

RESEARCH ARTICLE

Linear PDE with constant coefficients

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Abstract

We discuss practical methods for computing the space of solutions to an arbitrary homogeneous linear system of partial differential equations with constant coefficients. These rest on the Fundamental Principle of Ehrenpreis–Palamodov from the 1960s. We develop this further using recent advances in computational commutative algebra.

1. Introduction

Our calculus class taught us how to solve ordinary differential equations (ODE) of the form

$$c_0\phi + c_1\phi' + c_2\phi'' + \cdots + c_m\phi^{(m)} = 0. \quad (1.1)$$

Here we seek functions $\phi = \phi(z)$ in one unknown z . The ODE is linear of order m , it has constant coefficients $c_i \in \mathbb{C}$, and it is homogeneous, meaning that the right-hand side is zero. The set of all solutions is a vector space of dimension m . A basis consists of m functions

$$\phi(z) = z^a \cdot \exp(u_i z). \quad (1.2)$$

Here u_i is a complex zero with multiplicity larger than $a \in \mathbb{N}$ of the characteristic polynomial

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_mx^m. \quad (1.3)$$

Thus solving the ODE (1.1) means finding all the zeros of (1.3) and their multiplicities.

We next turn to a partial differential equation (PDE) for functions $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ that is familiar from the undergraduate curriculum, namely the one-dimensional wave equation

$$\phi_{tt}(z, t) = c^2 \phi_{zz}(z, t), \quad \text{where } c \in \mathbb{R} \setminus \{0\}. \quad (1.4)$$

D’Alembert found in 1747 that the general solution is the superposition of traveling waves,

$$\phi(z, t) = f(z + ct) + g(z - ct), \quad (1.5)$$

where f and g are twice differentiable functions in one variable. For the special parameter value $c = 0$, the PDE (1.4) becomes $\phi_{tt} = 0$, and the general solution has still two summands

$$\phi(z, t) = f(z) + t \cdot h'(z). \quad (1.6)$$

We get this from (1.5) by replacing $g(z - ct)$ with $\frac{1}{2c}(h(z+ct) - h(z-ct))$ and taking the limit $c \rightarrow 0$. Here, the role of the characteristic polynomial (1.3) is played by the quadratic form

$$q_c(u, v) = v^2 - c^2u^2 = (v - cu)(v + cu). \quad (1.7)$$

The solutions (1.5) and (1.6) mirror the algebraic geometry of the conic $\{q_c = 0\}$ for any $c \in \mathbb{R}$.

Our third example is a system of three PDE for unknown functions

$$\psi : \mathbb{R}^4 \rightarrow \mathbb{C}^2, (x, y, z, w) \mapsto (\alpha(x, y, z, w), \beta(x, y, z, w)).$$

Namely, we consider the following linear PDE with constant coefficients:

$$\alpha_{xx} + \beta_{yy} = \alpha_{yz} + \beta_{zz} = \alpha_{xz} + \beta_{yw} = 0. \tag{1.8}$$

The general solution to this system has nine summands, labeled a, b, \dots, h and $(\tilde{\alpha}, \tilde{\beta})$:

$$\begin{aligned} \alpha &= a_z(y, z, w) - b_y(x, y) + c(y, w) + xd(y, w) + xg(z, w) - xyh_z(z, w) + \tilde{\alpha}(x, y, z, w), \\ \beta &= -a_y(y, z, w) + b_x(x, y) + e(x, w) + zf(x, w) + xh(z, w) + \tilde{\beta}(x, y, z, w). \end{aligned} \tag{1.9}$$

Here, a is any function in three variables, b, c, d, e, f, g, h are functions in two variables, and $\tilde{\psi} = (\tilde{\alpha}, \tilde{\beta})$ is any function $\mathbb{R}^4 \rightarrow \mathbb{C}^2$ that satisfies the following four linear PDE of first order:

$$\tilde{\alpha}_x + \tilde{\beta}_y = \tilde{\alpha}_y + \tilde{\beta}_z = \tilde{\alpha}_z - \tilde{\alpha}_w = \tilde{\beta}_z - \tilde{\beta}_w = 0. \tag{1.10}$$

We note that all solutions to (1.10) also satisfy (1.8), and they admit the integral representation

$$\tilde{\alpha} = \int t(\exp(s^2x + sty + t^2(z+w)))d\mu(s, t), \quad \tilde{\beta} = -\int s(\exp(s^2x + sty + t^2(z+w)))d\mu(s, t), \tag{1.11}$$

where μ is a measure on \mathbb{C}^2 . All functions in (1.9) are assumed to be suitably differentiable.

Our aim is to present methods for solving arbitrary systems of homogeneous linear PDE with constant coefficients. The input is a system like (1.1), (1.4), (1.8), or (1.10). We seek to compute the corresponding output (1.2), (1.5), (1.9), or (1.11), respectively. We present techniques that are based on the Fundamental Principle of Ehrenpreis and Palamodov, as discussed in the classical books [7, 17, 23, 32]. We utilize the theory of differential primary decomposition [12]. While deriving (1.5) from (1.4) is easy by hand, getting from (1.8) to (1.9) requires a computer.

This article is primarily expository. One goal is to explain the findings in [8–12], such as the differential primary decompositions of minimal size, from the viewpoint of analysis and PDE. In addition to these recent advances, our development rests on a considerable body of earlier work. The articles [15, 29, 31] are especially important. However, there are also some new contributions in the present article, mostly in Sections 4, 5, and 6. We describe the first universally applicable algorithm for computing Noetherian operators.

This presentation is organized as follows. Section 2 explains how linear PDE are represented by polynomial modules. The Fundamental Principle (Theorem 2.2) is illustrated with concrete examples. In Section 3, we examine the support of a module and how it governs exponential solutions (Proposition 3.7) and polynomial solutions (Proposition 3.9). Theorem 3.8 characterizes PDE whose solution space is finite dimensional. Section 4 features the theory of differential primary decomposition [9, 12]. Theorem 4.4 shows how this theory yields the integral representations promised by Ehrenpreis–Palamodov. This result appeared implicitly in the analysis literature, but the present algebraic form is new. It is the foundation of our algorithm for computing a minimal set of Noetherian multipliers. This is presented in Section 5, along with its implementation in the command `solvePDE` in `Macaulay2` [21].

The concepts of schemes and coherent sheaves are central to modern algebraic geometry. In Section 6, we argue that linear PDE are an excellent tool for understanding these concepts and for computing their behaviors in families. Hilbert schemes and Quot schemes make an appearance along the lines of [9, 11]. Section 7 is devoted to directions for further study and research in the subject area of this paper. It also features more examples and applications.

2. PDE and polynomials

Our point of departure is the observation that homogeneous linear partial differential equations with constant coefficients are the same as vectors of polynomials. The entries of the vectors are elements in

the polynomial ring $R = K[\partial_1, \partial_2, \dots, \partial_n]$, where K is a subfield of the complex numbers \mathbb{C} . In all our examples, we use the field $K = \mathbb{Q}$ of rational numbers. This has the virtue of being amenable to exact symbolic computation, e.g. in Macaulay2 [21].

For instance, in (1.1), we have $n = 1$. Writing $\partial = \frac{\partial}{\partial z}$ for the generator of R , our ODE is given by one polynomial $p(\partial) = c_0 + c_1\partial + \dots + c_m\partial^m$, where $c_0, c_1, \dots, c_m \in K$. For $n \geq 2$, we write $\mathbf{z} = (z_1, \dots, z_n)$ for the unknowns in the functions we seek, and the partial derivatives that act on these functions are $\partial_i = \partial_{z_i} = \frac{\partial}{\partial z_i}$. With this notation, the wave equation in (1.4) corresponds to the polynomial $q_c(\partial) = \partial_2^2 - c^2\partial_1^2 = (\partial_2 - c\partial_1)(\partial_2 + c\partial_1)$ with $n = 2$. Finally, the PDE in (1.8) has $n = 4$ and is encoded in three polynomial vectors

$$\begin{pmatrix} \partial_1^2 \\ \partial_1\partial_2 \end{pmatrix}, \quad \begin{pmatrix} \partial_2\partial_3 \\ \partial_3^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \partial_1^2\partial_3 \\ \partial_1\partial_2\partial_4 \end{pmatrix}. \tag{2.1}$$

The system (1.8) corresponds to the submodule of R^2 that is generated by these three vectors.

We shall study PDE that describe vector-valued functions from n -space to k -space. To this end, we need to specify a space \mathcal{F} of sufficiently differentiable functions such that \mathcal{F}^k contains our solutions. The scalar-valued functions in \mathcal{F} are either real-valued functions $\psi : \Omega \rightarrow \mathbb{R}$ or complex-valued functions $\psi : \Omega \rightarrow \mathbb{C}$, where Ω is a suitable subset of \mathbb{R}^n or \mathbb{C}^n . Later we will be more specific about the choice of \mathcal{F} . One requirement is that the space \mathcal{F}^k should contain the exponential functions

$$q(\mathbf{z}) \cdot \exp(\mathbf{u}'\mathbf{z}) = q(z_1, \dots, z_n) \cdot \exp(u_1z_1 + \dots + u_nz_n). \tag{2.2}$$

Here $\mathbf{u} \in \mathbb{C}^n$ and q is any vector of length k whose entries are polynomials in n unknowns.

Remark 2.1 ($k = 1$) A differential operator $p(\partial)$ in R annihilates the function $\exp(\mathbf{u}'\mathbf{z})$ if and only if $p(\mathbf{u}) = 0$. This is the content of [27, Lemma 3.25]. See also Lemma 3.26. If $p(\partial)$ annihilates a function $q(\mathbf{z}) \cdot \exp(\mathbf{u}'\mathbf{z})$, where q is a polynomial of positive degree, then \mathbf{u} is a point of higher multiplicity on the hypersurface $\{p = 0\}$. In the case $n = 1$, when p is the characteristic polynomial (1.3), we have a solution basis of exponential functions (1.2).

Another requirement for the space \mathcal{F} is that it is closed under differentiation. In other words, if $\phi = \phi(z_1, \dots, z_n)$ lies in \mathcal{F} , then so does $\partial_i \bullet \phi = \frac{\partial \phi}{\partial z_i}$ for $i = 1, 2, \dots, n$. The elements of \mathcal{F}^k are vector-valued functions $\psi = \psi(\mathbf{z})$. Their coordinates ψ_i are scalar-valued functions in \mathcal{F} . All in all, \mathcal{F} should be large, in the sense that it furnishes enough solutions. Formulated algebraically, we want \mathcal{F} to be an injective R -module [25]. A more precise desideratum, formulated by Oberst [28–30], is that \mathcal{F} should be an injective cogenerator.

Examples of injective cogenerators include the ring $\mathbb{C}[[z_1, \dots, z_n]]$ of formal power series, the space $C^\infty(\mathbb{R}^n)$ of smooth complex-valued functions over \mathbb{R}^n , or more generally, the space $\mathcal{D}'(\mathbb{R}^n)$ of complex-valued distributions on \mathbb{R}^n . If Ω is any open convex domain in \mathbb{R}^n , then we can also take \mathcal{F} to be $C^\infty(\Omega)$ or $\mathcal{D}'(\Omega)$. In this paper, we focus on algebraic methods. Analytic difficulties are mostly swept under the rug.

Our PDE are elements in the free R -module R^k , that is, they are column vectors of length k whose entries are polynomials in $\partial = (\partial_1, \dots, \partial_n)$. Such a vector acts on \mathcal{F}^k by coordinate-wise application of the differential operator and then adding up the results in \mathcal{F} . In this manner, each element in R^k defines an R -linear map $\mathcal{F}^k \rightarrow \mathcal{F}$. For instance, the third vector in (2.1) is an element in R^2 that acts on functions $\psi : \mathbb{R}^4 \rightarrow \mathbb{C}^2$ in \mathcal{F}^2 as follows:

$$\begin{pmatrix} \partial_1^2\partial_3 \\ \partial_1\partial_2\partial_4 \end{pmatrix} \bullet (\psi_1(\mathbf{z}), \psi_2(\mathbf{z})) = \frac{\partial^3 \psi_1}{\partial z_1^2 \partial z_3} + \frac{\partial^3 \psi_2}{\partial z_1 \partial z_2 \partial z_4}. \tag{2.3}$$

The right-hand side is a scalar-valued function $\mathbb{R}^4 \rightarrow \mathbb{C}$, that is, it is an element of \mathcal{F} .

Our systems of PDE are submodules M of the free module R^k . By Hilbert’s Basis Theorem, every module M is finitely generated, so we can write $M = \text{image}_R(A)$, where A is a $k \times l$ matrix with entries in R . Each column of A is a generator of M , and it defines a differential operator that maps \mathcal{F}^k to \mathcal{F} . The solution space to the PDE given by M equals

$$\text{Sol}(M) := \{ \psi \in \mathcal{F}^k : m \bullet \psi = 0 \text{ for all } m \in M \}. \tag{2.4}$$

It suffices to take the operators m from a generating set of M , such as the l columns of A . The case $k = 1$ is of special interest, since we often consider PDE for scalar-valued functions. In that case, the submodule is an ideal in the polynomial ring R , and we denote this by I . The solution space $\text{Sol}(I)$ of the ideal $I \subseteq R$ is the set of functions ϕ in \mathcal{F} such that $p(\partial) \bullet \phi = 0$ for all $p \in I$. Thus, ideals are instances of modules, with their own notation.

