A Determinantal Expansion for a Class of Definite Integral

Part 3. Generalised Continued Fractions

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1. We have shown in [1] that under certain conditions the definite integral $\int_a^b \frac{A^2(x)\,w(x)}{C(x)}\,dx$ may be approximated by a determinantal ratio. It is our object now to develop the theory when $C(x)$ is a polynomial, showing the relation to the continued fraction form for $\int_a^b \frac{w(x)\,dx}{z-x}$. In particular we shall give various forms for the approximants, and an integral form for the numerator.

2. From [1] we have the expansion

$$F(z_1, z_2, \ldots, z_n) = \int_a^b \frac{w(x)\,dx}{\prod_1^n (z_j - x)} = \sum_{s=0}^{\infty} \frac{|\,\alpha_0, \gamma_{01}, \gamma_{12}, \ldots, \gamma_{s-1,s}\,|^2}{\Delta_{s-1}\,\Delta_s} \tag{1}$$

$$= -\lim_{s\to\infty} \begin{vmatrix} 0 & \alpha_0 & \alpha_1 & \ldots & \alpha_s \\ \alpha_0 & \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0s} \\ \alpha_1 & \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1s} \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \alpha_s & \gamma_{s0} & \gamma_{s1} & \cdots & \gamma_{ss} \end{vmatrix} \div \Delta_s \tag{2}$$

$$= \lim_{s\to\infty} \frac{N_{s+1}(z_1, z_2, \ldots, z_n)}{D_{s+1}(z_1, z_2, \ldots, z_n)} \tag{3}$$

where [1]

$$\alpha_s = \int_a^b \theta_s(z)\,w(x)\,dx, \qquad \gamma_{r,s} = \int_a^b \theta_r(x)\,\theta_s(x) w(x) \prod_1^n (z_j - x)\,dx,$$

$$\Delta_s = D_{s+1}(z) = |\,\gamma_{00}, \gamma_{11}, \ldots, \gamma_{ss}\,|,$$
$$w(x) \geqq 0 \text{ for } a \leqq x \leqq b \qquad (a, b \text{ finite}),$$

$$\int_a^b w(x)\,dx \text{ exists and is positive,}$$

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[1] $N_s(z_1, z_2, \ldots, z_n)$ and $D_s(z, z_2, \ldots z_n)$ etc. will be abbreviated to $N_s(z)$ and $D_s(z)$ when ambiguity is unlikely.

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$\prod_1^n (z_j - x) \equiv (z_1 - x)(z_2 - x) \ldots (z_n - x) > 0$ and the z's are distinct,

$\theta_s(x)$ is an arbitrary polynomial of precise degree s with highest coefficient λ_s.

We shall refer to (1) and (2) as continued fractions (C.F.'s) of the nth order, and $R_s(z_1, z_2, \ldots, z_n) = N_s/D_s$ as the sth approximant or convergent.

The expansions (1) and (2) arise from a consideration of the minimum value of

$$S^2 = \int_a^b w(x)\,\Pi(z-x)\left\{\frac{1}{\Pi(z-x)} - \sum_{s=0}^{r-1} A_s q_s(x)\right\}^2 dx \tag{4}$$

where $q_s(x) = \mid \theta_0(x), \gamma_{01}, \gamma_{12}, \ldots, \gamma_{s-1,s} \mid /\{(-)^s\lambda_s\Delta_{s-1}\}$, (5)
and $\{q_s(x)\}$ is an orthogonal system with respect to the weight function $w(x)\Pi(z-x)$, the highest coefficient in $q_s(x)$ being unity. Indeed if we write

$$\phi_r = \int_a^b w(x)\prod_1^n (z_j - x)\, q_r^2(x)\, dx \tag{6}$$

then
$$A_r\phi_r = \int_a^b q_r(x) w(x)\, dx \tag{7}$$

and
$$S^2{}_{\text{Min}} = F(z) - \sum_0^{r-1} A_s^2\phi_s. \tag{8}$$

It may be remarked in passing that a consideration of the minimum value of $\int_a^b (z-x)\,w(x)\ \{(z-x)^{-1} - \Sigma A_s q_s(x)\}^2 dx$ and of $\int_a^b x(z-x)w(x)\{(z-x)^{-1} - \Sigma A^1_s\, q^1_s\,(x)\}^2\, dx$ leads to continued fractions for $\int_a^b (z-x)^{-1}\, w(x)\, dx$ related to the 'even' and 'odd' parts of a Stieltjes type of continued fraction. The present approach shows immediately the central part played by orthogonal polynomials, and although in essence both these expressions were considered by Stieltjes [2], it is only at a later stage that the orthogonality property emerges.

