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ENTIRE SOLUTIONS OF A CURVATURE FLOW IN AN UNDULATING CYLINDER

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Abstract

We consider a curvature flow $V = \kappa + A$ in a two-dimensional undulating cylinder Ω described by $\Omega := \{(x, y) \in \mathbb{R}^2 \mid -g_1(y) < x < g_2(y), y \in \mathbb{R}\}$, where V is the normal velocity of a moving curve contacting the boundaries of Ω perpendicularly, κ is its curvature, $A > 0$ is a constant and $g_1(y), g_2(y)$ are positive smooth functions. If g_1 and g_2 are periodic functions and there are no stationary curves, Matano *et al.* ['Periodic traveling waves in a two-dimensional cylinder with saw-toothed boundary and their homogenization limit', *Netw. Heterog. Media* **1** (2006), 537–568] proved the existence of a periodic travelling wave. We consider the case where g_1, g_2 are general nonperiodic positive functions and the problem has some stationary curves. For each stationary curve Γ unstable from above/below, we construct an entire solution growing out of it, that is, a solution curve Γ_t which increases/decreases monotonically, converging to Γ as $t \to -\infty$ and converging to another stationary curve or to $+\infty/-\infty$ as $t \to \infty$.

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1. Introduction

We consider the curvature-driven motion of a plane curve in a two-dimensional cylinder Ω with undulating boundaries. The law of motion of the curve is given by

$$V = \kappa + A, \tag{1.1}$$

where V denotes the normal velocity of the curve, κ denotes its curvature and $A > 0$ is a constant representing a driving force. The domain Ω is defined by

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid -g_1(y) < x < g_2(y), y \in \mathbb{R}\}$$

(see Figure 1), where $g_1(y)$ and $g_2(y)$ are smooth and positive functions.

By a solution of (1.1) we mean a time-dependent simple curve γ_t in Ω which satisfies (1.1) and contacts the left/right boundary $\partial_1\Omega/\partial_2\Omega$ perpendicularly. Equation (1.1) appears as a certain singular limit of an Allen–Cahn-type nonlinear diffusion equation under the Neumann boundary conditions. The curve γ_t represents the

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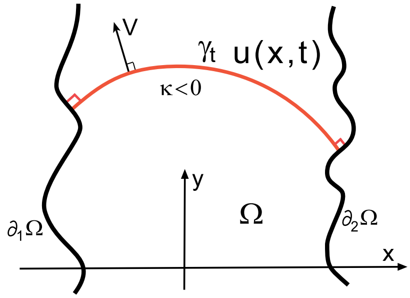

FIGURE 1. Ω and the curve γ_t.

interface between two different phases. See, for example, [1, 7] for details and also [2] and references therein for curvature flow from a geometrical point of view. To avoid sign confusion, the normal to the curve γ_t is chosen toward the upper region, and the signs of V and κ are understood in accordance with this choice of the normal direction. Consequently, κ is negative at those points where the curve is concave (see Figure 1).

We will mainly consider the case where γ_t is expressed as a graph of a certain function $y = u(x,t)$ at each time t. Let $\zeta_i(t)$ $(i = 1, 2)$ be the x-coordinates of the end points of γ_t lying on $\partial_i\Omega$. Now (1.1) is equivalent to

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \quad \zeta_1(t) < x < \zeta_2(t),\ t > 0, \tag{1.2}$$

with the Neumann boundary conditions

$$\begin{cases} u_x(\zeta_1(t),t) = g_1'(u(\zeta_1(t),t)), & (\zeta_1(t), u(\zeta_1(t),t)) \in \partial_1\Omega, \\ u_x(\zeta_2(t),t) = -g_2'(u(\zeta_2(t),t)), & (\zeta_2(t), u(\zeta_2(t),t)) \in \partial_2\Omega. \end{cases} \tag{1.3}$$

Throughout this paper we assume the slope condition

$$|g_1'(y)|, \quad |g_2'(y)| < 1 \quad \text{for all } y \in \mathbb{R}, \tag{1.4}$$

which is used to prevent γ_t from developing singularities near the boundaries. A function $u(x,t)$ defined for $\zeta_1(t) \le x \le \zeta_2(t)$, $t \ge 0$, is called a time-global classical solution of (1.2)–(1.3) if:

(a) u, u_x are continuous for $\zeta_1(t) \le x \le \zeta_2(t)$, $t \ge 0$, and u_{xx}, u_t are continuous for $\zeta_1(t) < x < \zeta_2(t)$, $t > 0$;
(b) u satisfies (1.2)–(1.3) for $\zeta_1(t) < x < \zeta_2(t)$, $t > 0$.

It is called a stationary solution of (1.2)–(1.3) if it is independent of t. It is easily seen that the graph of a stationary solution is a concave circular arc of radius A^{-1} which contacts $\partial_i\Omega$ $(i = 1, 2)$ perpendicularly.