The solution spaces $\text{Sol}(M)$ and $\text{Sol}(I)$ are \mathbb{C} -vector spaces and R -modules. Indeed, any \mathbb{C} -linear combination of solutions is again a solution. The R -module action means applying the same differential operator $p(\partial)$ to each coordinate, which leads to another vector in \mathcal{F}^k . This action takes solutions to solutions because the ring of differential operators with constant coefficients $R = \mathbb{C}[\partial_1, \dots, \partial_n]$ is commutative.

The purpose of this paper is to present practical methods for the following task:

$$\begin{aligned} & \text{Given a } k \times l \text{ matrix } A \text{ with entries in } R = K[\partial_1, \dots, \partial_n], \text{ compute a good} \\ & \text{representation for the solution space } \text{Sol}(M) \text{ of the module } M = \text{image}_R(A). \end{aligned} \tag{2.5}$$

If $k = 1$ then we consider the ideal I generated by the entries of A and we compute $\text{Sol}(I)$.

This raises the question of what a “good representation” means. The formulas in (1.2), (1.5), (1.9) and (1.11) are definitely good. They guide us to what is desirable. Our general answer stems from the following important result at the crossroads of analysis and algebra. It involves two sets of unknowns $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$. Here \mathbf{x} gives coordinates on certain irreducible varieties V_i in \mathbb{C}^n that are parameter spaces for solutions. Our solutions ψ are functions in \mathbf{z} . We take $\mathcal{F} = C^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open, convex, and bounded.

Theorem 2.2 (Ehrenpreis–Palamodov Fundamental Principle). *Consider a module $M \subseteq R^k$, representing linear PDE for a function $\psi : \Omega \rightarrow \mathbb{C}^k$. There exist irreducible varieties V_1, \dots, V_s in \mathbb{C}^n and finitely many vectors B_{ij} of polynomials in $2n$ unknowns (\mathbf{x}, \mathbf{z}) , all independent of the set Ω , such that any solution $\psi \in \mathcal{F}$ admits an integral representation*

$$\psi(\mathbf{z}) = \sum_{i=1}^s \sum_{j=1}^{m_i} \int_{V_i} B_{ij}(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x}'\mathbf{z}) d\mu_{ij}(\mathbf{x}). \tag{2.6}$$

Here m_i is a certain invariant of (M, V_i) and each μ_{ij} is a bounded measure supported on the variety V_i .

Theorem 2.2 appears in different forms in the books by Björk [7, Theorem 8.1.3], Ehrenpreis [17], Hörmander [23, Section 7.7], and Palamodov [32]. Other references with different emphases include [5, 25, 29]. For a perspective from commutative algebra see [11, 12].

In the next sections, we will study the ingredients in Theorem 2.2. Given the module M , we compute each associated variety V_i , the arithmetic length m_i of M along V_i , and the Noetherian multipliers $B_{i,1}, B_{i,2}, \dots, B_{i,m_i}$. We shall see that not all n of the unknowns z_1, \dots, z_n appear in the polynomials B_{ij} but only a subset of $\text{codim}(V_i)$ of them.

The most basic example is the ODE in (1.1), with $l = n = k = 1$. Here $V_i = \{u_i\}$ is the i th root of the polynomial (1.3), which has multiplicity m_i , and $B_{ij} = z^{j-1}$. The measure μ_{ij} is a scaled Dirac measure on u_i , so the integrals in (2.6) are multiples of the basis functions (1.2).

In light of Theorem 2.2, we now refine our computational task in (2.5) as follows:

$$\begin{aligned} & \text{Given a } k \times l \text{ matrix } A \text{ with entries in } R = K[\partial_1, \dots, \partial_n], \text{ compute the varieties } V_i \\ & \text{and the Noetherian multipliers } B_{ij}(\mathbf{x}, \mathbf{z}). \text{ This encodes } \text{Sol}(M) \text{ for } M = \text{image}_R(A). \end{aligned} \tag{2.7}$$

In our introductory examples, we gave formulas for the general solution, namely (1.5) and (1.9). We claim that such formulas can be read off from the integrals in (2.6). For instance, for the wave equation (1.4), we have $s = 2$, $B_{1,1} = B_{1,2} = 1$, and (1.5) is obtained by integrating $\exp(\mathbf{x}'\mathbf{z})$ against measures $d\mu_{i1}(\mathbf{x})$ on two lines V_1 and V_2 in \mathbb{C}^2 . For the system (1.8), we find $s = 6$, with $m_1 = m_2 = m_3 = 1$ and $m_4 = m_5 = m_6 = 2$, and the nine integrals in (2.6) translate into (1.9). We shall explain such a translation in full detail for two other examples.

Example 2.3 ($n = 3, k = 1, l = 2$) The ideal $I = \langle \partial_1^2 - \partial_2\partial_3, \partial_3^2 \rangle$ represents the PDE

$$\frac{\partial^2 \phi}{\partial z_1^2} = \frac{\partial^2 \phi}{\partial z_2 \partial z_3} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial z_3^2} = 0 \tag{2.8}$$

for a scalar-valued function $\phi = \phi(z_1, z_2, z_3)$. This is [10, Example 4.2]. A Macaulay2 computation as in Section 5 shows that $s = 1, m_1 = 4$. It reveals the Noetherian multipliers

$$B_1 = 1, B_2 = z_1, B_3 = z_1^2 x_2 + 2z_3, B_4 = z_1^3 x_2 + 6z_1 z_3.$$

Arbitrary functions $f(z_2) = \int \exp(tz_2)dt$ are obtained by integrating against suitable measures on the line $V_1 = \{(0, t, 0) : t \in \mathbb{C}\} \subset \mathbb{C}^3$. Their derivatives are found by differentiating under the integral sign, namely $f'(z_2) = \int t \cdot \exp(tz_2)dt$. Consider four functions a, b, c, d , each specified by a different measure. Thus, the sum of the four integrals in (2.6) evaluates to

$$\phi(\mathbf{z}) = a(z_2) + z_1 b(z_2) + (z_1^2 c'(z_2) + 2z_3 c(z_2)) + (z_1^3 d'(z_2) + 6z_1 z_3 d(z_2)). \tag{2.9}$$

According to Ehrenpreis–Palamodov, this sum is the general solution of the PDE (2.8).

Our final example uses concepts from primary decomposition, to be reviewed in Section 3.

Example 2.4 ($n = 4, k = 2, l = 3$). Let $M \subset R^4$ be the module generated by the columns of

$$A = \begin{bmatrix} \partial_1 \partial_3 & \partial_1 \partial_2 & \partial_1^2 \partial_2 \\ \partial_1^2 & \partial_2^2 & \partial_1^2 \partial_4 \end{bmatrix}. \tag{2.10}$$

Computing $\text{Sol}(M)$ means solving $\frac{\partial^2 \psi_1}{\partial z_1 \partial z_3} + \frac{\partial^2 \psi_2}{\partial z_1^2} = \frac{\partial^2 \psi_1}{\partial z_1 \partial z_2} + \frac{\partial^2 \psi_2}{\partial z_2^2} = \frac{\partial^3 \psi_1}{\partial z_1^2 \partial z_2} + \frac{\partial^3 \psi_2}{\partial z_1^2 \partial z_4} = 0$. Two solutions are $\psi(\mathbf{z}) = (\phi(z_2, z_3, z_4), 0)$ and $\psi(\mathbf{z}) = \exp(s^2 t z_1 + s t^2 z_2 + s^3 z_3 + t^3 z_4) \cdot (t, -s)$.

We apply Theorem 2.2 to derive the general solution to (2.10). The module M has six associated primes, namely $P_1 = \langle \partial_1 \rangle, P_2 = \langle \partial_2, \partial_4 \rangle, P_3 = \langle \partial_2, \partial_3 \rangle, P_4 = \langle \partial_1, \partial_3 \rangle, P_5 = \langle \partial_1, \partial_2 \rangle$, and $P_6 = \langle \partial_1^2 - \partial_2 \partial_3, \partial_1 \partial_2 - \partial_3 \partial_4, \partial_2^2 - \partial_1 \partial_4 \rangle$. Four of them are minimal and two are embedded. We find that $m_1 = m_2 = m_3 = m_4 = m_6 = 1$ and $m_5 = 4$. A minimal primary decomposition

$$M = M_1 \cap M_2 \cap M_3 \cap M_4 \cap M_5 \cap M_6 \tag{2.11}$$

is given by the following primary submodules of R^4 , each of which contains M :

$$M_1 = \text{im}_R \begin{bmatrix} \partial_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2 = \text{im}_R \begin{bmatrix} \partial_2 & \partial_4 & 0 & 0 & \partial_3 \\ 0 & 0 & \partial_2 & \partial_4 & \partial_1 \end{bmatrix}, \quad M_3 = \text{im}_R \begin{bmatrix} \partial_2 & \partial_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_4 = \text{im}_R \begin{bmatrix} \partial_3^5 & \partial_1 & 0 & 0 \\ 0 & \partial_2 & \partial_1 & \partial_3 \end{bmatrix}, \quad M_5 = \text{im}_R \begin{bmatrix} \partial_1 & \partial_2^5 & 0 & 0 \\ 0 & 0 & \partial_1^2 & \partial_2^2 \end{bmatrix}, \quad M_6 = \text{im}_R \begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & \partial_4 & \partial_1 \end{bmatrix}.$$

The number of Noetherian multipliers B_{ij} is $\sum_{i=1}^6 m_i = 9$. We choose them to be

$$B_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_{2,1} = \begin{bmatrix} x_1 \\ -x_3 \end{bmatrix}, B_{3,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_{4,1} = \begin{bmatrix} x_2 z_1 \\ -1 \end{bmatrix}, B_{5,i} = \begin{bmatrix} 0 \\ z_1 z_2 \end{bmatrix}, \begin{bmatrix} 0 \\ z_1 \end{bmatrix}, \begin{bmatrix} 0 \\ z_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{6,1} = \begin{bmatrix} x_4 \\ -x_2 \end{bmatrix}.$$

These nine vectors describe all solutions to our PDE. For instance, $B_{3,1}$ gives the solutions $\begin{bmatrix} \alpha(z_1, z_4) \\ 0 \end{bmatrix}$, and $B_{5,1}$ gives the solutions $\begin{bmatrix} 0 \\ z_1 z_2 \beta(z_3, z_4) \end{bmatrix}$, where α, β are bivariate functions. Furthermore, $B_{1,1}$ and $B_{6,1}$ encode the two families of solutions mentioned after (2.10).

For the latter, we note that $V_6 = V(P_6)$ is the surface in \mathbb{C}^4 with parametric representation $(x_1, x_2, x_3, x_4) = (s^2 t, s t^2, s^3, t^3)$ for $s, t \in \mathbb{C}$. This surface is the cone over the twisted cubic curve, in the same notation as in [11, Section 1]. The kernel under the integral in (2.6) equals

$$\begin{bmatrix} x_4 \\ -x_2 \end{bmatrix} \exp(x_1 z_1 + x_2 z_2 + x_3 z_3 + x_4 z_4) = t^2 \begin{bmatrix} t \\ -s \end{bmatrix} \exp(s^2 t z_1 + s t^2 z_2 + s^3 z_3 + t^3 z_4).$$

This is a solution to M_6 , and hence to M , for any values of s and t . Integrating the left-hand side over $\mathbf{x} \in V_6$ amounts to integrating the right-hand side over $(s, t) \in \mathbb{C}^2$. Any such integral is also a solution. Ehrenpreis–Palamodov tells us that these are all the solutions.

3. Modules and varieties

Our aim is to offer practical tools for solving PDE. The input is a $k \times l$ matrix A with entries in $R = K[\partial_1, \dots, \partial_n]$, and $M = \text{image}_R(A)$ is the corresponding submodule of $R^k = \bigoplus_{j=1}^k R e_j$. The output is the description of $\text{Sol}(M)$ sought in (2.7). That description is unique up to basis change, in the sense of [12, Remark 3.8], by the discussion in Section 4. Our method is implemented in a Macaulay2 command, called `solvePDE` and to be described in Section 5.

We now explain the ingredients of Theorem 2.2 coming from commutative algebra (cf. [18]). For a vector $m \in R^k$, the quotient $(M : m)$ is the ideal $\{f \in R : fm \in M\}$. A prime ideal $P_i \subseteq R$ is associated to M if there exists $m \in R^k$ such that $(M : m) = P_i$. Since R is Noetherian, the list of associated primes of M is finite, say P_1, \dots, P_s . If $s = 1$ then the module M is called primary or P_1 -primary. A primary decomposition of M is a list of primary submodules $M_1, \dots, M_s \subseteq R^k$ where M_i is P_i -primary and $M = M_1 \cap M_2 \cap \dots \cap M_s$.

Primary decomposition is a standard topic in commutative algebra. It is usually presented for ideals ($k = 1$), as in [27, Chapter 3]. The case of modules is analogous. The latest version of Macaulay2 has an implementation of primary decomposition for modules, as described in [9, Section 2]. Given M , the primary module M_i is not unique if P_i is an embedded prime.

