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CORRIGENDUM

AVERAGING FORMULA FOR NIELSEN NUMBERS OF MAPS ON INFRA-SOLVMANIFOLDS OF TYPE (R) – CORRIGENDUM

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The authors gave an example showing an error in [2, Lemma 3.3], and below offer at least a partial correction for that error under the unimodularity assumption. This makes all of the remaining results in [2] valid.

Consider the three-dimensional solvable non-unimodular Lie algebra $\mathfrak{S}$:

$$\mathfrak{S} = \mathbb{R}^2 \rtimes_\sigma \mathbb{R}, \quad \text{where } \sigma(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}.$$

This Lie algebra has a faithful matrix representation as follows:

$$\begin{bmatrix} s & 0 & 0 & a \\ 0 & s & 0 & b \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can choose an ordered (linear) basis for $\mathfrak{S}$:

$$\mathbf{b}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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They satisfy $[\mathbf{b}_1, \mathbf{b}_2] = \mathbf{0}$, $[\mathbf{b}_3, \mathbf{b}_1] = \mathbf{b}_1$ and $[\mathbf{b}_3, \mathbf{b}_2] = \mathbf{b}_2$. The connected and simply connected solvable Lie group S associated with the Lie algebra $\mathfrak{S}$ is

$$S = \left\{ \begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^t & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \, x, y, t \in \mathbb{R} \right\}.$$

Let $g = ((x, y), t)$ denote an element of S. Because $\mathrm{Ad}(g) : \mathfrak{S} \to \mathfrak{S}$ is given by $\mathrm{Ad}(g)(A) = gAg^{-1}$ for $A \in \mathfrak{S}$, a simple computation shows that the adjoint of g is given by

$$\mathrm{Ad}(g) = \begin{bmatrix} e^t & 0 & -x \\ 0 & e^t & -y \\ 0 & 0 & 1 \end{bmatrix}.$$

Let φ be a Lie algebra homomorphism on $\mathfrak{S}$. Since $[\mathfrak{S}, \mathfrak{S}]$ is generated by $\mathbf{e}_1$ and $\mathbf{e}_2$, we have

$$\begin{aligned} \varphi(\mathbf{b}_1) &= m_{11}\mathbf{b}_1 + m_{21}\mathbf{b}_2, \\ \varphi(\mathbf{b}_2) &= m_{12}\mathbf{b}_1 + m_{22}\mathbf{b}_2, \\ \varphi(\mathbf{b}_3) &= p\mathbf{b}_1 + q\mathbf{b}_2 + m\mathbf{b}_3 \end{aligned}$$

for some $m_{ij}, p, q, m \in \mathbb{R}$. Since φ preserves the bracket operations $[\mathbf{b}_3, \mathbf{b}_1] = \mathbf{b}_1$ and $[\mathbf{b}_3, \mathbf{b}_2] = \mathbf{b}_2$, it follows easily that

$$\begin{aligned} m_{11}(m-1) &= 0, & m_{12}(m-1) &= 0, \\ m_{21}(m-1) &= 0, & m_{22}(m-1) &= 0. \end{aligned}$$

Therefore, with respect to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathfrak{S}$, φ is one of the following:

$$\text{Type (I)} \quad \begin{bmatrix} m_{11} & m_{12} & p \\ m_{21} & m_{22} & q \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Type (II)} \quad \begin{bmatrix} 0 & 0 & p \\ 0 & 0 & q \\ 0 & 0 & m \end{bmatrix} \quad \text{with } m \neq 1.$$

Now we can easily check that

$$\det(\varphi - I) = \begin{cases} 0 & \text{when } \varphi \text{ is of type (I)}, \\ m-1 & \text{when } \varphi \text{ is of type (II)}; \end{cases}$$

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$$\det(\varphi - \mathrm{Ad}(g)) = \begin{cases} 0 & \text{when } \varphi \text{ is of type (I)}, \\ e^{2t}(m-1) & \text{when } \varphi \text{ is of type (II)}. \end{cases}$$

This example shows that [2, Lemma 3.3] is not true in general. We remark also that S is not unimodular, and hence, as can be expected, $\det(\mathrm{Ad}(g)) = e^{2t} \neq 1$ for all $t \neq 0$. We prove, however, that the lemma is true under the unimodularity assumption of the connected Lie group. That is, the following theorem.

THEOREM 1. *Let S be a connected and simply connected solvable Lie group, and let $D : S \to S$ be a Lie group homomorphism. If S is unimodular, then for any $x \in S$,*

$$\det(I - D_*) = \det(I - \mathrm{Ad}(x)D_*).$$

REMARK 2. It is known that if a Lie group admits a lattice (discrete cocompact subgroup), then it is unimodular. Consequently, the remaining results of [2] are valid.

LEMMA 3. *Let S be a connected and simply connected solvable Lie group, and let $D : S \to S$ be a Lie group homomorphism. Then, for any $x \in S$, $I - D_*$ is an isomorphism if and only if $I - \mathrm{Ad}(x)D_*$ is an isomorphism.*

Proof. Because $I - \mathrm{Ad}(x^{-1})\mathrm{Ad}(x)D_* = I - D_*$, it suffices to show the only if.