In [6], the authors considered this problem in case $g_1 = g_2$ are positive 1-periodic functions and proved that if:

(H1) the problem (1.2)–(1.3) has no stationary solutions,

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then there is a time-global classical solution $u(x,t)$ for any smooth initial data u_0 with $|u_0'(x)|$ small, which propagates to infinity and converges to a *periodic travelling wave* $U(x,t)$ with $U(x,t+T) = U(x,t)+1$. In [4], they extended the results to the case when g_1 and g_2 are positive recurrent functions and showed that there is always an *entire solution* (that is, a solution defined for all $t \in \mathbb{R}$) propagating from $-\infty$ to ∞ when (H1) holds. On the other hand, in case:

(H2) the problem (1.2)–(1.3) has a stationary solution,

rather than the existence of entire solutions moving from $-\infty$ to ∞, there are some solutions converging to stationary ones, called pinning phenomena.

In this paper we continue to consider the (H2) case. For any suitable function u_0, the global existence of the classical solution for (1.2)–(1.3) with initial data $u(x,0) = u_0(x)$ can be derived as in [4, 6]. (In fact, one can convert the problem into another one with Neumann boundary conditions by straightening the boundaries $\partial_i\Omega$ with isothermic coordinates as in [4, 6]. If necessary, one can first consider the problem in a piece $\Omega \cap \{(x,y) \mid -L < y < L\}$ of Ω for each $L > 0$ and then regard the problem as one with $2L$-periodic boundaries. Thus, one can apply the argument in [4, 6] to derive the global existence of the classical solutions.) We will show that although there are no entire solutions propagating from $-\infty$ to ∞, the problem (1.2)–(1.3) still has some (bounded or unbounded) entire solutions growing out of the stationary solution, even for more general boundary functions g_1 and g_2. More precisely, assume that $u = v(x)$ is a stationary solution of (1.2)–(1.3). If it is unstable from above in the sense that any circular arc γ above v (near v and contacting the boundaries of Ω perpendicularly) has curvature larger than $-A$, then there is an entire solution $\mathcal{U}(x,t)$ propagating upward monotonically, converging to $v(x)$ as $t \to -\infty$ and converging to another stationary solution or to $+\infty$ as $t \to \infty$ (see details in Theorem 3.3). Similarly, if $v(x)$ is unstable from below in the sense that any circular arc γ below v (near v and contacting the boundaries of Ω perpendicularly) has curvature smaller than $-A$, then there is an entire solution $\mathcal{U}(x,t)$ propagating downward monotonically, converging to $v(x)$ as $t \to -\infty$ and converging to another stationary solution or to $-\infty$ as $t \to \infty$ (see details in Theorem 3.5).

Any entire solution connecting two stationary ones corresponds to a heteroclinic orbit in the phase space. For semilinear parabolic equations in a fixed domain, such solutions have been well studied more than thirty years ago (see, for example, [5, Theorems 1, 3 and 8]). In some sense, our main results, Theorems 3.3 and 3.5, can be regarded as the curvature flow version of Matano's results.

In Section 2 we present a necessary and sufficient condition for the existence of stationary solutions. In Section 3.2 we will consider a sequence of solutions $u(x,t)$ of (1.2)–(1.3) with initial data close to $v(x)$ (like γ) and then use the renormalisation method to construct entire solutions. For this purpose, we need to show that the time for u to travel a certain distance can be arbitrarily long, provided γ is sufficiently close to v. In order to estimate this time, we construct a complicated upper solution in Section 3.1.

2. Stationary solutions

A function $v(x)$ defined on $[\xi_1, \xi_2]$ for some ξ_1, ξ_2 with $\xi_1 < 0 < \xi_2$ is called a stationary solution of (1.2)–(1.3) if it solves the problem

$$\begin{cases} v_{xx} + A(1+v_x^2)^{3/2} = 0, & \xi_1 \le x \le \xi_2, \\ v_x(\xi_1) = g_1'(v(\xi_1)), & (\xi_1, v(\xi_1)) \in \partial_1\Omega, \\ v_x(\xi_2) = -g_2'(v(\xi_2)), & (\xi_2, v(\xi_2)) \in \partial_2\Omega. \end{cases} \tag{2.1}$$

In this section we present a necessary and sufficient condition for the existence of such solutions.

Assume that $P_1(-g_1(y_1), y_1) \in \partial_1\Omega$ and $P_2(g_2(y_2), y_2) \in \partial_2\Omega$. Assume further that there is a *concave* circular arc Γ with centre (x_0, y_0) which contacts $\partial_i\Omega$ perpendicularly at P_i $(i = 1, 2)$. Then the function of Γ (denoted by $r(x; y_1, y_2)$) satisfies

$$g_1'(y_1) = r_x(-g_1(y_1)) = \tan\theta_1(y_1), \quad -g_2'(y_2) = r_x(g_2(y_2)) = \tan\theta_2(y_2)$$

for some $\theta_1(y_1)$, $\theta_2(y_2) \in (-\pi/2, \pi/2)$. It is easily seen that

$$\sin\theta_1(y_1) = \frac{g_1'(y_1)}{\sqrt{1+(g_1'(y_1))^2}}, \quad \sin\theta_2(y_2) = \frac{-g_2'(y_2)}{\sqrt{1+(g_2'(y_2))^2}}.$$

The curvature of Γ can be expressed in terms of $g_i(y_i)$ and $\theta_i(y_i)$ in the following way.