The contribution of the primary module M_i to M is quantified by a positive integer m_i , called the arithmetic length of M along P_i . To define this, we consider the localization $(R_{P_i})^k / M_{P_i}$. This is a module over the local ring R_{P_i} . The arithmetic length is the length of the largest submodule of finite length in $(R_{P_i})^k / M_{P_i}$; in symbols, $m_i = \text{length}(H_{P_i}^0((R_{P_i})^k / M_{P_i}))$. The sum $m_1 + \dots + m_s$ is an invariant of the module M , denoted $\text{mult}(M)$, and known as the arithmetic multiplicity of M . These numbers can be computed in Macaulay2 as in [12, Remark 5.1]. We return to these invariants in Theorem 4.3.

To make the connection to Theorem 2.2, we set $V_i = V(P_i)$ for $i = 1, 2, \dots, s$. Thus, V_i is the irreducible variety in \mathbb{C}^n defined by the prime ideal P_i in $R = K[\partial_1, \dots, \partial_n]$. The integer m_i is an invariant of the pair (M, V_i) : it measures the thickness of the module M along V_i .

By taking the union of the irreducible varieties V_i we obtain the variety

$$V(M) := V_1 \cup V_2 \cup \dots \cup V_s \subset \mathbb{C}^n.$$

Algebraists refer to $V(M)$ as the support of M , while analysts call it the characteristic variety of M . The support is generally reducible, with $\leq s$ irreducible components. For instance, the module M in Example 2.4 has six associated primes, and an explicit primary decomposition was given in (2.11).

However, the support $V(M)$ has only four irreducible components in \mathbb{C}^4 , namely one hyperplane, two-dimensional planes, and one nonlinear surface (twisted cubic).

Remark 3.1. *If $k = 1$ and $M = I$, then the support $V(M)$ coincides with the variety $V(I)$ attached as usual to an ideal I , namely the common zero set in \mathbb{C}^n of all polynomials in I .*

The relationship between modules and ideals mirrors the relationship between PDE for vector-valued functions and related PDE for scalar-valued functions. To pursue this a bit further, we now define two ideals that are naturally associated with a given module $M \subseteq R^k$.

The first ideal is the *annihilator* of the quotient module $R^k/M = \text{coker}_R(A)$, which is

$$I := \text{Ann}_R(R^k/M) = \{f \in R : fm \in M \text{ for all } m \in R^k\}.$$

The second is the zeroth *Fitting ideal* of R^k/M , which is the ideal in R generated by the $k \times k$ minors of the presentation matrix A . It is independent of the choice of A , and we write

$$J := \text{Fitt}_0(R^k/M) = \langle k \times k \text{ subdeterminants of } A \rangle.$$

We are interested in the affine varieties in \mathbb{C}^n defined by these ideals. They are denoted by $V(I)$ and $V(J)$, respectively. The following is a standard result in commutative algebra.

Proposition 3.2. *The three varieties above are equal for every submodule M of R^k , that is,*

$$V(M) = V(I) = V(J) \subseteq \mathbb{C}^n. \tag{3.1}$$

Proof. This follows from [18, Proposition 20.7]. □

Remark 3.3. *It can happen that $\text{rank}(A) < k$, for instance when $k > l$. In that case, $I = J = \{0\}$ and $V(M) = \mathbb{C}^n$. Geometrically, the module M furnishes a coherent sheaf that is supported on the entire space \mathbb{C}^n . For instance, let $k = n = 2, l = 1$, and $A = \begin{pmatrix} \partial_1 \\ -\partial_2 \end{pmatrix}$. The PDE asks for pairs (ψ_1, ψ_2) such that $\partial\psi_1/\partial z_1 = \partial\psi_2/\partial z_2$. We see that $\text{Sol}(M)$ consists of all pairs $(\partial\alpha/\partial z_2, \partial\alpha/\partial z_1)$, where $\alpha = \alpha(z_1, z_2)$ runs over functions in two variables. In general, the left kernel of A furnishes differential operators for creating solutions to M .*

The following example shows that (3.1) is not true at the level of schemes (cf. Section 6).

Example 3.4. *($n = k = 3, l = 5$) Let $R = \mathbb{C}[\partial_1, \partial_2, \partial_3]$ and M the submodule of R^3 given by*

$$A = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 0 \\ 0 & \partial_1^2 & \partial_2 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 & \partial_3 \end{pmatrix}.$$

We find $I = \langle \partial_1^2, \partial_1\partial_2 \rangle \supset J = \langle \partial_1^4, \partial_1^3\partial_3, \partial_1^2\partial_2, \partial_1\partial_2\partial_3 \rangle$. The sets of associated primes are

$$\begin{aligned} \text{Ass}(I) &= \{ \langle \partial_1 \rangle, \langle \partial_1, \partial_2 \rangle \} && \text{with } \text{amult}(I) = 2 \\ \subset \text{Ass}(M) &= \{ \langle \partial_1 \rangle, \langle \partial_1, \partial_2 \rangle, \langle \partial_1, \partial_3 \rangle \} && \text{with } \text{amult}(M) = 4 \\ \subset \text{Ass}(J) &= \{ \langle \partial_1 \rangle, \langle \partial_1, \partial_2 \rangle, \langle \partial_1, \partial_3 \rangle, \langle \partial_1, \partial_2, \partial_3 \rangle \} && \text{with } \text{amult}(J) = 5 \end{aligned}$$

The support $V(M)$ is a plane in 3-space, on which I and J define different scheme structures. Our module M defines a coherent sheaf on that plane that lives between these two schemes. We consider the PDE in each of the three cases, we compute the Noetherian multipliers, and from this we derive the general solution. To begin with, functions in $\text{Sol}(J)$ have the form

$$\alpha(z_2, z_3) + z_1\beta(z_3) + z_1^2\gamma(z_3) + z_1\delta(z_2) + c \cdot z_1^3.$$

The first two terms give functions in the subspace $\text{Sol}(I)$. Elements in $\text{Sol}(M)$ are vectors

$$(\rho(z_2, z_3), \sigma(z_3) + z_1\tau(z_3), \omega(z_2)).$$

These represent all functions $\mathbb{C}^3 \rightarrow \mathbb{C}^3$ that satisfy the five PDE given by the matrix A .

Remark 3.5. The quotient R/I embeds naturally into the direct sum of k copies of R^k/M , via $1 \mapsto e_j$. This implies $\text{Ass}(I) \subseteq \text{Ass}(M)$. It would be worthwhile to understand how the differential primary decompositions of I, J and M are related, and to study implications for the solution spaces $\text{Sol}(I)$, $\text{Sol}(J)$, and $\text{Sol}(M)$. What relationships hold between these?

Lemma 3.6. Fix a $k \times l$ matrix $A(\partial)$ and its module $M \subseteq R^k$ as above. A point $\mathbf{u} \in \mathbb{C}^n$ lies in $V(M)$ if and only if there exist constants $c_1, \dots, c_k \in \mathbb{C}$, not all zero, such that

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \exp(u_1 z_1 + \dots + u_n z_n) \in \text{Sol}(M). \tag{3.2}$$

More precisely, (3.2) holds if and only if $(c_1, \dots, c_k) \cdot A(\mathbf{u}) = 0$.

Proof. Let $a_{ij}(\partial)$ denote the entries of the matrix $A(\partial)$. Then (3.2) holds if and only if

$$\sum_{i=1}^k a_{ij}(\partial) \bullet (c_i \exp(u_1 z_1 + \dots + u_n z_n)) = 0 \quad \text{for all } j = 1, \dots, l.$$

This is equivalent to

$$\sum_{i=1}^k c_i a_{ij}(\mathbf{u}) \exp(u_1 z_1 + \dots + u_n z_n) = 0 \quad \text{for all } j = 1, \dots, l.$$

This condition holds if and only if $(c_1, \dots, c_k) \cdot A(\mathbf{u})$ is the zero vector in \mathbb{C}^l . We conclude that, for any given $\mathbf{u} \in \mathbb{C}^n$, the previous condition is satisfied for some $c \in \mathbb{C}^k \setminus \{0\}$ if and only if $\text{rank}(A(\mathbf{u})) < k$ if and only if $\mathbf{u} \in V(M) = V(I)$. Here we use Proposition 3.2. \square

Here is an alternative way to interpret the characteristic variety of a system of PDE:

Proposition 3.7. The solution space $\text{Sol}(M)$ contains an exponential solution $q(\mathbf{z}) \cdot \exp(\mathbf{u}'\mathbf{z})$ if and only if $\mathbf{u} \in V(M)$. Here q is some vector of k polynomials in n unknowns, as in (2.2).

Proof. One direction is clear from Lemma 3.6. Next, suppose $q(\mathbf{z}) \exp(\mathbf{u}'\mathbf{z}) \in \text{Sol}(M)$. The partial derivative of this function with respect to any unknown z_i is also in $\text{Sol}(M)$. Hence,

$$\partial_i \bullet (q(\mathbf{z}) \exp(\mathbf{u}'\mathbf{z})) = (\partial_i \bullet q(\mathbf{z})) \exp(\mathbf{u}'\mathbf{z}) + u_i q(\mathbf{z}) \exp(\mathbf{u}'\mathbf{z}) \in \text{Sol}(M) \quad \text{for } i = 1, \dots, n.$$

Hence, the exponential function $(\partial_i \bullet q(\mathbf{z})) \exp(\mathbf{u}'\mathbf{z})$ is in $\text{Sol}(M)$. Since the degree of $\partial_i \bullet q(\mathbf{z})$ is less than that of $q(\mathbf{z})$, we can find a sequence $D = \partial_{i_1} \partial_{i_2} \dots \partial_{i_s}$ such that $D \bullet q$ is a nonzero constant vector and $(D \bullet q) \exp(\mathbf{u}'\mathbf{z}) \in \text{Sol}(M)$. Lemma 3.6 now implies that $\mathbf{u} \in V(M)$. \square

The solution space $\text{Sol}(M)$ to a submodule $M \subseteq R^k$ is a vector space over \mathbb{C} . It is infinite-dimensional whenever $V(M)$ is a variety of positive dimension. This follows from Lemma 3.6 because there are infinitely many points \mathbf{u} in $V(M)$. However, if $V(M)$ is a finite subset of \mathbb{C}^n , then $\text{Sol}(M)$ is finite-dimensional. This is the content of the next theorem.

Theorem 3.8. Consider a module $M \subseteq R^k$, viewed as a system of linear PDE. Its solution space $\text{Sol}(M)$ is finite-dimensional over \mathbb{C} if and only if $V(M)$ has dimension 0. In this case, $\dim_{\mathbb{C}} \text{Sol}(M) =$

$\dim_K(R^k/M) = \text{amult}(M)$. There is a basis of $\text{Sol}(M)$ given by vectors $q(\mathbf{z})\exp(\mathbf{u}'\mathbf{z})$, where $\mathbf{u} \in V(M)$ and $q(\mathbf{z})$ runs over a finite set of polynomial vectors, whose cardinality is the length of M along the maximal ideal $\langle x_1 - u_1, \dots, x_n - u_n \rangle$. There exist polynomial solutions if and only if $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ is an associated prime of M . The polynomial solutions are found by solving the PDE given by the \mathfrak{m} -primary component of M .

Proof. This is the main result in Oberst’s article [30], proved in the setting of injective cogenerators \mathcal{F} . The same statement for $\mathcal{F} = C^\infty(\Omega)$ appears in [7, Ch. 8, Theorem 7.1]. The scalar case ($k = 1$) is found in [27, Theorem 3.27]. The proof given there uses solutions in the power series ring, which is an injective cogenerator, and it generalizes to modules. □

By a *polynomial solution* we mean a vector $q(\mathbf{z})$ whose coordinates are polynomials. The \mathfrak{m} -primary component in Theorem 3.8 is computed by a double saturation step. When $M = I$ is an ideal, then this double saturation is $I : (I : \mathfrak{m}^\infty)$, as seen in [27, Theorem 3.27]. For submodules M of R^k with $k \geq 2$, we would compute $M : \text{Ann}(R^k/(M : \mathfrak{m}^\infty))$. The inner colon $(M : \mathfrak{m}^\infty)$ is the intersection of all primary components of M whose variety V_i does not contain the origin 0. It is computed as $(M : f) = \{m \in R^k : fm \in M\}$, where f is a random homogeneous polynomial of large degree. The outer colon is the module $(M : g)$, where g is a general polynomial in the ideal $\text{Ann}(R^k/(M : f))$. See also [9, Proposition 2.2].

It is an interesting problem to identify polynomial solutions when $V(M)$ is no longer finite and to decide whether these are dense in the infinite-dimensional space of all solutions. Here “dense” refers to the topology on \mathcal{F} used by Lomadze in [26]. The following result gives an algebraic characterization of the closure in $\text{Sol}(M)$ of the subspace of polynomial solutions.

Proposition 3.9. *The polynomial solutions are dense in $\text{Sol}(M)$ if and only if the origin 0 lies in every associated variety V_i of the module M . If this fails, then the topological closure of the space of polynomial solutions $q(\mathbf{z})$ to M is the solution space of $M : \text{Ann}(R^k/(M : \mathfrak{m}^\infty))$.*

Proof. This proposition is our reinterpretation of Lomadze’s result in [26, Theorem 3.1]. □

The result gives rise to algebraic algorithms for answering analytic questions about a system of PDE. The property in the first sentence can be decided by running the primary decomposition algorithm in [9]. For the second sentence, we need to compute a double saturation as above. This can be carried out in Macaulay2 as well.