Let $G = [S, S]$; then G is nilpotent, and $S/G \cong \mathbb{R}^k$ for some k. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & S & \longrightarrow & \mathbb{R}^k & \longrightarrow & 1 \\ & & \downarrow D' & & \downarrow D & & \downarrow \bar{D} & & \\ 1 & \longrightarrow & G & \longrightarrow & S & \longrightarrow & \mathbb{R}^k & \longrightarrow & 1 \end{array}$$

This induces the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{G} & \longrightarrow & \mathfrak{S} & \longrightarrow & \mathbb{R}^k & \longrightarrow & 1 \\ & & \downarrow I-D'_* & & \downarrow I-D_* & & \downarrow I-\bar{D}_* & & \\ 1 & \longrightarrow & \mathfrak{G} & \longrightarrow & \mathfrak{S} & \longrightarrow & \mathbb{R}^k & \longrightarrow & 1 \end{array}$$

For $x \in S$, we denote by τ_x the inner automorphism on S whose differential is $\mathrm{Ad}(x)$. This induces an automorphism on G, and we still

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denote it by τ_x and its differential is $\mathrm{Ad}'(x)$. Then we can express $I - D_*$ and $I - \mathrm{Ad}(x)D_*$ as

$$I - D_* = \begin{bmatrix} I - \bar{D}_* & 0 \\ * & I - D'_* \end{bmatrix},$$
$$I - \mathrm{Ad}(x)D_* = \begin{bmatrix} I - \bar{D}_* & 0 \\ * & I - \mathrm{Ad}'(x)D'_* \end{bmatrix}$$

with respect to some linear basis for $\mathfrak{S}$.

Assume that $I - \bar{D}_*$ is an isomorphism. We claim that $I - D'_*$ is an isomorphism if and only if $I - \mathrm{Ad}'(x)D'_*$ is an isomorphism.

Since $I - \bar{D}$ is an isomorphism on $\mathbb{R}^k$, $\mathrm{fix}(\bar{D}) = \ker(I - \bar{D})$ is a trivial group. For any $x \in S$, we consider the exact sequence of the Reidemeister sets

$$\mathcal{R}[\tau_x D'] \xrightarrow{\hat{i}^x} \mathcal{R}[\tau_x D] \xrightarrow{\hat{p}^x} \mathcal{R}[\bar{D}] \longrightarrow 1;$$

$\hat{p}^x$ is surjective, and $(\hat{p}^x)^{-1}([\bar{1}]) = \mathrm{im}(\hat{i}^x)$. If $\hat{i}^x([g_1]) = \hat{i}^x([g_2])$ for some $g_1, g_2 \in G$, then by definition there is $y \in S$ such that $g_2 = y g_1 (\tau_x D(y))^{-1}$. The image in S/G is then $\bar{g}_2 = \bar{y}\bar{g}_1 \bar{D}(\bar{y})^{-1}$, which yields that $\bar{y} \in \mathrm{fix}(\bar{D}) = \{\bar{1}\}$, and so $y \in G$. This shows that $\hat{i}^x$ is injective for all $x \in S$. Because there is a bijection between the Reidemeister sets $\mathcal{R}[D]$ and $\mathcal{R}[\tau_x D]$ given by $[g] \mapsto [gx^{-1}]$, it follows that $R(D') = R(\tau_x D')$. On the other hand, by [1, Lemma 3.4], since $I - \bar{D}_*$ is an isomorphism, $R(\bar{D}) < \infty$, and

$$I - \mathrm{Ad}'(x)D'_* \text{ is an isomorphism} \Longleftrightarrow R(\tau_x D') < \infty,$$
$$I - D'_* \text{ is an isomorphism} \Longleftrightarrow R(D') < \infty.$$

This proves our claim.

Now assume that $I - D_*$ is an isomorphism. Then it follows that $I - \bar{D}_*$ and $I - D'_*$ are isomorphisms. By the above claim, $I - \mathrm{Ad}'(x)D'_*$, and hence $I - \mathrm{Ad}(x)D_*$ are isomorphisms. □

Proof of Theorem 1. If S is Abelian, then $\mathrm{Ad}(x)$ is the identity and hence there is nothing to prove. We may assume that S is non-Abelian. Further, by Lemma 3, we may assume that $I - D_*$ is an isomorphism. Hence, $I - \bar{D}_*$ and $I - \mathrm{Ad}(x)D_*$ are isomorphisms for all $x \in S$.