Case 1. $\theta_1(y_1) > 0 > \theta_2(y_2)$. By a simple geometrical observation we see that, in this case, $-g_1(y_1) < x_0 < g_2(y_2)$ and

$$\frac{r_{xx}}{(1+r_x^2)^{3/2}} = -\frac{\sin\theta_1(y_1) - \sin\theta_2(y_2)}{g_1(y_1) + g_2(y_2)} < 0. \tag{2.2}$$

Case 2. $\theta_1(y_1) > \theta_2(y_2) \ge 0$. Again, we see by a simple geometrical observation that $-g_1(y_1) < g_2(y_2) \le x_0$ and (2.2) holds.

Case 3. $0 \ge \theta_1(y_1) > \theta_2(y_2)$. In this case, $x_0 \le -g_1(y_1) < g_2(y_2)$ and (2.2) holds.

In accordance with the sign of $(r_{xx}/(1+r_x^2)^{3/2}) + A$, we can decide whether r is a solution, a lower solution or an upper solution of (2.1) as the following theorem shows.

THEOREM 2.1. *The function $r(x; y_1, y_2)$ is a solution (respectively lower solution, upper solution) of (2.1) according as*

$$K(y_1, y_2) := A[g_1(y_1) + g_2(y_2)] - [\sin\theta_1(y_1) - \sin\theta_2(y_2)] = 0$$

(respectively $K(y_1, y_2) > 0$, $K(y_1, y_2) < 0$).

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3. Entire solutions

3.1. Upper and lower solutions. In this subsection we construct some upper and lower solutions of the problem (1.2)–(1.3), which will be used to construct entire solutions in the next subsection (see Remark 3.4 below).

Step 1. Stationary solutions. Our construction is based on the stationary solutions from Section 2. We first present some further properties of stationary solutions. Assume, for some $Y_1, Y_2 \in \mathbb{R}$, that $r(x; Y_1, Y_2)$ is a solution of (2.1). Then there is an upper semicircle with radius A^{-1} and centre (x_0, y_0), whose function is

$$v(x; x_0, y_0) := y_0 + \sqrt{A^{-2} - (x - x_0)^2}, \quad |x - x_0| < A^{-1},$$

such that $r(x)$ is a truncation of $v(x)$ over some interval $[\xi_1, \xi_2]$:

$$r(x; Y_1, Y_2) \equiv v(x; x_0, y_0)|_{[\xi_1,\xi_2]},$$

where ξ_1, ξ_2 are defined by

$$\xi_1 := -g_1(Y_1) = -g_1(v(\xi_1)) < 0, \quad \xi_2 := g_2(Y_2) = g_2(v(\xi_2)) > 0.$$

Set $J := [y_0,\ y_0 + A^{-1} + 1]$. By the slope condition (1.4), we can find a constant $G \in (0, 1)$ such that

$$|g_1'(y)| \le G, \quad |g_2'(y)| \le G \quad \text{for } y \in J.$$

By the boundary conditions in (2.1),

$$|v_x(\xi_i)| = \frac{|\xi_i - x_0|}{\sqrt{A^{-2} - |\xi_i - x_0|^2}} = |g_i'(Y_i)| \le G \quad (i = 1, 2).$$

Hence,

$$|\xi_i - x_0| \le X := \frac{G}{A\sqrt{1 + G^2}} \quad (i = 1, 2).$$

Set

$$\widetilde{G} := \frac{1 + G}{2} \in (G, 1), \quad \widetilde{X} := \frac{\widetilde{G}}{A\sqrt{1 + \widetilde{G}^2}} \in (X, A^{-1}).$$

Then $v(x)$ is well defined over $I := \{x \mid |x - x_0| \le \widetilde{X}\}$ and $|v_x(x)| \le \widetilde{G}$ for $x \in I$.

Define

$$h_i := \min_{y \in J} g_i(y), \quad K_i := \max_{y \in J} |g_i''(y)| \quad (i = 1, 2).$$

Then $h_1, h_2 > 0$ by the positivity of $g_i(y)$ $(i = 1, 2)$. We will construct an upper solution based on $v(x; x_0, y_0)$ in the case that x_0 satisfies

$$-h_1 < x_0 < h_2. \tag{3.1}$$

(Note that this condition holds in particular in the following cases:

- $v(x)$ is a symmetric function and $x_0 = 0$;
- $g_1(y) \ge H$, $g_2(y) \ge H$ for some $H > 0$ (as in [4, 6]) and $x_0 \in (-H, H)$.)