4. Differential primary decomposition

We now shift gears and pass to a setting that is dual to the one we have seen so far. Namely, we discuss differential primary decompositions [9, 12]. That duality is subtle and can be confusing at first sight. To mitigate this, we introduce new notation. We set $x_i = \partial_i = \partial_{z_i}$ for $i = 1, \dots, n$. Thus, R is now the polynomial ring $K[x_1, \dots, x_n]$. This is the notation we are used to from algebra courses (such as [27]). We write $\partial_{x_1}, \dots, \partial_{x_n}$ for the differential operators corresponding to x_1, \dots, x_n . Later on, we also identify $z_i = \partial_{x_i}$, and we think of the unknowns \mathbf{x} and \mathbf{z} in the multipliers $B_i(\mathbf{x}, \mathbf{z})$ as dual in the sense of the Fourier transform.

The ring of differential operators on the polynomial ring R is the Weyl algebra

$$D_n = R\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle = K\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n} \rangle.$$

The $2n$ generators commute, except for the n relations $\partial_{x_i}x_i - x_i\partial_{x_i} = 1$, which expresses the Product Rule from Calculus. Elements in the Weyl algebra D_n are linear differential operators with polynomial coefficients. We write $\delta \bullet p$ for the result of applying $\delta \in D_n$ to a polynomial $p = p(\mathbf{x})$ in R . For instance,

$x_i \bullet p = x_i p$ and $\partial_{x_i} \bullet p = \partial p / \partial x_i$. Let D_n^k denote the k -tuples of differential operators in D_n . These operate on the free module R^k as follows:

$$D_n^k \times R^k \rightarrow R : (\delta_1, \dots, \delta_k) \bullet (p_1, \dots, p_k) = \sum_{j=1}^k \delta_j \bullet p_j.$$

Fix a submodule M of R^k and let P_1, \dots, P_s be its associated primes, as in Section 3. A differential primary decomposition of M is a list $\mathcal{A}_1, \dots, \mathcal{A}_s$ of finite subsets of D_n^k such that

$$M = \{m \in R^k : \delta \bullet m \in P_i \text{ for all } \delta \in \mathcal{A}_i \text{ and } i = 1, 2, \dots, s\}. \tag{4.1}$$

This is a membership test for the module M using differential operators. This test is geometric since the polynomial $\delta \bullet m$ lies in P_i if and only if it vanishes on the variety $V_i = V(P_i)$.

Theorem 4.1 *Every submodule M of R^k has a differential primary decomposition. We can choose the sets $\mathcal{A}_1, \dots, \mathcal{A}_s$ such that $|\mathcal{A}_i|$ is the arithmetic length of M along the prime P_i .*

Proof and discussion. The result is proved in [12] and further refined in [9]. These sources also develop an algorithm. We shall explain this in Section 5, along with a discussion of the Macaulay2 command `solvePDE`, which computes differential primary decompositions. □

The differential operators in $\mathcal{A}_1, \dots, \mathcal{A}_s$ are known as *Noetherian operators* in the literature; see [10, 11, 15, 31]. Theorem 4.1 says that we can find a collection of $\text{amult}(M) = m_1 + \dots + m_s$ Noetherian operators in D_n^k to characterize membership in the module M .

Remark 4.2 *The construction of Noetherian operators is studied in [7, 8, 10, 11, 23, 31]. Some of these sources offer explicit methods, while others remain at an abstract level. All previous methods share one serious shortcoming, namely they yield operators separately for each primary component M_i of M . They do not take into account how one primary component is embedded into another. This leads to a number of operators that can be much larger than $\text{amult}(M)$. We refer to [12, Example 5.6] for an instance from algebraic statistics where the previous methods require 1044 Noetherian operators, while $\text{amult}(M) = 207$ suffice.*

While Theorem 4.1 makes no claim of minimality, it is known that $\text{amult}(M)$ is the minimal number of Noetherian operators required for a differential primary decomposition of a certain desirable form. To make this precise, we begin with a few necessary definitions. For any given subset \mathcal{S} of $\{x_1, \dots, x_n\}$, the *relative Weyl algebra* is defined as the subring of the Weyl algebra D_n using only differential operators corresponding to variables not in \mathcal{S} :

$$D_n(\mathcal{S}) := R \langle \partial_{x_i} : x_i \notin \mathcal{S} \rangle \subseteq D_n. \tag{4.2}$$

Thus, if $\mathcal{S} = \emptyset$, then $D_n(\mathcal{S}) = D_n$, and if $\mathcal{S} = \{x_1, \dots, x_n\}$, then $D_n(\mathcal{S}) = R = K[x_1, \dots, x_n]$.

For any prime ideal P_i in R we fix a set $\mathcal{S}_i \subseteq \{x_1, \dots, x_n\}$ that satisfies $K[\mathcal{S}_i] \cap P_i = \{0\}$ and is maximal with this property. Thus, \mathcal{S}_i is a maximal independent set of coordinates on the irreducible variety $V(P_i)$. Equivalently, \mathcal{S}_i is a basis of the algebraic matroid defined by the prime P_i ; cf. [27, Example 13.2]. The cardinality of \mathcal{S}_i equals the dimension of $V(P_i)$.

Theorem 4.3 *The differential primary decomposition in Theorem 4.1 can be chosen so that $\mathcal{A}_i \subseteq D_n(\mathcal{S}_i)^k$. The arithmetic length of M along P_i is a lower bound for the cardinality of \mathcal{A}_i in any differential primary decomposition of M such that $\mathcal{A}_i \subseteq D_n(\mathcal{S}_i)^k$ for $i = 1, \dots, s$.*

Proof and discussion. This was shown in [12, Theorem 4.6]. The case of ideals ($k = 1$) appears in [12, Theorem 3.6]. See also [9]. The theory developed in [12] is more general in that R can be any Noetherian

K -algebra. In this paper, we restrict to polynomial rings $R = K[x_1, \dots, x_n]$ where K is a subfield of \mathbb{C} . That case is treated in detail in [9]. □

We next argue that Theorems 2.2 and 4.1 are really two sides of the same coin. Every element A in the Weyl algebra D_n acts as a differential operator with polynomial coefficients on functions in the unknowns $\mathbf{x} = (x_1, \dots, x_n)$. Such a differential operator has a unique representation where all derivatives are moved to the right of the polynomial coefficients:

$$A(\mathbf{x}, \partial_{\mathbf{x}}) = \sum_{\mathbf{r}, \mathbf{s} \in \mathbb{N}^n} c_{\mathbf{r}, \mathbf{s}} x_1^{r_1} \cdots x_n^{r_n} \partial_{x_1}^{s_1} \cdots \partial_{x_n}^{s_n}, \quad \text{where } c_{\mathbf{r}, \mathbf{s}} \in K. \tag{4.3}$$

There is a natural K -linear isomorphism between the Weyl algebra D_n and the polynomial ring $K[\mathbf{x}, \mathbf{z}]$ which takes the operator A in (4.3) to the following polynomial B in $2n$ variables:

$$B(\mathbf{x}, \mathbf{z}) = \sum_{\mathbf{r}, \mathbf{s} \in \mathbb{N}^n} c_{\mathbf{r}, \mathbf{s}} x_1^{r_1} \cdots x_n^{r_n} z_1^{s_1} \cdots z_n^{s_n}. \tag{4.4}$$

In Sections 1, 2, and 3, polynomials in $R = K[x_1, \dots, x_n] = K[\partial_1, \dots, \partial_n]$ act as differential operators on functions in the unknowns $\mathbf{z} = (z_1, \dots, z_n)$. For such operators, polynomials in \mathbf{x} are constants. By contrast, in the current section, we introduced the Weyl algebra D_n . Its elements act on functions in $\mathbf{x} = (x_1, \dots, x_n)$, with polynomials in \mathbf{z} being constants. These two different actions of differential operators, by D_n and R on scalar-valued functions, extend to actions by D_n^k and R^k on vector-valued functions. We highlight the following key point:

Our distinction between the \mathbf{z} -variables and \mathbf{x} -variables is absolutely essential. (4.5)

The following theorem is the punchline of this section. It allows us to identify Noetherian operators (4.3) with Noetherian multipliers (4.4). This was assumed tacitly in [11, Section 3].

Theorem 4.4 *Consider any differential primary decomposition of the module M as in Theorem 4.3. Then this translates into an Ehrenpreis–Palamodov representation of the solution space $\text{Sol}(M)$. Namely, if we replace each operator $A(\mathbf{x}, \partial_{\mathbf{x}})$ in \mathcal{A}_i by the corresponding polynomial $B(\mathbf{x}, \mathbf{z})$, then these $\text{amult}(M)$ polynomials satisfy the conclusion of Theorem 2.2.*

Example 4.5 ($k = l = n = 1$) *We illustrate Theorem 4.4 and the warning (4.5) for an ODE (1.1) with $m = 3$. Set $p(x) = x^3 + 3x^2 - 9x + 5 = (x - 1)^2(x + 5)$ in (1.3). The ideal $I = \langle p \rangle$ has $s = 2$ associated primes in $R = \mathbb{Q}[x]$, namely $P_1 = \langle x - 1 \rangle$ and $P_2 = \langle x + 5 \rangle$, with $m_1 = 2$ and $m_2 = 1$, so $\text{amult}(I) = 3$. A differential primary decomposition of I is given by $\mathcal{A}_1 = \{1, \partial_x\}$ and $\mathcal{A}_2 = \{1\}$. The three Noetherian operators translate into the Noetherian multipliers $B_{11} = 1, B_{12} = z, B_{21} = 1$. The integrals in (2.6) now furnish the general solution $\phi(z) = \alpha \exp(z) + \beta z \exp(z) + \gamma \exp(-5z)$ to the differential equation $\phi''' + 3\phi'' - 9\phi' + 5\phi = 0$.*

The derivation of Theorem 4.4 rests on the following lemma on duality between \mathbf{x} and \mathbf{z} .

Lemma 4.6 *Let p and q be polynomials in n unknowns with coefficients in K . We have*

$$q(\partial_{\mathbf{z}}) \bullet (p(\mathbf{z}) \exp(\mathbf{x}'\mathbf{z})) = p(\partial_{\mathbf{x}}) \bullet (q(\mathbf{x}) \exp(\mathbf{x}'\mathbf{z})). \tag{4.6}$$

Proof. The parenthesized expression on the left equals $p(\partial_{\mathbf{x}}) \bullet \exp(\mathbf{x}'\mathbf{z})$, while that on the right equals $q(\partial_{\mathbf{z}}) \bullet \exp(\mathbf{x}'\mathbf{z})$. Therefore, the expression in (4.6) is the result of applying the operator $p(\partial_{\mathbf{x}})q(\partial_{\mathbf{z}}) = q(\partial_{\mathbf{z}})p(\partial_{\mathbf{x}})$ to $\exp(\mathbf{x}'\mathbf{z})$, when viewed as a function in $2n$ unknowns. □

We now generalize this lemma to $k \geq 2$, we replace p by a polynomial vector that depends on both \mathbf{x} and \mathbf{z} , and we rename that vector using the identification between (4.3) and (4.4).

Proposition 4.7 *Let $B(\mathbf{x}, \mathbf{z})$ be a k -tuple of polynomials in $2n$ variables and $A(\mathbf{x}, \partial_{\mathbf{x}}) \in D_n^k$ the corresponding k -tuple of differential operators in the Weyl algebra. Then we have*

$$q(\partial_{\mathbf{z}}) \bullet (B(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x}'\mathbf{z})) = A(\mathbf{x}, \partial_{\mathbf{x}}) \bullet (q(\mathbf{x}) \exp(\mathbf{x}'\mathbf{z})). \tag{4.7}$$

Proof. If $k = 1$, we write $A(\mathbf{x}, \partial_{\mathbf{x}}) = \sum_{\alpha} c_{\alpha}(\mathbf{x})\partial_{\mathbf{x}}^{\alpha}$ as in (4.3) and $B(\mathbf{x}, \mathbf{z}) = \sum_{\alpha} c_{\alpha}(\mathbf{x})\mathbf{z}^{\alpha}$ as in (4.4). Only finitely many of the polynomials $c_{\alpha}(\mathbf{x})$ are nonzero. Applying Lemma 4.6 gives

$$A(\mathbf{x}, \partial_{\mathbf{x}}) \bullet (q(\mathbf{x}) \exp(\mathbf{x}'\mathbf{z})) = \sum_{\alpha} c_{\alpha}(\mathbf{x})q(\partial_{\mathbf{z}}) \bullet (\mathbf{z}^{\alpha} \exp(\mathbf{x}'\mathbf{z})) = q(\partial_{\mathbf{z}}) \bullet (B(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x}'\mathbf{z})).$$

The extension from $k = 1$ to $k \geq 2$ follows because the differential operation \bullet is K -linear. □