Denote $G = [S, S]$ and $\Lambda_0 = S/G$. Then G is nilpotent, and $\Lambda_0 \cong \mathbb{R}^{k_0}$ for some $k_0 > 0$. Consider the lower central series of G:

$$G = \delta_1(G) \supset \delta_2(G) \supset \cdots \supset \delta_c(G) \supset \delta_{c+1}(G) = 1,$$

where $\delta_{i+1}(G)=[G,\delta_i(G)]$. Let $\Lambda_i=\delta_i(G)/\delta_{i+1}(G)$. Then $\Lambda_i\cong\mathbb{R}^{k_i}$ for some $k_i>0$. For each $x\in S$, the conjugation τ_x by x induces an automorphism on G. Since each $\delta_i(G)$ is a characteristic subgroup of G, $\tau_x\in\mathrm{Aut}(G)$ restricts to an automorphism on $\delta_i(G)$, and hence on Λ_i. Now, if $x\in G$, then we have observed that the induced action on Λ_i is trivial. Consequently, there is a well-defined action of $\Lambda_0=S/G$ on Λ_i. Hence, there is a well-defined action of Λ_0 on Λ_i. This action can be viewed as a homomorphism $\mu_i:\Lambda_0\to\mathrm{Aut}(\Lambda_i)$. Note that μ_0 is trivial. Moreover, for any $x\in S$ denoting its image under $S\to\Lambda_0$ by $\bar{x}$, the differential of conjugation τ_x by x can be expressed as a matrix of the form

$$\mathrm{Ad}(x)(=\tau_{x*})=\begin{bmatrix} I & 0 & \cdots & 0\\ * & \mu_1(\bar{x}) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & \mu_c(\bar{x})\end{bmatrix}$$

by choosing a suitable basis of the Lie algebra $\mathfrak{S}$ of S.

The homomorphism $D:S\to S$ induces homomorphisms $D_i:\delta_i(G)\to\delta_i(G)$ and hence homomorphisms $\bar{D}_i:\Lambda_i\to\Lambda_i$, so that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \delta_{i+1}(G) & \longrightarrow & \delta_i(G) & \longrightarrow & \Lambda_i & \longrightarrow & 0\\ & & \big\downarrow{\scriptstyle D_{i+1}} & & \big\downarrow{\scriptstyle D_i} & & \big\downarrow{\scriptstyle \bar{D}_i} & & \\ 1 & \longrightarrow & \delta_{i+1}(G) & \longrightarrow & \delta_i(G) & \longrightarrow & \Lambda_i & \longrightarrow & 0 \end{array}$$

Hence, the differential of D can be expressed as a matrix of the form

$$D_*=\begin{bmatrix} \bar{D}_0 & 0 & \cdots & 0\\ * & \bar{D}_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & \bar{D}_c\end{bmatrix}$$

with respect to the same basis for $\mathfrak{S}$ chosen as above.

Furthermore, the above commutative diagram produces the following identities:

$$\bar{D}_i\circ\mu_i(\bar{x})=\mu_i(\bar{D}_0(\bar{x}))\circ\bar{D}_i,\quad \forall\bar{x}\in\Lambda_0,\ \forall i=0,1,\ldots,c.$$

Let $x\in S$ with $\bar{x}\in\Lambda_0=\mathbb{R}^{k_0}$. Since $I-\bar{D}_0:\mathbb{R}^{k_0}\to\mathbb{R}^{k_0}$ is invertible, we can choose $\bar{y}\in\Lambda_0$ so that $(I-\bar{D}_0)(\bar{y})=\bar{x}$. Now, using the above identities, we observe that

$$\begin{aligned}\det(I-\mu_i(\bar{x})\bar{D}_i) &= \det(\mu_i(\bar{y})\mu_i(\bar{y})^{-1}-\mu_i(\bar{x})\mu_i(\bar{D}_0(\bar{y}))\bar{D}_i\mu_i(\bar{y})^{-1}) \\ &= \det(\mu_i(\bar{y})\mu_i(\bar{y})^{-1}-\mu_i(\bar{x}+\bar{D}_0(\bar{y}))\bar{D}_i\mu_i(\bar{y})^{-1}) \\ &= \det(\mu_i(\bar{y})\mu_i(\bar{y})^{-1}-\mu_i(\bar{y})\bar{D}_i\mu_i(\bar{y})^{-1}) \\ &= \det(\mu_i(\bar{y}))\det(I-\bar{D}_i)\det(\mu_i(\bar{y}))^{-1} \\ &= \det(I-\bar{D}_i).\end{aligned}$$

Consequently, we have

$$\begin{aligned}\det(I-\mathrm{Ad}(x)D_*) &= \det(I-\bar{D}_0)\prod_{i=1}^{c}\det(I-\mu_i(\bar{x})\bar{D}_i) \\ &= \det(I-\bar{D}_0)\prod_{i=1}^{c}\det(I-\bar{D}_i)=\det(I-D_*).\end{aligned}$$

This completes the proof of our theorem. □

References

[1] K. Dekimpe and P. Penninckx, *The finiteness of the Reidemeister number of morphisms between almost-crystallographic groups*, J. Fixed Point Theory Appl. **9** (2011), 257–283.

[2] J. B. Lee and K. B. Lee, *Averaging formula for Nielsen numbers of maps on infra-solvmanifolds of type* (R), Nagoya Math. J. **196** (2009), 117–134; doi:10.1017/S0027763000009818.

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