Since the graph of v contacts $\partial_i\Omega$ ($i = 1, 2$) perpendicularly, we can find a small number $\varepsilon_0 \in (0, 1)$ such that, for any function $w(x)$ defined over I satisfying

$$v(x) \le w(x) \le v(x) + \varepsilon_0, \quad |w_x(x) - v_x(x)| \le \varepsilon_0 \quad \text{for } x \in I, \tag{3.2}$$

its graph contacts $\partial_1\Omega$ (respectively $\partial_2\Omega$) at exactly one point $(\zeta_1, w(\zeta_1))$ (respectively $(\zeta_2, w(\zeta_2))$) with $\zeta_1, \zeta_2 \in I$. Moreover, since $g_1'(\xi_1) = v_x(\xi_1) > 0$, $g_2'(\xi_2) = -v_x(\xi_2) > 0$ in case (3.1) holds, we can choose ε_0 sufficiently small such that $g_1'(\zeta_1) > 0$, $g_2'(\zeta_2) > 0$ and so

$$x_0 - \widetilde{X} \le \zeta_1 \le \xi_1 \le -h_1 < x_0 < h_2 \le \xi_2 \le \zeta_2 \le x_0 + \widetilde{X}.$$

Step 2. Construction of an upper solution. To construct an upper solution we need some parameters. Set

$$M_i := \frac{G[A(1+\widetilde{G}^2)^{3/2} + K_i\widetilde{G}]}{1 - G\widetilde{G}} + K_i \quad (i = 1, 2).$$

Choose $\beta > M_1 + M_2 + 1$ sufficiently large such that

$$M_1 + \beta + (M_1 - \beta)e^{\beta(h_1+x_0)} < 0 < (\beta - M_2)e^{\beta(h_2-x_0)} - M_2 - \beta. \tag{3.3}$$

For such a β, choose a large t_0 such that

$$T := t_0 - \frac{1}{\beta^2}\Big\{\Big|\ln\frac{\varepsilon_0}{\beta}\Big| + \Big|\ln\frac{A}{\beta^2}\Big| + \Big|\ln\frac{(\widetilde{X} - X)(1 - G\widetilde{G})}{G}\Big| + \ln(1 + e^{\beta\widetilde{X}})\Big\} > 0.$$

Using these parameters we define two functions

$$\rho(x, t) := e^{\beta^2(t-t_0)}(e^{\beta(x-x_0)} + e^{-\beta(x-x_0)}), \quad (x, t) \in D := I \times [0, T]$$

and

$$\bar{u}(x, t) := v(x; x_0, y_0) + \rho(x, t), \quad (x, t) \in D. \tag{3.4}$$

We will show that $\bar{u}$ is an upper solution.

Step 3. Some a priori *estimates on* $\bar{u}$. By the choice of t_0,

$$\beta^2(T - t_0) \le \ln\frac{\varepsilon_0}{\beta(1 + e^{\beta\widetilde{X}})} < \ln\frac{\varepsilon_0}{1 + e^{\beta\widetilde{X}}} < 0.$$

Hence,

$$|\bar{u}(x, t) - v(x)| = |\rho(x, t)| \le e^{\beta^2(T-t_0)}(1 + e^{\beta\widetilde{X}}) \le \varepsilon_0, \quad (x, t) \in D$$

and

$$|\bar{u}_x(x, t) - v_x(x)| = |\rho_x(x, t)| \le \beta e^{\beta^2(T-t_0)}(1 + e^{\beta\widetilde{X}}) \le \varepsilon_0, \quad (x, t) \in D.$$

These inequalities imply that $\bar{u}(\cdot, t)$ satisfies the conditions for w in (3.2) for each $t \in [0, T]$ and so its graph contacts $\partial_1\Omega$ (respectively $\partial_2\Omega$) at exactly one point, whose x-coordinate satisfies

$$x_0 - \widetilde{X} \le \bar{\zeta}_1(t) \le \xi_1 \le -h_1 < x_0 \quad (\text{respectively } x_0 < h_2 \le \xi_2 \le \bar{\zeta}_2(t) \le x_0 + \widetilde{X}).$$