We now take a step toward proving Theorem 4.4 in the case $s = 1$. Let M be a primary submodule of R^k with $\text{Ass}(M) = \{P\}$. Its support $V(M) = V(P)$ is an irreducible affine variety in \mathbb{C}^n . Consider the sets of all Noetherian operators and all Noetherian multipliers:

$$\begin{aligned} \mathfrak{A} &:= \{A \in D_n^k : A \bullet m \in P \text{ for all } m \in M\} && \text{and} \\ \mathfrak{B} &:= \{B \in K[\mathbf{x}, \mathbf{z}] : B(\mathbf{x}, \mathbf{z})\exp(\mathbf{x}'\mathbf{z}) \in \text{Sol}(M) \text{ for all } \mathbf{x} \in V(P)\}. \end{aligned} \tag{4.8}$$

Proposition 4.8 *The bijection between D_n^k and $K[\mathbf{x}, \mathbf{z}]^k$, given by identifying the operator A in (4.3) with the polynomial B in (4.4), restricts to a bijection between the sets \mathfrak{A} and \mathfrak{B} .*

Proof. Let $m_1, \dots, m_l \in K[\mathbf{x}]^k$ be generators of M . Suppose $A \in \mathfrak{A}$. Then

$$\sum_{i=1}^k A_i(\mathbf{x}, \partial_{\mathbf{x}}) \bullet \sum_{j=1}^l m_{ij}(\mathbf{x})f_j(\mathbf{x})$$

vanishes for all $\mathbf{x} \in V(P)$ and all polynomials $f_1, \dots, f_l \in \mathbb{C}[\mathbf{x}]$. Since the space of complex-valued polynomials is dense in the space of all entire functions on \mathbb{C}^n , the preceding implies

$$\sum_{i=1}^k A_i(\mathbf{x}, \partial_{\mathbf{x}}) \bullet m_{ij}(\mathbf{x}) \exp(\mathbf{x}'\mathbf{z}) = 0 \quad \text{for all } \mathbf{z} \in \mathbb{C}^n, \mathbf{x} \in V(P) \text{ and } j = 1, \dots, l.$$

Using Proposition 4.7, this yields

$$\sum_{i=1}^k m_{ij}(\partial_{\mathbf{z}}) \bullet B_i(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x}'\mathbf{z}) = 0 \quad \text{for all } \mathbf{z} \in \mathbb{C}^n, \mathbf{x} \in V(P) \text{ and } j = 1, \dots, l.$$

We conclude that the polynomial vector $B(\mathbf{x}, \mathbf{z})$ corresponding to $A(\mathbf{x}, \partial_{\mathbf{x}})$ lies in \mathfrak{B} .

To prove the converse, we note that the implications above are reversible. Thus, if $B(\mathbf{x}, \mathbf{z})$ is in \mathfrak{B} , then $A(\mathbf{x}, \partial_{\mathbf{x}})$ is in \mathfrak{A} . This uses the fact that linear combinations of the exponential functions $\mathbf{x} \rightarrow \exp(\mathbf{x}'\mathbf{z})$, for $\mathbf{z} \in \mathbb{C}^n$, are also dense in the space of entire functions. □

Proof of Theorem 4.4. Let \mathcal{A} be any finite subset of \mathfrak{A} which gives a differential primary decomposition of the P -primary module M . This exists and can be chosen to have cardinality equal to the length of M along P . Let \mathcal{B} be the set of Noetherian multipliers (4.4) corresponding to the set \mathcal{A} of Noetherian operators (4.3). Proposition 4.8 shows that the exponential function $\mathbf{z} \rightarrow B(\mathbf{x}, \mathbf{z})\exp(\mathbf{x}'\mathbf{z})$ is in $\text{Sol}(M)$ whenever $\mathbf{x} \in V(P)$ and $B \in \mathcal{B}$. Hence all \mathbb{C} -linear combinations of such functions are in $\text{Sol}(M)$. More generally, by differentiating under the integral sign, we find that all functions of the following form are solutions of M :

$$\psi(\mathbf{z}) = \sum_{B \in \mathcal{B}} \int_{V(P)} B(\mathbf{x}, \mathbf{z})\exp(\mathbf{x}'\mathbf{z})d\mu_B(\mathbf{x}).$$

We need to argue that all solutions in $\mathcal{F} = C^\infty(\Omega)$ admit such an integral representation. Suppose first that all associated primes of M are minimal. Then each \mathcal{A}_i spans a bimodule in the sense of [9, Theorem 3.2 (d)]. Hence, for each associated prime P_i , the module

$$M_i = \{m \in R^k : \delta \bullet m \in P_i \text{ for all } \delta \in \mathcal{A}_i\}$$

is P_i -primary, and $M = M_1 \cap \dots \cap M_s$ is a minimal primary decomposition. The operators in \mathcal{A}_i are in the relative Weyl algebra $D_n(\mathcal{S}_i)$ and fully characterize the P_i -primary component of M . We may thus follow the classical analytical constructions in the books [7, 23, 32] to patch together the integral representation of $\text{Sol}(M_i)$ for $i = 1, \dots, s$, under the correspondence of Noetherian operators and Noetherian multipliers. Therefore, all solutions have the form (2.6).

Things are more delicate when M has embedded primes. Namely, if P_i is embedded, then the operators in \mathcal{A}_i only characterize the contribution of the P_i -primary component relative to all other components contained in P_i . We see this in Section 5. One argues by enlarging \mathcal{A}_i to vector space generators of the relevant bimodule. Then the previous patching argument applies. And, afterward one shows that the added summand in the integral representation are redundant because they are covered by associated varieties $V(P_j)$ containing $V(P_i)$. □

5. Software and algorithm

In this section, we present an algorithm for solving linear PDE with constant coefficients. It is based on the methods for ideals given in [8, 10, 11]. The case of modules appears in [9]. We note that the computation of Noetherian operators has a long history, going back to work in the 1990's by Ulrich Oberst [28, 29, 30, 31], who developed a construction of Noetherian operators for primary modules. This was further developed by Damiano, Sabadini and Struppa [15] who presented the first Gröbner-based algorithm. It works for primary ideals under the restrictive assumption that the characteristic variety has a rational point after passing to a (algebraically nonclosed) field of fractions. Their article also points to an implementation in CoCoA, but we were unable to access that code. Since these early approaches rely on the ideals or modules being primary, using them in practice requires first computing a primary decomposition. If there are embedded primes, the number of Noetherian operators output by these methods will not be minimal either.

We here present a new algorithm that is universally applicable, to all ideals and modules over a polynomial ring. There are no restrictions on the input and the output is minimal. The input is a submodule M of R^k , where $R = K[x_1, \dots, x_n]$. The output is a differential primary decomposition of size $\text{amult}(M)$ as in Theorem 4.3. A first step is to find $\text{Ass}(M) = \{P_1, \dots, P_s\}$. For each associated prime P_i , the elements $A(\mathbf{x}, \partial_{\mathbf{x}})$ in the finite set $\mathcal{A}_i \subset D_n(\mathcal{S}_i)$ are rewritten as polynomials $B(\mathbf{x}, \mathbf{z})$, using the identification of (4.3) with (4.4). Only the $\text{codim}(P_i)$ many variables z_i with $x_i \notin \mathcal{S}_i$ appear in these Noetherian multipliers B .

We now describe our implementation for (2.7) in Macaulay2 [21]. The command is called `solvePDE`, as in [12, Section 5]. It is distributed with Macaulay2 starting from version 1.18 in the package `NoetherianOperators` [8]. The user begins by fixing a polynomial ring $R = K[x_1, \dots, x_n]$. Here K is usually the rational numbers `QQ`. Fairly arbitrary variable names x_i are allowed. The argument of `solvePDE` is an ideal in R or a submodule of R^k . The output is a list of pairs $\{P_i, \{B_{i1}, \dots, B_{i,m_i}\}\}$ for $i = 1, \dots, s$, where P_i is a prime ideal given by generators in R , and each B_{ij} is a vector over a newly created polynomial ring $K[x_1, \dots, x_n, z_1, \dots, z_n]$. The new variables z_i are named internally by Macaulay2. The system writes dx_i for z_i . To be precise, each new variable is created from an old variable by prepending the character `d`. This notation can be confusing at first, but one gets used to it. The logic comes from the differential primary decompositions described in [12, Section 5].

Each B_{ij} in the output of `solvePDE` encodes an exponential solution $B_{ij}(\mathbf{x}, \mathbf{z})\exp(\mathbf{x}'\mathbf{z})$ to M . Here \mathbf{x} are the old variables chosen by the user, and \mathbf{x} denotes points in the irreducible variety $V(P_i) \subseteq \mathbb{C}^n$. The solution is a function in the new unknowns $\mathbf{z} = (dx_1, \dots, dx_n)$. For instance, if $n = 3$ and the input is in the ring `QQ[u, v, w]`, then the output lives in the ring `QQ[u, v, w, du, dv, dw]`. Each solution to the PDE

is a function $\psi(du, dv, dw)$, and these functions are parametrized by a variety $V(P_i)$ in a 3-space whose coordinates are (u, v, w) .

We now demonstrate how this works for two examples featured in the introduction.

Example 5.1. Consider the third order ODE (1.1) in Example 4.5. We solve this as follows:

```
R = QQ[x]; I = ideal(x^3 + 3*x^2 - 9*x + 5); solvePDE(I)
  {{ideal(x - 1), { | 1 |, | dx |}}, {ideal(x + 5), { | 1 |}}}
```

The first line is the input. The second line is the output created by solvePDE. This list of $s = 2$ pairs encodes the general solution $\phi(z)$. Remember: z is the newly created symbol dx.

Example 5.2. We solve the PDE (1.8) by typing the 2×3 matrix whose columns are (2.1):

```
R = QQ[x1,x2,x3,x4];
M = image matrix {{x1^2,x2*x3,x1^2*x3},{x1*x2,x3^2,x1*x2*x4}}; solvePDE(M)
```

The reader is encouraged to run this code and to check that the output is the solution (1.9).

The method in solvePDE is described in Algorithm 1 below. A key ingredient is a translation map. We now explain this in the simplest case, when the module is supported in one point. Suppose $V(M) = \{\mathbf{u}\}$ for some $\mathbf{u} \in K^n$. We set $\mathfrak{m}_{\mathbf{u}} = \langle x_1 - u_1, \dots, x_n - u_n \rangle$ and

$$\gamma_{\mathbf{u}} : R \rightarrow R, x_i \mapsto x_i + u_i \quad \text{for } i = 1, \dots, n. \tag{5.1}$$

The following two results are straightforward. We will later use them when M is any primary module, \mathbf{u} is the generic point of $V(M)$, and $\mathbb{K} = K(\mathbf{u})$ is the associated field extension of K .

Proposition 5.3 A constant coefficient operator $A(\partial_x)$ is a Noetherian operator for the $\mathfrak{m}_{\mathbf{u}}$ -primary module M if and only if $A(\partial_x)$ is a Noetherian operator for the \mathfrak{m}_0 -primary module $\hat{M} := \gamma_{\mathbf{u}}(M)$. Dually, $B(\mathbf{z}) \exp(\mathbf{u}'\mathbf{z})$ is in $\text{Sol}(M)$ if and only if $B(\mathbf{z})$ is in $\text{Sol}(\hat{M})$.

We note that all Noetherian operators over a K -rational point can be taken to have constant coefficients. This follows from Theorem 3.8. This observation reduces the computation of solutions for a primary module to finding the polynomial solutions of the translated module. Next, we bound the degrees of these polynomials.

Proposition 5.4 Let $\hat{M} \subseteq R^k$ be an \mathfrak{m}_0 -primary module. There exists an integer r such that $\mathfrak{m}_0^{r+1}R^k \subseteq \hat{M}$. The space $\text{Sol}(\hat{M})$ consists of k -tuples of polynomials of degree $\leq r$.

Propositions 5.3 and 5.4 furnish a method for computing solutions of an $\mathfrak{m}_{\mathbf{u}}$ -primary module M . We start by translating M so that it becomes the \mathfrak{m}_0 -primary module \hat{M} . The integer r provides an ansatz $\sum_{j=1}^k \sum_{|\alpha| \leq r} v_{\alpha,j} \mathbf{z}^{\alpha} e_j$ for the polynomial solutions. The coefficients $v_{\alpha,j}$ are computed by linear algebra over the ground field K . Here are the steps:

1. Let r be the smallest integer such that $\mathfrak{m}_0^{r+1}R^k \subseteq \hat{M}$.
2. Let $\text{Diff}(\hat{M})$ be the matrix whose entries are the polynomials $\hat{m}_i \bullet (\mathbf{z}^{\alpha} e_j) \in R$. The row labels are the generators $\hat{m}_1, \dots, \hat{m}_l$ of \hat{M} , and the column labels are the $\mathbf{z}^{\alpha} e_j$.
3. Let $\ker_K(\text{Diff}(\hat{M}))$ denote the K -linear subspace of the R -module $\ker_R(\text{Diff}(\hat{M}))$ consisting of vectors $(v_{\alpha,j})$ with all entries in K . Every such vector gives a solution

$$\sum_{j=1}^k \sum_{|\alpha| \leq r} v_{\alpha,j} \mathbf{z}^{\alpha} \exp(\mathbf{u}'\mathbf{z}) e_j \in \text{Sol}(M). \tag{5.2}$$

Algorithm 1 SolvePDE

Input: An arbitrary submodule M of R^k
Output: List of associated primes with corresponding Noetherian multipliers.