3. We shall now consider various forms for $N_s(z)$ and $D_s(z)$. These arise by taking (a) $\theta_s(x) = (\nu - x)^s$;

(b) $\theta_s(x) = p_s(x)$,

where $\{p_s(x)\}$ is an orthonormal system with respect to $w(x)$, and $p_s(x)$ has highest coefficient k_s;

$$(c)\ \theta_s(x) = q_s(x) = \frac{|\,p_s(x),\ p_{s+1}(z_1),\ p_{s+2}(z_2), \ldots, p_{s+n}(z_n)\,|}{k_{s+n}\Pi(z-x)\ |\,p_s(z_1),\ p_{s+1}(z_2), \ldots p_{s+n-1}(z_n)\,|} \tag{9}$$

and the system $\{q_s(x)\}$ is orthogonal with respect to $w(x)\Pi(z-x)$.

(*a*) Here $\alpha_s = \int_a^b (\nu - x)^s w(x)\,dx = m_s$ say, (10)

$$\gamma_{r,s} = \int_a^b (\nu - x)^{r+s} w(x)\Pi\,(z-x)\,dx = M_{r+s+n} \text{ say.} \tag{11}$$

For particular choices of $w(x)$, m_s is an Appell polynomial. Further, if $z_1 = z_2 = \ldots = z_n = \nu$ then $\gamma_{r,s} = m_{r+s+n}$. From (5) we have, in the notation of persymmetric determinants,

$$\phi_r = \frac{P_{r+1}(M_n, M_{n+1}, \ldots, M_{n+2r})}{P_r(M_n, M_{n+1}, \ldots, M_{n+2r-2})} \tag{12}$$

and $\sum\limits_{s=0}^{r-1} A_s^2 \phi_s$

$$= -\begin{vmatrix} 0 & m_0 & m_1 \ldots & m_{r-1} \\ m_0 & M_n & M_{n+1} \ldots & M_{n+r-1} \\ m_1 & M_{n+1} & M_{n+2} \ldots & M_{n+r-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m_{r-1} & M_{n+r-1} & M_{n+r} \ldots & M_{n+2r-2} \end{vmatrix} \div P_r(M_n, M_{n+1}, \ldots M_{n+2r-2}). \tag{13}$$

If the roots z_j are equal and $z_j = \nu$, M_s is to be replaced by m_s.

(*b*) The polynomials $p_s(x)$ follow a recurrence relation

$$\begin{aligned} p_s(x) &= (A_s x + B_s)\ p_{s-1}(x) - C_s p_{s-2}(x), & s &= 2, 3, \ldots \\ p_1(x) &= (A_1 x + B_1)p_0, & p_0 &= k_0 \\ A_r &= k_r/k_{r-1} > 0, & C_r &= A_r/A_{r-1} > 0, \end{aligned} \tag{14}$$

which may be written

$$(x - z_1)p_{s-1}(x) = A_s^{-1}p_s(x) - (z_1 + B_s A_s^{-1})p_{s-1}(x) + A_{s-1}^{-1}p_{s-2}(x). \tag{15}$$