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Now we prove that

$$(\bar{u}_x)^2 = (v_x + \rho_x)^2 \le v_x^2, \quad (x,t) \in D. \tag{3.5}$$

First, for $(x,t) \in D$ with $x \le x_0$, we have $\rho_x(x,t) \le 0$ and, with $z := x_0 - x \in [0, \widetilde{X}]$,

$$\begin{aligned} v_x(x) + \rho_x(x,t) &= \frac{z}{\sqrt{A^{-2} - z^2}} - \beta e^{\beta^2(t-t_0)}(e^{\beta z} - e^{-\beta z}) \\ &\ge \frac{z}{\sqrt{A^{-2} - z^2}} - \beta e^{\beta^2(T-t_0)}(e^{\beta z} - e^{-\beta z}) \\ &\ge F(z) := \frac{z}{\sqrt{A^{-2} - z^2}} - \frac{A}{\beta(1 + e^{\beta \widetilde{X}})}(e^{\beta z} - e^{-\beta z}), \end{aligned}$$

since $\beta e^{\beta^2(T-t_0)} \le A/[\beta(1 + e^{\beta\widetilde{X}})]$ by the choice of t_0. Note that $F(0) = 0$ and

$$F'(z) = \frac{A^{-2}}{(A^{-2} - z^2)^{3/2}} - \frac{A}{1 + e^{\beta\widetilde{X}}}(e^{\beta z} + e^{-\beta z}) \ge 0 \quad \text{for } z \in [0, \widetilde{X}].$$

We conclude that $F(z) \ge 0$ and so

$$v_x \ge v_x + \rho_x \ge \rho_x \ge -v_x \quad \text{for } (x,t) \in D \text{ with } x \le x_0.$$

In a similar way one can show that

$$\rho_x \ge 0 \quad \text{and} \quad v_x \le v_x + \rho_x \le \rho_x \le -v_x \quad \text{for } (x,t) \in D \text{ with } x \ge x_0.$$

The inequality (3.5) then follows from these estimates.

Next, by the choice of t_0, we can prove that

$$v_{xx} + \rho_{xx} \le -A + \beta^2 \rho \le -A + \beta^2 e^{\beta^2(t-t_0)}(1 + e^{\beta\widetilde{X}}) \le -A + A = 0, \quad (x,t) \in D. \tag{3.6}$$

Furthermore, for $t \in [0, T]$, there exists θ lying between $\bar{\zeta}_1(t)$ and ξ_1 such that

$$\begin{aligned} |\bar{\zeta}_1(t) - \xi_1| &= |g_1(\bar{u}(\bar{\zeta}_1(t), t)) - g_1(v(\xi_1))| \le G|\bar{u}(\bar{\zeta}_1(t), t) - v(\xi_1)| \\ &\le G[|\bar{u}(\bar{\zeta}_1(t), t) - \bar{u}(\xi_1, t)| + |\bar{u}(\xi_1, t) - v(\xi_1)|] \\ &\le G|\bar{u}_x(\theta, t)| \cdot |\bar{\zeta}_1(t) - \xi_1| + G\rho(\xi_1, t) \\ &\le G \cdot \max_{x \in I} |v_x| \cdot |\bar{\zeta}_1(t) - \xi_1| + G\rho(\xi_1, t) \\ &\le G\widetilde{G}|\bar{\zeta}_1(t) - \xi_1| + G\rho(\xi_1, t) \end{aligned}$$

and so, by the choice of t_0,

$$|\bar{\zeta}_1(t) - \xi_1| \le \frac{G}{1 - G\widetilde{G}}\rho(\xi_1, t) \le \frac{G}{1 - G\widetilde{G}} e^{\beta^2(T-t_0)}(1 + e^{\beta\widetilde{X}}) \le \widetilde{X} - X. \tag{3.7}$$

Step 4. Verification of the upper solution.

LEMMA 3.1. *Let $\bar{u}$ be defined as in (3.4). If (3.1) holds, then $\bar{u}$ is an upper solution of (1.2)–(1.3) in the time interval* $[0, T]$.

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Proof. First, for $(x,t) \in \{(x,t) \mid \bar\zeta_1(t) < x < \bar\zeta_2(t), t \in [0,T]\} \subset D$, by (3.5) and (3.6),

$$\bar u_t - \frac{\bar u_{xx}}{1+\bar u_x^2} - A\sqrt{1+\bar u_x^2} \geq \beta^2\rho - \frac{v_{xx}+\rho_{xx}}{1+v_x^2} - A\sqrt{1+v_x^2} = \beta^2\rho - \frac{\beta^2\rho}{1+v_x^2} \geq 0.$$

Next we consider the boundary conditions. For $t \in [0,T]$, there are some θ_1, θ_3 lying between $\bar\zeta_1(t)$ (for simplicity, we write it as $\bar\zeta_1$) and ξ_1 and some θ_2 lying between $v(\xi_1)$ and $\bar u(\bar\zeta_1, t)$ such that