- 1: **for** each associated prime ideal P of M **do**
- 2: $U \leftarrow MR_P^k \cap R^k$
- 3: $V \leftarrow (U : P^\infty)$
- 4: $r \leftarrow$ the smallest number such that $V \cap P^{r+1}R^k$ is a subset of U
- 5: $S \leftarrow$ a maximal set of independent variables modulo P
- 6: $\mathbb{K} \leftarrow \text{Frac}(R/P)$
- 7: $T \leftarrow \mathbb{K}[y_i : x_i \notin S]$
- 8: $\gamma \leftarrow$ the map defined in (5.5)
- 9: $\mathfrak{m} \leftarrow$ the irrelevant ideal in T
- 10: $\hat{U} \leftarrow \gamma(U) + \mathfrak{m}^{r+1}T^k$
- 11: $\hat{V} \leftarrow \gamma(V) + \mathfrak{m}^{r+1}T^k$
- 12: $N \leftarrow$ a \mathbb{K} -vector space basis of the space of k -tuples of polynomials of degree $\leq r$
- 13: $\text{Diff}(\hat{U}) \leftarrow$ the matrix given by the \bullet -product of generators of \hat{U} with elements of N
- 14: $\text{Diff}(\hat{V}) \leftarrow$ the matrix given by the \bullet -product of generators of \hat{V} with elements of N
- 15: $\mathcal{K} \leftarrow \ker_{\mathbb{K}}(\text{Diff}(\hat{U})) / \ker_{\mathbb{K}}(\text{Diff}(\hat{V}))$
- 16: $\mathcal{A} \leftarrow$ a \mathbb{K} -vector space basis of \mathcal{K}
- 17: $\mathcal{B} \leftarrow$ lifts of the vectors in $\mathcal{A} \subset T^k$ to vectors in $R[\text{d}x_1, \dots, \text{d}x_n]^k$
- 18: **return** the pair (P, \mathcal{B})

Example 5.5 [$n = k = r = 2$] *The following module is \mathfrak{m}_0 -primary of multiplicity three:*

$$M = \text{image}_R \begin{bmatrix} \partial_1^3 & \partial_2 - c_1\partial_1^2 - c_2\partial_1 & c_3\partial_1^2 + c_4\partial_1 + c_5 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.3}$$

Here c_1, c_2, c_3, c_4, c_5 are arbitrary constants in K . The matrix $\text{Diff}(M)$ has three rows, one for each generator of M , and it has 12 columns, indexed by $e_1, z_1e_1, \dots, z_2^2e_1, e_2, z_1e_2, \dots, z_2^2e_2$. The space $\ker_K(\text{Diff}(M))$ is 3-dimensional. A basis furnishes the three polynomial solutions

$$\begin{bmatrix} -1 \\ c_5 \end{bmatrix}, \begin{bmatrix} -(z_1 + c_2z_2) \\ c_5z_1 + c_2c_5z_2 + c_4 \end{bmatrix}, \begin{bmatrix} -((z_1 + c_2z_2)^2 + 2c_1z_2) \\ c_5(z_1 + c_2z_2)^2 + 2c_4z_1 + 2(c_1c_5 + c_2c_4)z_2 + 2c_3 \end{bmatrix}. \tag{5.4}$$

We now turn to Algorithm 1. The input and output are as described in (2.7). The method was introduced in [9, Algorithm 4.6] for computing differential primary decompositions. We use it for solving PDE. It is implemented in Macaulay2 under the command `solvePDE`. In our discussion, the line numbers refer to the corresponding lines of pseudocode in Algorithm 1.

Line 1 We begin by finding all associated primes of M . These define the irreducible varieties V_i in (2.7). By [19, Theorem 1.1], the associated primes of codimension i coincide with the minimal primes of $\text{Ann Ext}_R^i(M, R)$. This reduces the problem of finding associated primes of a module to the more familiar problem of finding minimal primes of a polynomial ideal. This method is implemented and distributed with Macaulay2 starting from version 1.17 via the command `associatedPrimes R^k/M`. See [9, Section 2].

The remaining steps are repeated for each $P \in \text{Ass}(M)$. For a fixed associated prime P , our goal is to identify the contribution to $\text{Sol}(M)$ of the P -primary component of M .

Lines 2–3 To achieve this goal, we study solutions for two different R -submodules of R^k . The first one, denoted U , is the intersection of all P_i -primary components of M , where P_i are the associated primes contained in P . Thus $U = MR_P^k \cap R^k$, which is the extension-contraction module of M under localization at P . It is computed as $U = (M : f^\infty)$, where $f \in R$ is contained in every associated prime P_j not contained in P .

The second module, denoted V , is the intersection of all P_i -primary components of M , where P_i is strictly contained in P . Hence, $V = (U : P^\infty)$ is the saturation of U at P . We have $U = V \cap Q$, where Q is a P -primary component of M . Thus, the difference between the solution spaces $\text{Sol}(U)$ and $\text{Sol}(V)$ is caused by the primary module Q .

When P is a minimal prime, U is the unique P -primary component of M , and $V = R^k$.

Line 4 The integer r bounds the degree of Noetherian multipliers associated to U but not V . Namely, if the function $\phi(\mathbf{z}) = B(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x}'\mathbf{z})$ lies in $\text{Sol}(U) \setminus \text{Sol}(V)$ for all $\mathbf{x} \in V(P)$, then the \mathbf{z} -degree of the polynomial $B(\mathbf{x}, \mathbf{z})$ is at most r . This will lead to an ansatz for the Noetherian multipliers responsible for the difference between $\text{Sol}(U)$ and $\text{Sol}(V)$.

Lines 5–8 The modules U and V are reduced to simpler modules \hat{U} and \hat{V} with similar properties. Namely, \hat{U} and \hat{V} are primary and their characteristic varieties are the origin. This reduction involves two new ingredients: a new polynomial ring T in fewer variables over a field \mathbb{K} that is a finite extension of K , and a ring map $\gamma : R \rightarrow T$.

Fix a maximal set $\mathcal{S} = \{x_{i_1}, \dots, x_{i_{n-c}}\}$ with $P \cap K[x_{i_1}, \dots, x_{i_{n-c}}] = \{0\}$. We define $T := \mathbb{K}[y_i : x_i \notin \mathcal{S}]$, where $\mathbb{K} = \text{Frac}(R/P)$. This is a polynomial ring in $n - |\mathcal{S}| = c$ new variables y_i , corresponding to the x_i not in the set \mathcal{S} of independent variables. Writing u_i for the image of x_i in $\mathbb{K} = \text{Frac}(R/P)$, the ring map γ is defined as follows:

$$\gamma : R \rightarrow T, \quad x_i \mapsto \begin{cases} y_i + u_i, & \text{if } x_i \notin \mathcal{S}, \\ u_i, & \text{if } x_i \in \mathcal{S}. \end{cases} \tag{5.5}$$

By abuse of notation, we denote by γ the extension of (5.5) to a map $R^k \rightarrow T^k$.

Lines 9–11 Let $\mathfrak{m} := \langle y_i : x_i \notin \mathcal{S} \rangle$ be the irrelevant ideal of T . We define the T -submodules

$$\hat{U} := \gamma(U) + \mathfrak{m}^{r+1}T^k \quad \text{and} \quad \hat{V} := \gamma(V) + \mathfrak{m}^{r+1}T^k \quad \text{of } T^k.$$

These modules are \mathfrak{m} -primary: their solutions are finite-dimensional \mathbb{K} -vector spaces consisting of polynomials of degree $\leq r$. The polynomials in $\text{Sol}(\hat{U}) \setminus \text{Sol}(\hat{V})$ capture the difference between \hat{U} and \hat{V} , and also the difference between U and V after lifting.

Lines 12–14 We construct matrices $\text{Diff}(\hat{U})$ and $\text{Diff}(\hat{V})$ with entries in $\mathbb{K}[z_i : x_i \notin \mathcal{S}]$. As in (5.2), their kernels over \mathbb{K} correspond to polynomial solutions of \hat{U} and \hat{V} . The set $N = \{\mathbf{z}^\alpha e_j : |\alpha| \leq r, j = 1, \dots, k\}$ is a \mathbb{K} -basis for elements of degree $\leq r$ in $\mathbb{K}[z_i : x_i \notin \mathcal{S}]^k$. The y_i -variables act on the z_i variables as partial derivatives, i.e. $y_i = \frac{\partial}{\partial z_i}$. We define the matrix $\text{Diff}(\hat{U})$

as follows. Let $\hat{U}_1, \dots, \hat{U}_\ell$ be generators of \hat{U} . The rows of $\text{Diff}(\hat{U})$ are indexed by these generators, the columns are indexed by N , and the entries are the polynomials $\hat{U}_i \bullet \mathbf{z}^\alpha e_j$. In the same way, we construct $\text{Diff}(\hat{V})$.

Lines 15–16 Let $\ker_{\mathbb{K}}(\text{Diff}(\hat{U}))$ be the space of vectors in the kernel of $\text{Diff}(\hat{U})$ whose entries are in \mathbb{K} . The \mathbb{K} -vector space $\ker_{\mathbb{K}}(\text{Diff}(\hat{U}))$ parametrizes the polynomial solutions

$$\sum_{j=1}^k \sum_{|\alpha| \leq r} v_{\alpha,j} \mathbf{z}^\alpha e_j \in \text{Sol}(\hat{U}).$$

The same holds for \hat{V} . The quotient space $\mathcal{K} := \ker_{\mathbb{K}}(\text{Diff}(\hat{U})) / \ker_{\mathbb{K}}(\text{Diff}(\hat{V}))$ characterizes excess solutions in $\text{Sol}(\hat{U})$ relative to $\text{Sol}(\hat{V})$. Write \mathcal{A} for a \mathbb{K} -basis of \mathcal{K} .

Lines 17–18 We interpret \mathcal{A} as a set of Noetherian multipliers for M by performing a series of lifts and transformations. For each element $\bar{\mathbf{v}} \in \mathcal{A}$, we choose a representative $\mathbf{v} \in \ker_{\mathbb{K}}(\text{Diff}(\hat{U}))$.

The entries of \mathbf{v} are in $\mathbb{K} = \text{Frac}(R/P)$ and may contain denominators. Multiplying \mathbf{v} by a common multiple of the denominators yields a vector with entries in R/P , indexed by N . We lift this to a vector $\mathbf{u} = (u_{\alpha_j})$ with entries in R . The Noetherian multiplier corresponding to \mathbf{u} is the following vector in $R[\mathrm{d}x_i : x_i \notin S]^k$:

$$B(\mathbf{x}, \mathrm{d}\mathbf{x}) = \sum_{j=1}^k \sum_{|\alpha| \leq r} u_{\alpha_j}(\mathbf{x}) \mathrm{d}\mathbf{x}^\alpha e_j.$$

Applying the map $\bar{\mathbf{v}} \mapsto \mathbf{u}$ to each $\bar{\mathbf{v}} \in \mathcal{A}$ yields a set \mathcal{B} of Noetherian multipliers. These multipliers describe the contribution of the P -primary component of M to $\text{Sol}(M)$.

The output of Algorithm 1 is a list of pairs (P, \mathcal{B}) , where P ranges over $\text{Ass}(M)$ and $\mathcal{B} = \{B_1, \dots, B_m\}$ is a subset of $R[\mathrm{d}x_1, \dots, \mathrm{d}x_n]^k$. The cardinality m is the multiplicity of M along P . The output describes the solutions to the PDE given by M . Consider the functions

$$\phi_P(\mathrm{d}x_1, \dots, \mathrm{d}x_n) = \sum_{i=1}^m \int_{V(P)} B_i(\mathbf{x}, \mathrm{d}\mathbf{x}) \exp(x_1 \mathrm{d}x_1 + \dots + x_n \mathrm{d}x_n) d\mu_i(\mathbf{x}).$$

Then the space of solutions to M consists of all functions

$$\sum_{P \in \text{Ass}(M)} \phi_P(\mathrm{d}x_1, \dots, \mathrm{d}x_n).$$

A differential primary decomposition of M is obtained from this by reading $\mathrm{d}x_i$ as ∂_{x_i} . Indeed, the command `differentialPrimaryDecomposition` described in [9] is identical to our command `solvePDE`. All examples in [9, Section 6] can be interpreted as solving PDE.

6. Schemes and coherent sheaves

The concepts of schemes and coherent sheaves are central to modern algebraic geometry. These generalize varieties and vector bundles, and they encode geometric structures with multiplicities. The point is that the supports of coherent sheaves and other schemes are generally nonreduced. We here argue that our linear PDE offer a useful way to think about the geometry of these objects. That perspective motivated the writing of [27, Section 3.3].

The affine schemes we consider are defined by ideals I in a polynomial ring R . Likewise, submodules M of R^k represent coherent sheaves on \mathbb{C}^n . We study the affine scheme $\text{Spec}(R/I)$ and the coherent sheaf given by the module R^k/M . The underlying geometric objects are the affine varieties $V(I)$ and $V(M)$ in \mathbb{C}^n . The latter was discussed in Section 3. The solution spaces $\text{Sol}(I)$ or $\text{Sol}(M)$ furnish nonreduced structures on these varieties, encoded in the integral representations due to Ehrenpreis–Palamodov. According to Section 4, these are dual to differential primary decompositions. Coherent sheaves were a classical tool in the analysis of linear PDE, but in the analytic category, where their role was largely theoretical. The Ehrenpreis–Palamodov Fundamental Principle appears in Hörmander’s book under the header *Coherent analytic sheaves on Stein manifolds* [23, Chapter VII]. Likewise, Treves’ exposition, in the title of [35, Section 3.2], calls for *Analytic sheaves to the rescue*. By contrast, sheaves in this paper are concrete and algebraic: they are modules in `Macaulay2`.