We require the following generalisation of (15):

$$\prod_{j=1}^{n}(x - z_j)p_{s-1}(x) = e_{s-1,n}p_{s+n-1} + e_{s-1,n-1}p_{s+n-2} + \ldots$$
$$+ e_{s-1,0}p_{s-1} + e_{s-2,1}p_{s-2} + e_{s-3,2}p_{s-3} + \ldots + e_{s-n-1,n}p_{s-n-1} \tag{16}$$

where $e_{s-1,n} = k_{s-1}/k_{s+n-1}$, and $e_{s,r}$ is to be taken as zero if $s < 0$. The notation is justified by the identity

$$\int_a^b [p_{s-1}(x)\Pi\,(x-z)]\,p_r(x)\,w(x)\,dx = \int_a^b [p_r(x)\,\Pi(x-z)]p_{s-1}(x)\,w(x)\,dx.$$

For example,

$$(x - z_1)(x - z_2)\,p_{s-1}(x) = f_{s-1}p_{s+1} + g_{s-1}p_s + h_{s-1}p_{s-1}$$
$$+ g_{s-2}p_{s-2} + f_{s-3}p_{s-3} \quad (s = 1, 2, \ldots), \tag{17}$$

where
$$\begin{cases} f_{s-1} = A_s^{-1} A_{s+1}^{-1}, \ s = 1, 2, \ldots, \\ -A_s g_{s-1} = B_{s+1} A_{s+1}^{-1} + z_1 + z_2 + B_s A_s^{-1}, \ s = 1, 2, \ldots \\ h_{s-1} = A_s^{-2} + (z_1 + B_s A_s^{-1})(z_2 + B_s A_s^{-1}) + A_{s-1}^{-2}, \ s = 2, 3, \ldots \\ h_0 = A_1^{-2} + (z_1 + B_1 A_1^{-1})(z_2 + B_1 A_1^{-1}). \end{cases}$$

From (6) and (7) we find, after using (16),

$$\phi_r = \frac{(-)^n}{k_r^2} \frac{K_{r+1}(e_{00}, e_{01}, e_{02}, \ldots, e_{0n})}{K_r(e_{00}, e_{01}, e_{02}, \ldots, e_{0n})}, \quad r = 0, 1, \ldots \tag{18}$$

and

$$\sum_{s=0}^{r-1} A_s^2 \phi_s = (-)^n k_0^{-2} \frac{K_{r-1}(e_{10}, e_{11}, \ldots, e_{1n})}{K_r(e_{00}, e_{01}, \ldots, e_{0n})},$$
$$r = 1, 2, \ldots, K_0 = 1, \tag{19}$$

where we have introduced the notation $K_r\ (e_{00}, e_{01}, \ldots, e_{0n})$ for a generalised continuant determinant of order r, symmetric, with elements $e_{00}, e_{10}, e_{20}, \ldots$ in the diagonal through (1, 1), $e_{01}, e_{11}, e_{21}, \ldots$ in the diagonal through (1, 2), and so on. We shall refer to these as continuants of the nth kind. Thus the ratio of continuants of the 1st kind is related to C.F's of the first order, the determinants concerned consisting of elements in three diagonals only. Similarly C.F.'s of the second order are associated with the ratio of continuants of the 2nd kind which in turn have elements in five diagonals only.

(c) Writing for simplicity

$$| \, p_s(x), p_{s+1}(z_1), \ldots, p_{s+n}(z_n) \, | = A_s(x, z_1, z_2, \ldots, z_n)$$

we have
$$q_s(x) = \frac{A_s(x, z_1, \ldots, z_n)}{k_{s+n} \prod_1^n (z_j - x) A_s(z_1, z_2, \ldots, z_n)}$$

$$= \frac{1}{k_s A_s(z_1, \ldots, z_n)} \sum_{r=0}^{s} | \, p_r(z_1), p_{s+1}(z_2), p_{s+2}(z_3), \ldots, p_{s+n-1}(z_n) \, | \, p_r(x) \tag{20}$$

by a generalisation of a theorem of Darboux quoted in [1], (17).