$$\begin{aligned}
\bar u_x(\bar\zeta_1,t) - g_1'(\bar u(\bar\zeta_1,t)) &= v_x(\bar\zeta_1) + \rho_x(\bar\zeta_1,t) - g_1'(\bar u(\bar\zeta_1,t))\\
&= v_x(\xi_1) + v_{xx}(\theta_1)(\bar\zeta_1 - \xi_1) - g_1'(v(\xi_1)) - g_1''(\theta_2)[\bar u(\bar\zeta_1,t) - v(\xi_1)] + \rho_x(\bar\zeta_1,t)\\
&= v_{xx}(\theta_1)(\bar\zeta_1 - \xi_1) - g_1''(\theta_2)[\rho(\bar\zeta_1,t) + v(\bar\zeta_1) - v(\xi_1)] + \rho_x(\bar\zeta_1,t)\\
&= [v_{xx}(\theta_1) - g_1''(\theta_2)v_x(\theta_3)](\bar\zeta_1 - \xi_1) - g_1''(\theta_2)\rho(\bar\zeta_1,t) + \rho_x(\bar\zeta_1,t)\\
&\leq [A(1+\widetilde G^2)^{3/2} + K_1\widetilde G]\cdot|\bar\zeta_1 - \xi_1| + K_1\rho(\bar\zeta_1,t) + \rho_x(\bar\zeta_1,t)\\
&\leq \frac{G[A(1+\widetilde G^2)^{3/2} + K_1\widetilde G]}{1 - G\widetilde G}\rho(\xi_1,t) + K_1\rho(\bar\zeta_1,t) + \rho_x(\bar\zeta_1,t)\\
&\leq M_1\rho(\bar\zeta_1,t) + \rho_x(\bar\zeta_1,t).
\end{aligned}$$

The penultimate inequality follows from (3.7) and the last follows from $\bar\zeta_1 \leq \xi_1 < x_0$. Therefore, for $t \in [0,T]$,

$$\begin{aligned}
e^{-\beta^2(t-t_0)}[\bar u_x(\bar\zeta_1,t) - g_1'(\bar u(\bar\zeta_1,t))] &\leq M_1(1 + e^{-\beta(\bar\zeta_1 - x_0)}) + \beta(1 - e^{-\beta(\bar\zeta_1 - x_0)})\\
&\leq M_1 + \beta + (M_1 - \beta)e^{\beta(h_1+x_0)} < 0,
\end{aligned}$$

by the choice of β in (3.3). Similarly, one can prove that

$$\bar u_x(\bar\zeta_2(t),t) \geq -g_2'(\bar u(\bar\zeta_2(t),t)), \quad t \in [0,T].$$

Therefore, $\bar u(x,t)$ is an upper solution of (1.2)–(1.3) in the time interval $[0,T]$. □

In a similar way, one can prove the following result.

Lemma 3.2. *Assume (3.1). For sufficiently large β and t_0, $\underline{u}(x,t) := v(x;x_0,y_0) - \rho(x,t)$ is a lower solution of (1.2)–(1.3) in the time interval $[0,T_1]$, provided $T_1 > 0$ is small.*

3.2. Entire solutions. Let $v(x;x_0,y_0)$ be a solution of (2.1), whose graph contacts $\partial_1\Omega$ perpendicularly at a point $(-g_1(Y_1), Y_1)$ and contacts $\partial_2\Omega$ perpendicularly at $(g_2(Y_2), Y_2)$. Then $K(Y_1,Y_2) = 0$ by Theorem 2.1. If, for some $\varepsilon > 0$,

$$K(y_1,y_2) > 0, \quad y_1 \in (Y_1, Y_1+\varepsilon),\ y_2 \in (Y_2, Y_2+\varepsilon), \tag{3.8}$$

then $v(x;x_0,y_0)$ is *unstable from above*. In fact, for $y_i \in (Y_i, Y_i+\varepsilon)$ $(i = 1,2)$, by Theorem 2.1, $r(x;y_1,y_2)$ is a lower solution and so its corresponding circular arc has a curvature larger than $-A$. This implies that the solution of (1.2)–(1.3) starting from $r(x;y_1,y_2)$ moves upward monotonically. Thus, v is unstable from above. Similarly, we can see that $v(x;x_0,y_0)$ is a stationary solution *unstable from below* if, for some $\varepsilon > 0$,

$$K(y_1,y_2) < 0, \quad y_1 \in (Y_1-\varepsilon, Y_1),\ y_2 \in (Y_2-\varepsilon, Y_2). \tag{3.9}$$

In this section we construct entire solutions starting from a stationary solution unstable from above or below.

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Theorem 3.3. *Let $v(x;x_0,y_0)\equiv r(x;Y_1,Y_2)$ be a solution of (2.1), which is unstable from above in the sense that (3.8) holds. Assume that $x_0\in(-g_1(Y_1),g_2(Y_2))$. Then there is an entire solution $\mathcal{U}(x,t)$ of (1.2)–(1.3) such that $\mathcal{U}_t(x,t)>0$, $\mathcal{U}(\cdot,t)\to v(\cdot)$ as $t\to-\infty$ and, as $t\to\infty$, $\mathcal{U}(\cdot,t)$ either goes upward to infinity or converges to another stationary solution.*

