{1}{2}}x + Sy \|^2$, from the proof of Lemma 4. Since $H_1 = (TH)^-$, we can choose $y_n \in H_2$ so that $Sy_n \to -(SS^*)^{\frac{1}{2}}x$. This gives $A_1 = 0$. Hence $\frac{1}{2}P = A = A_2 \leqq C$, which contradicts Lemma 4.

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