Hence from (6) and (7) we find

$$\phi_r = \frac{1}{k_r k_{r+n}} \frac{A_{r+1}(z_1, z_2, \ldots, z_n)}{A_r(z_1, z_2, \ldots, z_n)} \tag{21}$$

and

$$\sum_0^{r-1} A_s^2 \phi_s = \sum_{s=0}^{r-1} \frac{k_{s+n} \, | \, z_1^0, p_{s+1}(z_2), p_{s+2}(z_3), \ldots, p_{s+n-1}(z_n) \, |^2}{k_s A_s(z_1, z_2, \ldots, z_n) A_{s+1}(z_1, z_2, \ldots, z_n)}. \tag{22}$$

4. If we now consider the value of $\prod_{0}^{r-1}\phi_j$, we have from (18) and (21)

$$D_r(z_1, z_2, \ldots, z_n) = (-)^{rn}K_r(e_{00}, e_{01}, \ldots, e_{0n})$$

$$= \frac{A_r(z_1, z_2, \ldots, z_n)}{\left| z_1^0, z_2^1, \ldots, z_n^{n-1} \right| \prod_{r}^{n+r-1} k_j} \tag{23}$$

since $A_0(z_1, \ldots, z_n) = \prod_{0}^{n-1} k_j \left| z_1^0, z_2^1, \ldots, z_n^{n-1} \right|$, and from (12) and (21)

$$D_r(z_1, z_2, \ldots, z_n) = \prod_{j=0}^{r-1} k_j^2 \, P_r(M_n, M_{n+1}, \ldots, M_{n+2r-2}) \tag{24}$$

and as a consequence

$$N_r(z_1, z_2, \ldots, z_n) = (-)^{rn} k_0^{-2} K_{r-1}(e_{10}, e_{11}, \ldots, e_{1n})$$

$$= -\prod_{j=0}^{r-1} k_j^2 \begin{vmatrix} 0 & m_0 & m_1 \ldots & m_{r-1} \\ m_0 & M_n & M_{n+1} \ldots & M_{n+r-1} \\ m_1 & M_{n+1} & & \\ \vdots & \vdots & & \\ m_{r-1} & M_{n+r-1} & & M_{n+2r-2} \end{vmatrix} \tag{25}$$

$$= \frac{A_r(z_1, \ldots, z_n)}{\prod_{r}^{r+n-1} k_j \left| z_1^0, z_2^1, \ldots, z_n^{n-1} \right|}$$

$$\times \sum_{s=0}^{r-1} \frac{k_{s+n} \left| z_1^0, p_{s+1}(z_2), \ldots, p_{s+n-1}(z_n) \right|^2}{k_s A_s(z_1, \ldots, z_n) \, A_{s+1}(z_1, \ldots, z_n)}, \tag{25a}$$

$$r = 1, 2, \ldots .$$

When the roots z_j are equal, the only change required in (23) is to replace

$$\frac{A_r(z_1, z_2, \ldots, z_n)}{\left| z_1^0, z_2^1, \ldots z_n^{n-1} \right|}$$

by

$$\frac{\left| p_r(z_1), p_{r+1}^{(1)}(z_1), p_{r+2}^{(2)}(z_1), \ldots, p_{r+n-1}^{(n-1)}(z_n) \right|}{(n-1)\,!\,!},$$

where superscripts refer to derivatives and

$$(n-1)\,!\,! = (n-1)!\,(n-2)!\ldots 1!\,0!.$$

A similar modification is required in (25a).

As an illustration we take $w(x) = 1/\sqrt{(1-x^2)}$, $a = -1$, $b = 1$,

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$$\prod_1^n (z_j - x) \equiv (z-x)^n, \qquad z > 1,$$

with
$$p_s(x) = \sqrt{\frac{2}{\pi}}\cos s\theta, \qquad \cos\theta = x, \qquad s = 1, 2, \ldots$$

$$p_0(x) = \sqrt{\frac{1}{\pi}} = k_0; \qquad k_s = 2^{s-1}\sqrt{\frac{2}{\pi}}, \qquad s = 1, 2, \ldots$$

and
$$m_s = \int_{-1}^{1} \frac{(z-x)^s}{\sqrt{(1-x^2)}}\,dx = \Pi\,(z^2-1)^{s/2}\bar{P}^s\left(\frac{z}{\sqrt{(z^2-1)}}\right)$$

where $\bar{P}_s(x)$ is Legendre's polynomial. With $t = z/\sqrt{(z^2-1)}$ we find from the modified form of (23), and (24),