Proof. From the assumption $x_0\in(-g_1(Y_1),g_2(Y_2))$,

$$g_1'(Y_1)=v_x(-g_1(Y_1))>0\quad\text{and}\quad g_2'(Y_2)=-v_x(g_2(Y_2))>0.$$

Hence, there exists $\varepsilon_1\in(0,\varepsilon/2)$ such that $g_i'(y)>0$ for $y\in[Y_i,Y_i+2\varepsilon_1]$ $(i=1,2)$. Modifying the functions g_1 and g_2 outside the interval $[Y_i,Y_i+2\varepsilon_1]$ $(i=1,2)$ if necessary, we may assume that

$$-g_1(Y_1)\le -h_1:=-\min_{y\in[y_0,y_0+A^{-1}+1]}g_1(y)<x_0<h_2:=\min_{y\in[y_0,y_0+A^{-1}+1]}g_2(y)\le g_2(Y_2).$$

The function $\bar{u}(x,t)$ constructed in Section 3.1 is an upper solution of (1.2)–(1.3) as long as $\bar{u}(x,t)-v(x)=\rho(x,t)\le 2\varepsilon_1$ (since this inequality implies that the graph of $\bar{u}$ contacts $\partial_i\Omega$ at some points where the functions g_i are still unmodified).

For any large positive integer n with $1/n<\varepsilon_1$, we consider the solution $u_n(x,t)$ of the problem (1.2)–(1.3) with initial data $u_n(x,0)=r_n(x):=r(x;Y_1+1/n,Y_2+1/n)$. By the condition (3.8),

$$u_{nt}(x,0)=\frac{r_{nxx}}{1+r_{nx}^2}+A\sqrt{1+r_{nx}^2}>0.$$

By the maximum principle, $u_n(x,t)$ is a monotonically (strictly) increasing solution, that is, $u_{nt}(x,t)>0$ for any (x,t) in the domain of definition of u_n. In addition, by (3.8), u_n will propagate upward over the points $(-g_1(Y_1+2\varepsilon_1),Y_1+2\varepsilon_1)$ and $(g_2(Y_2+2\varepsilon_1),Y_2+2\varepsilon_1)$ before it tends to another possible stationary solution. For each large n, define t_n as follows:

$$t_n:=\sup\{s\mid u_n(\zeta_1(t),t)<Y_1+2\varepsilon_1,\ u_n(\zeta_2(t),t)<Y_2+2\varepsilon_1,\ t\in[0,s]\}.$$

We now use Lemma 3.1 to prove the following claim.

Claim. $t_n\to\infty$ as $n\to\infty$.

Proof. In fact, for any given $\tau>0$, we can take $\varepsilon_0\le 2\varepsilon_1$ and take t_0 sufficiently large such that $\tau-t_0\ll -1$ and $\tau<T$. Then the function $\bar{u}(x,t)$ defined in Section 3.1 satisfies

$$\begin{aligned}\bar{u}(\bar{\zeta}_1(\tau),\tau)&=v(\bar{\zeta}_1(\tau))+\rho(\bar{\zeta}_1(\tau),\tau)\le v(-g_1(Y_1))+e^{\beta^2(\tau-t_0)}(1+e^{\beta\widetilde{X}})\\&=Y_1+e^{\beta^2(\tau-t_0)}(1+e^{\beta\widetilde{X}})<Y_1+2\varepsilon_1,\end{aligned}$$

thanks to $\tau-t_0\ll-1$. Similarly, $\bar{u}(\bar{\zeta}_2(\tau),\tau)<Y_2+2\varepsilon_1$. Given such a $\bar{u}(x,t)$, we see that, when n is sufficiently large, $u_n(x,0)=r_n(x)<\bar{u}(x,0)$ on their common domain. By Lemma 3.1, $\bar{u}$ is an upper solution on the time interval $[0,T]$, so, by the maximum principle,

$$u_n(\zeta_i(\tau),\tau)\le\bar{u}(\bar{\zeta}_i(\tau),\tau)<Y_i+2\varepsilon_1\quad(i=1,2).$$

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By the definition of t_n, either $u_n(\zeta_1(t_n), t_n) = Y_1 + 2\varepsilon_1$ or $u_n(\zeta_2(t_n), t_n) = Y_2 + 2\varepsilon_1$. Hence,

$$\text{either} \quad u_n(\zeta_1(\tau), \tau) < u_n(\zeta_1(t_n), t_n) \quad \text{or} \quad u_n(\zeta_2(\tau), \tau) < u_n(\zeta_2(t_n), t_n).$$

Since u_n is strictly increasing in t, it follows that $t_n > \tau$. This proves the claim. □