One purpose of this section is to explore how PDE and their solutions behave under degenerations. We consider ideals and modules whose generators depend on a parameter ϵ . This is modelled algebraically by working over the field $K = \mathbb{C}(\epsilon)$ of rational functions in the variable ϵ . Algorithm 1 can be applied to the polynomial ring $R = K[x_1, \dots, x_n]$ over that field. We think of ϵ as a small quantity and we are interested in what happens when $\epsilon \rightarrow 0$.

Our discussion in this section is very informal. This is by design. We present a sequence of examples that illustrates the geometric ideas. The only formal result is Theorem 6.6, which concerns the role of the Quot scheme in parametrizing systems of linear PDE.

Example 6.1 ($n = 2$) Consider the prime ideal $I_\epsilon = \langle \partial_1^2 - \epsilon^2 \partial_2 \rangle$. For nonzero parameters ϵ , by Theorem 2.2, the solutions to this PDE are represented as one-dimensional integrals

$$\alpha_\epsilon(z_1, z_2) = \int \exp(\epsilon t z_1 + t^2 z_2) dt \in \text{Sol}(I).$$

By taking the limit for $\epsilon \rightarrow 0$, this yields arbitrary functions $a(z_2)$. These are among the solutions to $I_0 = \langle \partial_1^2 \rangle$. Other limit solutions are obtained via the reflection $t \mapsto -t$. Set

$$\beta_\epsilon(z_1, z_2) = \int \exp(-\epsilon t z_1 + t^2 z_2) dt \in \text{Sol}(I).$$

Note the similarity to the one-dimensional wave equation (1.5) with $c = \epsilon$. The solution for $\epsilon = 0$ is given in (1.6). This is found from the integrals above by taking the following limit:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} (\alpha_\epsilon(z_1, z_2) - \beta_\epsilon(z_1, z_2)) &= \int \lim_{\epsilon \rightarrow 0} \frac{\exp(\epsilon t z_1 + t^2 z_2) - \exp(-\epsilon t z_1 + t^2 z_2)}{2\epsilon} dt \\ &= \int t z_1 \exp(t^2 z_2) dt \\ &= z_1 b(z_2). \end{aligned} \tag{6.1}$$

We conclude that the general solution to I_0 equals $\phi(z_1, z_2) = a(z_2) + z_1 b(z_2)$, where b is any function in one variable. The calculus limit in (6.1) realizes a scheme-theoretic limit in the sense of algebraic geometry. Namely, two lines in (1.7) converge to a double line in \mathbb{C}^2 .

Example 6.2 ($n = 3$). For $\epsilon \neq 0$ consider the curve $t \mapsto (\epsilon t^3, t^4, \epsilon^2 t^2)$ in \mathbb{C}^3 . Its prime ideal equals $I_\epsilon = \langle \partial_1^2 - \partial_2 \partial_3, \partial_3^2 - \epsilon^4 \partial_2 \rangle$. The solution space $\text{Sol}(I_\epsilon)$ consists of the functions

$$\phi(z_1, z_2, z_3) = \int \exp(\epsilon t^3 z_1 + t^4 z_2 + \epsilon^2 t^2 z_3) dt. \tag{6.2}$$

What happens to these functions when ϵ tends to zero? We address this question algebraically. The scheme-theoretic limit of the given ideal I_ϵ is the ideal in Example 2.3. This is verified by a Gröbner basis computation (cf. [18, Section 15.8]). Passing from ideals to their varieties, we see a toric curve in \mathbb{C}^3 that degenerates to a line with multiplicity four.

We claim that the formula in (2.9) arises from (6.2), just as in Example 6.1. Namely, set $i = \sqrt{-1}$ and let $\phi_\epsilon \in \text{Sol}(I_\epsilon)$ be the function that is obtained from ϕ in (6.2) by replacing the parameter t with $i^k t$. Then the following four functions on the left are solutions to I_ϵ :

$$\begin{aligned} \phi_0 + \phi_1 + \phi_2 + \phi_3 &\longrightarrow a(z_2), \\ \epsilon^{-1}(\phi_0 + i\phi_1 + i^2\phi_2 + i^3\phi_3) &\longrightarrow z_1 b(z_2), \\ \epsilon^{-2}(\phi_0 + i^2\phi_1 + i^4\phi_2 + i^6\phi_3) &\longrightarrow z_1^2 c'(z_2) + 2z_3 c(z_2), \\ \epsilon^{-3}(\phi_0 + i^3\phi_1 + i^6\phi_2 + i^9\phi_3) &\longrightarrow z_1^3 d'(z_2) + 6z_1 z_3 d(z_2). \end{aligned}$$

The functions obtained as limits on the right are precisely the four summands seen in (2.9). Thus, the solution spaces to this family of PDE reflect the degeneration of the toric curve.

Such limits make sense also for modules. If a module $M_\epsilon \subseteq R^k$ depends on a parameter ϵ then we study its solution space $\text{Sol}(M_\epsilon)$ as ϵ tends to zero. Geometrically, we examine flat families of coherent sheaves on \mathbb{C}^n or on \mathbb{P}^{n-1} . A typical scenario comes from the action of the torus $(\mathbb{C}^*)^n$, where Gröbner degenerations arise as limits under one-parameter subgroups. The limit objects are monomial ideals (for

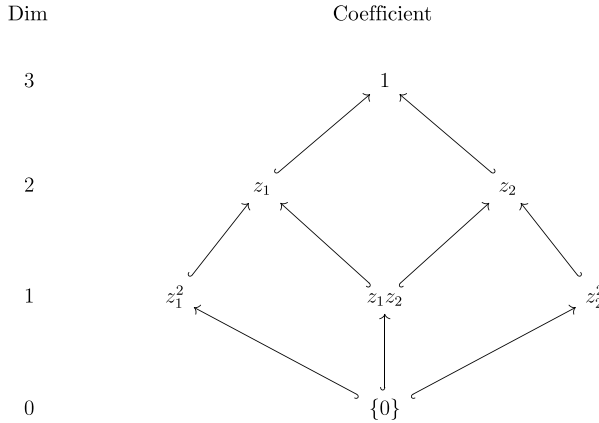


Figure 1. The coefficient vectors of the solutions to the PDE in Example 6.3 correspond to the above linear spaces with the given inclusions. We obtain two complete flags in \mathbb{C}^3 , along with one interaction between the two. Experts on quiver representations will take note.

$k = 1$) or torus-fixed submodules (for $k \geq 2$). The next example illustrates their rich structure with an explicit family of torus-fixed submodules.

Example 6.3 ($n = 2, k = 3, l = 6$) Given a 3×6 matrix A with random real entries, we set

$$M = \text{image}_R(A \cdot \text{diag}(\partial_1, \partial_1^2, \partial_1^3, \partial_2, \partial_2^2, \partial_2^3)) \subset R^3.$$

Then M is torus-fixed and \mathfrak{m} -primary, where $\mathfrak{m} = \langle \partial_1, \partial_2 \rangle$, and $\text{amult}(M) = 10$. A basis of $\text{Sol}(M)$ is given by ten polynomial solutions, namely the standard basis vectors e_1, e_2, e_3 , four vectors that are multiples of z_1, z_1, z_2, z_2 , and three vectors that are multiples z_1^2, z_1z_2, z_2^2 . The reader is invited to verify this with `Macaulay2`. Here is the input for one concrete instance:

```
R = QQ[x1,x2]
M = image matrix {{7*x1,5*x1^2,8*x1^3, 5*x2,9*x2^2,5*x2^3},
                 {8*x1,9*x1^2,8*x1^3, 4*x2,2*x2^2,4*x2^3},
                 {3*x1,2*x1^2,6*x1^3, 4*x2,4*x2^2,7*x2^3}}
solvePDE(M)
```

By varying the matrix A , and by extracting the vector multipliers of $1, z_1$ and z_1^2 , we obtain any complete flag of subspaces in \mathbb{C}^3 . The vector multipliers of $1, z_2$, and z_2^2 give us another complete flag of subspaces in \mathbb{C}^3 , and the multiplier of z_1z_2 gives us the intersection line of the planes corresponding to the multipliers of z_1 and z_2 . This is illustrated in Figure 1. Thus flag varieties, with possible additional structure, appear naturally in such families.

The degenerations of ideals and modules we saw point us to Hilbert schemes and Quot schemes. Let us now also take a fresh look at Example 5.5. The modules M in that example form a flat family over the affine space \mathbb{C}^5 with coordinates $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5)$. For $\mathbf{c} = 0$ we obtain the PDE whose solution space equals $\mathbb{C}\{e_1, z_1e_1, z_1^2e_1\}$. But, what happens when one of the coordinates of \mathbf{c} tends to infinity? That limit exists in the Quot scheme.

In our context, Hilbert schemes and Quot schemes serve as parameter spaces for primary ideals and primary modules. This was shown for ideals in [11] and for modules in [9]. In what follows we shall discuss the latter case. Fix a prime ideal P of codimension c in $R = K[x_1, \dots, x_n]$. Write \mathbb{K} for the field of fractions of the integral domain R/P , as in Line 6 of Algorithm 1. We write u_1, \dots, u_n for the images in \mathbb{K} of the variables x_1, \dots, x_n in R . After possibly permuting these variables, we shall assume that

$P \cap K[x_{c+1}, \dots, x_n] = \{0\}$. The set $\{u_{c+1}, \dots, u_n\}$ is algebraically independent over K , so it serves as \mathcal{S} in Line 5.

Consider the formal power series ring $S = \mathbb{K}[[y_1, \dots, y_c]]$ where y_1, \dots, y_c are new variables. This is a local ring with maximal ideal $\mathfrak{m} = (y_1, \dots, y_c)$. We are interested in \mathfrak{m} -primary submodules L of S^k . The quotient module S^k/L is finite-dimensional as a \mathbb{K} -vector space, and we write $\nu = \dim_{\mathbb{K}}(S^k/L)$ for its dimension. The *punctual Quot scheme* is a parameter space whose points are precisely those modules. We denote the Quot scheme by

$$\text{Quot}^\nu(S^k) = \{L \subset S^k : L \text{ submodule with } \text{Ass}(L) = \mathfrak{m} \text{ and } \dim_{\mathbb{K}}(S^k/L) = \nu\}. \tag{6.3}$$

This is a quasiprojective scheme over \mathbb{K} , i.e. it can be defined by a finite system of polynomial equations and inequations in a large but finite set of variables. Each solution to that system is one submodule L . This construction goes back to Grothendieck, and it plays a fundamental role in parametrizing coherent sheaves in algebraic geometry. While a constructive approach to Quot schemes exists, thanks to Skjelnes [34], the problem remains to write defining equations for $\text{Quot}^\nu(S^k)$ in a computer-readable format, for small values of c, k, ν . A natural place to start would be the case $c = 2$, given that coherent sheaves supported at a smooth point on a surface are of considerable interest in geometry and physics [1, 3, 20, 22].

The next two examples offer a concrete illustration of the concept of Quot schemes. We exhibit the Quot schemes that parametrize two families of linear PDE we encountered before.

Example 6.4 ($c = 2, k = 3, \nu = 10$) Consider the formal power series ring $S = \mathbb{K}[[y_1, y_2]]$ where \mathbb{K} is any field. Replacing ∂_1, ∂_2 with y_1, y_2 in Example 6.3, every 3×6 matrix A over \mathbb{K} defines a submodule L of S^3 . The quotient S^3/L is a 10-dimensional \mathbb{K} -vector space, so L corresponds to a point in the Quot scheme $\text{Quot}^{10}(S^3)$. By varying A , we obtain a closed subscheme of $\text{Quot}^{10}(S^3)$, which contains the complete flag variety we saw in Example 6.3.

Example 6.5 For $S = \mathbb{K}[[y_1, y_2]]$, the scheme $\text{Quot}^\nu(S^k)$ is an irreducible variety of dimension $k\nu - 1$, by [3, Theorem 2.2]. If $k = 2, \nu = 3$ then this dimension is five. The affine space with coordinates \mathbf{c} in Example 5.5 is a dense open subset W of $\text{Quot}^3(S^2)$, by [3, Section 7].

For $k = 1$, the Quot scheme is the *punctual Hilbert scheme* $\text{Hilb}^\nu(S)$; see [6]. The points on this Hilbert scheme represent \mathfrak{m} -primary ideals of length ν in $S = \mathbb{K}[[y_1, \dots, y_c]]$. It was shown in [11, Theorem 2.1] that $\text{Hilb}^\nu(S)$ parametrizes the set of all P -primary ideals in R of multiplicity ν . This means that we can encode P -primary ideals in R by \mathfrak{m} -primary ideals in S , thus reducing scheme structures on any higher-dimensional variety to a scheme structure on a single point. This was generalized from ideals to submodules ($k \geq 2$) by Chen and Cid-Ruiz [9]. Geometrically, we encode coherent sheaves by those supported at one point, namely the generic point of $V(P)$, corresponding to the field extension \mathbb{K}/K . Here is the main result from [9], stated for the polynomial ring R , analogously to [11, Theorem 2.1].