$$\begin{vmatrix} b_r & b_{r+1} & \ldots b_{r+n-1} \\ rc_r & (r+1)c_{r+1} & \ldots (r+n-1)\,c_{r+n-1} \\ r^2 b_r & (r+1)^2 b_{r+1} & \ldots (r+n-1)^2 b_{r+n-1} \\ r^3 c_r & (r+1)^3 c_{r+1} & \ldots (r+n-1)^3 c_{r+n-1} \\ \vdots & \vdots & \vdots \end{vmatrix}_{(n)} = \frac{2^\chi (n-1)!!}{(t^2-1)^{r(r-1)/2}} P_r\{\bar{P}_n, \bar{P}_{n+1}, \ldots, \bar{P}_{n+2r-2}\}$$

$$n, r = 1, 2, \ldots$$

where
$$\begin{cases} b_r = (t+1)^r + (t-1)^r \\ c_r = (t+1)^r - (t-1)^r \\ 2\chi = (n+r)(n+r-1) + (r-1)(r-2) \end{cases}$$

and $\bar{P}_s$ stands for $\bar{P}_s(t)$. The result for $n = 1$ has been given by Geronimus [3].

5. *A formula for the numerators.* Consider the identity

$$|\, x^0, p\,(z_1), p_{r+1}(z_2), \ldots, p_{r+n-1}(z_n)\,| = \prod_1^n (z_j - x) \sum_{s=0}^{r-1} B_s q_s(x) \tag{27}$$

where
$$B_s\phi_s = \frac{|\, z_1^0, p_{s+1}(z_2), \ldots, p_{s+n-1}(z_n)\,|}{k_s}\,\frac{A_r(z_1, \ldots, z_n)}{A_s(z_1, \ldots, z_n)}$$

$$s = 0, 1, \ldots, r-1.$$

Then

$$\int_a^b \frac{|\, x^0, p_r(z_1), \ldots, p_{r+n-1}(z_n)\,|}{\prod_1^n (z_j - x) A_r(z_1, \ldots, z_n)}\, w(x)\,dx = \sum_{s=0}^{r-1} \frac{k_{s+n}\,|\, z_1^0, p_{s+1}(z_2), \ldots, p_{s+n-1}(z_n)\,|^2}{k_s\; A_s(z_1, \ldots, z_n)\; A_{s+1}(z_1, \ldots, z_n)}.$$

Hence from (22) it follows that

$$N_r(z_1, z_2, \ldots, z_n) = \int_a^b \frac{|\, x^0, p_r(z_1).p_{r+1}(z_2), \ldots, p_{r+n-1}(z_n)\,|}{\prod_r^{r+n-1} k_j\,|\, x^0, z_1^1, z_2^2, \ldots, z_n^n\,|}\, w(x)dx \tag{28}$$

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since $\prod_1^n (z_j - x) \mid z_1^0, z_2^1, \ldots, z_n^{n-1} \mid \; = \; \mid x^0, z_1^1, z_2^2, \ldots, z_n^n \mid$.

When $z_1 = z_2 = \ldots = z_n = z$, the determinant in the numerator of (28) is to be replaced by $\mid x^0, p_r(z), p_{r+1}^{(1)}(z), \ldots, p_{r+n-1}^{(n-1)}(z) \mid$, and that in the denominator by $(z-x)^n(n-1)!!$.

In the particular case $n = 1$ we have the well-known formula

$$N_r(z) = k_r^{-1} \int_a^b \frac{p_r(z) - p_r(x)}{z - x} w(x)\, dx,$$

and from (23) $D_r(z) = k_r^{-1} \; p_r(z)$, the ratio of these being the rth convergent of the C.F.

$$\frac{A_1 k_0^{-2}}{A_1 z + B_1} - \frac{C_2}{A_2 z + B_2} - \frac{C_3}{A_3 z + B_3} - \ldots$$

We shall consider the recurrence relations for the numerators and denominators of generalised C.F.'s, and some special properties of second order C.F.'s in Part 4.

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