For each large n, define

$$U_n(x, t) := u_n(x, t + t_n) \quad \text{for } t \geq -t_n.$$

Then U_n solves the following problem:

$$\begin{cases} U_{nt} = \dfrac{U_{nxx}}{1 + U_{nx}^2} + A\sqrt{1 + U_{nx}^2}, & \zeta_1(t + t_n) < x < \zeta_2(t + t_n),\ t > -t_n, \\ U_{nx}(\zeta_1(t + t_n), t) = g_1'(U_n(\zeta_1(t + t_n), t)), & t > -t_n, \\ U_{nx}(\zeta_2(t + t_n), t) = -g_2'(U_n(\zeta_2(t + t_n), t)), & t > -t_n, \\ U_n(\zeta_i(t_n), 0) = Y_i + 2\varepsilon_1, & i = 1 \text{ or } i = 2. \end{cases}$$

This problem can be converted into another quasilinear parabolic equation with Neumann boundary conditions by straightening the boundaries $\partial_i\Omega$ with isothermic coordinates as in [4, 6] (if necessary, one can first consider the problem in a piece $\Omega \cap \{(x, y) \mid \min\{Y_1, Y_2\} - L \leq y \leq \max\{Y_1, Y_2\} + L\}$ of Ω for each $L > 0$ and then regard the problem as one with $2L$-periodic boundaries). Thus, one can use the parabolic estimates and Cantor's diagonal argument as in [4, 6] to conclude that a subsequence of the triple $\{(U_n(x, t), \zeta_1(t + t_n), \zeta_2(t + t_n))\}$ converges to a triple $\{(\mathcal{U}(x, t), \widetilde{\zeta}_1(t), \widetilde{\zeta}_2(t))\}$ (all of the components are defined in $\mathbb{R}$, since $t_n \to \infty$) such that

$$\begin{cases} \mathcal{U}_t = \dfrac{\mathcal{U}_{xx}}{1 + \mathcal{U}_x^2} + A\sqrt{1 + \mathcal{U}_x^2}, & \widetilde{\zeta}_1(t) < x < \widetilde{\zeta}_2(t),\ t \in \mathbb{R}, \\ \mathcal{U}_x(\widetilde{\zeta}_1(t), t) = g_1'(\mathcal{U}(\widetilde{\zeta}_1(t), t)), & t \in \mathbb{R}, \\ \mathcal{U}_x(\widetilde{\zeta}_2(t), t) = -g_2'(\mathcal{U}(\widetilde{\zeta}_2(t), t)), & t \in \mathbb{R}, \\ \mathcal{U}(\widetilde{\zeta}_i(0), 0) = Y_i + 2\varepsilon_1, & i = 1 \text{ or } i = 2. \end{cases}$$

Since u_n is monotonically increasing in t, we have $\mathcal{U}_t(x, t) > 0$ for all $t \in \mathbb{R}$ by the maximum principle. Therefore, as $t \to -\infty$, $\mathcal{U}(\cdot, t)$ goes downward and converges to the stationary solution $v(x)$. As $t \to \infty$, $\mathcal{U}(\cdot, t)$ goes upward to infinity (when there are no other stationary solutions lying above $v(x)$) or converges to another stationary solution (when there is a stationary solution lying above $v(x)$). □

REMARK 3.4. Note that the complicated upper solution $\bar{u}(x, t)$ constructed in Section 3.1 is only used to prove that $t_n \to \infty$ in the claim in the above proof. This limit means that a solution u starting from the neighbourhood of an unstable stationary solution v takes a very long time to leave v. This fact guarantees that the limiting function $\mathcal{U}(x, t)$ is defined for all $t < 0$ and so is an entire solution.

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In a similar way we can prove the following result.

THEOREM 3.5. *Let $v(x; x_0, y_0) \equiv r(x; Y_1, Y_2)$ be a solution of (2.1), which is unstable from below in the sense that (3.9) holds. Assume that $x_0 \in (-g_1(Y_1), g_2(Y_2))$. Then there is an entire solution $\mathcal{U}(x,t)$ of (1.2)–(1.3) such that $\mathcal{U}_t(x,t) < 0$, $\mathcal{U}(\cdot,t) \to v(\cdot)$ as $t \to -\infty$ and, as $t \to \infty$, $\mathcal{U}(\cdot,t)$ either goes downward to infinity or converges to another stationary solution.*

REMARK 3.6. Finally, we comment on Yau's famous problem (cf. [3]): *is it possible to evolve a closed plane curve γ_1 to converge to another one γ_2 (perhaps, up to an isometry), either in finite time or in infinite time, using a parabolic curvature flow?* In [3], the authors constructed a curvature flow and gave a positive answer to this interesting problem, in case both γ_1 and γ_2 are simple, convex, closed curves. Note that our results are related to and different from this problem. First, each solution $\mathcal{U}$ obtained in Theorems 3.3 and 3.5 is an entire one, satisfying $\mathcal{U}(x, -\infty) = \Gamma_1$ rather than $\mathcal{U}(x, 0) = \Gamma_1$. Second, any entire solution connects two stationary ones rather than two arbitrarily given curves. Third, the curves in this paper are not closed ones but with end points on the boundaries of Ω.

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