Theorem 6.6 The following four sets of objects are in a natural bijective correspondence:

- (a) P -primary submodules M in R^k of multiplicity ν over P ,
- (b) \mathbb{K} -points in the punctual Quot scheme $\text{Quot}^\nu(\mathbb{K}[[y_1, \dots, y_c]]^k)$,
- (c) ν -dimensional \mathbb{K} -subspaces of $\mathbb{K}[z_1, \dots, z_c]^k$ that are closed under differentiation,
- (d) ν -dimensional \mathbb{K} -subspaces of the Weyl–Noether module $\mathbb{K} \otimes_R D_{n,c}^k$ that are R -bimodules.

Moreover, any basis of the \mathbb{K} -subspace (d) can be lifted to a finite subset \mathcal{A} of $D_{n,c}^k$ such that

$$M = \{m \in R^k : \delta \bullet m \in P \text{ for all } \delta \in \mathcal{A}\}. \tag{6.4}$$

Here $D_{n,c}$ is the subalgebra of the Weyl algebra D_n consisting of all operators (4.3) with $s_{c+1} = \dots = s_n = 0$. This is a special case of (4.2). Elements in $D_{n,c}$ are differential operators in $\partial_{x_1}, \dots, \partial_{x_c}$ whose coefficients are polynomials in x_1, \dots, x_n . Note that $D_{n,0} = R$ and $D_{n,n} = D_n$. Equation (6.4) says

that \mathcal{A} is a differential primary decomposition for the primary module M . The Noetherian operators in \mathcal{A} characterize membership in M . In this paper, however, we focus on the \mathbb{K} -linear subspaces in item (c). By clearing denominators, we can represent such a subspace by a basis \mathcal{B} of elements in $K[x_1, \dots, x_n][z_1, \dots, z_c]$. These are precisely the Noetherian multipliers needed for the integral representation of $\text{Sol}(M)$. In summary, Theorem 6.6 may be understood as a theoretical counterpart to Algorithm 1. The following example elucidates the important role played by the Quot scheme in our algorithm.

Example 6.7 ($n = 4, c = 2, k = 3, v = 10$) *Let P be the prime ideal in [11, equation (1)]. Equivalently, P is the prime P_6 in Example 2.4. The surface $V(P) \subset \mathbb{C}^4$ is the cone over the twisted cubic curve. Consider the point in $\text{Quot}^{10}(\mathbb{K}[[y_1, y_2]]^3)$ given by a matrix A as in Example 6.4. The bijection from (b) to (a) in Theorem 6.6 yields a P -primary submodule M of multiplicity 10 in $K[x_1, \dots, x_4]^3$. Generators for the module M are found by computing the inverse image under the map γ , as shown in [9, equation (2)]. This step is the analogue for modules of the elimination that creates a large P -primary ideal Q from [11, equation (5)]. Geometrically speaking, the 10-dimensional space of polynomial vectors that are solutions to the PDE in Example 6.3 encodes a coherent sheaf of rank 3 on the singular surface $V(P)$.*

The ground field K in Section 5 need not be algebraically closed. In particular, we usually take $K = \mathbb{Q}$ when computing in Macaulay2. But this requires some adjustments in our results. For instance, Theorem 3.8 does not apply when the coordinates of $\mathbf{u} \in \mathbb{C}^n$ are not in K . In such situations, we may take \mathbb{K} to be an algebraic extension of K . We close with an example that shows the effect of the choice of ground field in a concrete computation.

Example 6.8 ($n = k = 2, l = 3$) *Consider the module M given in Macaulay2 as follows:*

```
R = QQ[x1,x2]; M = image matrix {{x1,x1*x2,x2},{x2,x1,x1*x2}};
dim(R^2/M), degree(R^2/M), amult(M)
```

The output shows that $\text{amult}(M) = 5$ when $K = \mathbb{Q}$, but $v = \text{amult}(M) = 6$ when $K = \mathbb{C}$: Applying now the command `solvePDE(M)`, we find the differential primary decomposition

```
{ideal (x2, x1), { | 1 |, | 0 |, | -dx1 |}}, {ideal (x2 - 1, x1 - 1), { |-1 |}},
| 0 | | 1 | | dx2 | | 1 |
2
{ideal (x1 + x2 + 1, x2 + x2 + 1), { | x2+1 |}}
| 1 |
```

The module M has three associated primes over $K = \mathbb{Q}$. The first gives three polynomial solutions, including $\begin{pmatrix} -z_1 \\ z_2 \end{pmatrix}$. The second prime contributes $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \exp(z_1 + z_2)$, and the third gives $\begin{pmatrix} x_2 + 1 \\ 1 \end{pmatrix} \exp(x_1 z_1 + x_2 z_2)$, where (x_1, x_2) is $\frac{1}{2}(-1 + \sqrt{3}i, -1 - \sqrt{3}i)$ or $\frac{1}{2}(-1 - \sqrt{3}i, -1 + \sqrt{3}i)$. Here $\mathbb{K} = \mathbb{Q}(\sqrt{3}i)$ is the field extension of $K = \mathbb{Q}$ defined by the third associated prime.

7. What next?

The results presented in this article suggest many directions for future study and research.

7.1. Special ideals and modules

One immediate next step is to explore the PDE corresponding to various specific ideals and modules that have appeared in the literature in commutative algebra and algebraic geometry.

One interesting example is the class of ideals studied recently by Conca and Tsakiris in [14], namely products of linear ideals. A minimal primary decomposition for such an ideal I is given in [14, Theorem 3.2]. It would be gratifying to find the arithmetic multiplicity $\text{amult}(I)$ and the solution spaces $\text{Sol}(I)$ in terms of matroidal data for the subspaces in $V(I)$.

A more challenging problem is to compute the solution space $\text{Sol}(J)$ when J is an ideal generated by n power sums in $R = \mathbb{Q}[\partial_1, \dots, \partial_n]$. This problem is nontrivial even for $n = 3$. To be more precise, we fix relatively prime integers $0 < a < b < c$, and we consider the ideal

$$J_{a,b,c} = \langle \partial_1^a + \partial_2^a + \partial_3^a, \partial_1^b + \partial_2^b + \partial_3^b, \partial_1^c + \partial_2^c + \partial_3^c \rangle.$$

If $(a, b, c) = (1, 2, 3)$ then $V(J_{1,2,3}) = \{0\}$ and $\text{Sol}(J_{1,2,3})$ is a six-dimensional space of polynomials, spanned by the discriminant $(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$ and its successive derivatives. In general, it is conjectured in [13] that $V(J_{a,b,c}) = \{0\}$ if and only if abc is a multiple of 6. If this holds then $\text{Sol}(J_{a,b,c})$ consists of polynomials. If this does not hold, then we must compute $V(J_{a,b,c})$ and extract the Noetherian multipliers from all associated primes of $J_{a,b,c}$. For instance, for $(a, b, c) = (2, 5, 8)$ with $K = \mathbb{Q}$, the arithmetic multiplicity is 38, one associated prime is $\langle \partial_1 + \partial_2 + \partial_3, \partial_2^2 + \partial_2\partial_3 + \partial_3^2 \rangle$, and the largest degree of a polynomial solution is 10.

It will be worthwhile to explore the solution spaces $\text{Sol}(M)$ for modules M with special combinatorial structure. One natural place to start are syzygy modules. For instance, take

$$A(\partial) = \begin{pmatrix} \partial_2 & \partial_3 & \partial_4 & 0 & 0 & 0 \\ -\partial_1 & 0 & 0 & \partial_3 & \partial_4 & 0 \\ 0 & -\partial_1 & 0 & -\partial_2 & 0 & \partial_4 \\ 0 & 0 & -\partial_1 & 0 & -\partial_2 & -\partial_3 \end{pmatrix}, \tag{7.1}$$

which is the first matrix in the Koszul complex for $n = 4$. Since $\text{rank}(A) = 3$, the module $M = \text{image}_R(A)$ is supported on the entire space, i.e. $V(M) = \mathbb{C}^4$. Its solutions are the gradient vectors $\nabla\alpha = \sum_{j=1}^4 \frac{\partial\alpha}{\partial z_j} e_j$, where $\alpha = \alpha(z_1, z_2, z_3, z_4)$ ranges over all functions in \mathcal{F} .

Toric geometry [27, Chapter 8] furnishes modules whose PDE should be interesting. The initial ideals of a toric ideal with respect to weight vectors are binomial ideals, so the theory of binomial primary decomposition applies, and it gives regular polyhedral subdivisions as in [27, Theorem 13.28]. Nonmonomial initial ideals should be studied from the differential point of view. Passing to coherent sheaves, we may examine the modules representing toric vector bundles and their Gröbner degenerations. In particular, the cotangent bundle of an embedded toric variety, in affine or projective space, is likely to encompass intriguing PDE.

7.2. Linear PDE with polynomial coefficients

We discuss an application to PDE with non-constant coefficients, here taken to be polynomials. Our setting is the Weyl algebra $D = \mathbb{C}\langle z_1, \dots, z_n, \partial_1, \dots, \partial_n \rangle$. A linear system of PDE with polynomial coefficients is a D -module. For instance, consider a D -ideal I , that is, a left ideal in the Weyl algebra D . The solution space of I is typically infinite-dimensional.

We construct solutions to I with the method of Gröbner deformations [33, Chapter 2]. Let $w \in \mathbb{R}^n$ be a general weight vector, and consider the initial D -ideal in ${}_{(-w,w)}(I)$. This is also a D -ideal, and it plays the role of Gröbner bases in solving polynomial equations. We know from [33, Theorem 2.3.3] that $\text{in}_{(-w,w)}(I)$ is fixed under the natural action of the n -dimensional algebraic torus $(\mathbb{C}^*)^n$ on the Weyl algebra D . This action is given in [33, equation (2.14)]. It gives rise to a Lie algebra action generated by the n Euler operators

$$\theta_i = z_i \partial_i \quad \text{for } i = 1, 2, \dots, n.$$

These Euler operators commute pairwise, and they generate a (commutative) polynomial subring $\mathbb{C}[\theta] = \mathbb{C}[\theta_1, \dots, \theta_n]$ of the Weyl algebra D . If J is any torus-fixed D -ideal, then it is generated by operators of the form $x^a p(\theta) \partial^b$ where $a, b \in \mathbb{N}^n$. We define the falling factorial

$$[\theta_b] := \prod_{i=1}^n \prod_{j=0}^{b_i-1} (\theta_i - j).$$

The *distraction* \tilde{J} is the ideal in $\mathbb{C}[\theta]$ generated by all polynomials $[\theta_b]p(\theta - b) = x^b p(\theta) \partial^b$, where $x^a p(\theta) \partial^b$ runs over a generating set of J . The space of classical solutions to J is equal to that of \tilde{J} . This was exploited in [33, Theorem 2.3.11] under the assumption that J is holonomic, which means that \tilde{J} is zero-dimensional in $\mathbb{C}[\theta]$. We here drop that assumption.

Given any D -ideal I , we compute its initial D -ideal $J = \text{in}_{(-w,w)}(I)$ for $w \in \mathbb{R}^n$ generic. Solutions to I degenerate to solutions of J under the Gröbner degeneration given by w . We can often reverse that construction: given solutions to J , we lift them to solutions of I . Now, to construct all solutions of J we study the *Frobenius ideal* $F = \tilde{J}$. This is an ideal in $\mathbb{C}[\theta]$.

We now describe all solutions to a given ideal F in $\mathbb{C}[\theta]$. This was done in [33, Theorem 2.3.11] for zero-dimensional F . Ehrenpreis–Palamodov allows us to solve the general case. Here is our algorithm. We replace each operator $\theta_i = z_i \partial_i$ by the corresponding ∂_i . We then apply `solvePDE` to get the general solution to the linear PDE with constant coefficients. In that general solution, we now replace each coordinate z_i by its logarithm $\log(z_i)$. In particular, each occurrence of $\exp(u_1 z_1 + \dots + u_n z_n)$ is replaced by a formal monomial $z_1^{u_1} \dots z_n^{u_n}$. The resulting expression represents the general solution to the Frobenius ideal F .

Example 7.1 *As a warm-up, we note that a function in one variable z_2 is annihilated by the squared Euler operator $\theta_2^2 = z_2 \partial_2 z_2 \partial_2$ if and only if it is a \mathbb{C} -linear combination of 1 and $\log(z_2)$. Consider the Frobenius ideal given by Palamodov’s system [11, Example 11]:*

$$F = \langle \theta_2^2, \theta_3^2, \theta_2 - \theta_1 \theta_3 \rangle.$$

To find all solutions to F , we consider the corresponding ideal $\langle \partial_2^2, \partial_3^2, \partial_2 - \partial_1 \partial_3 \rangle$ in $\mathbb{C}[\partial_1, \partial_2, \partial_3]$. By `solvePDE`, the general solution to that constant coefficient system equals

$$\alpha(z_1) + z_2 \cdot \beta'(z_1) + z_3 \cdot \beta(z_1),$$

where α and β are functions in one variable. We now replace z_i by $\log(z_i)$ and we abbreviate $A(z_1) = \alpha(\log(z_1))$ and $B(z_1) = \beta(\log(z_1))$. Thus, A and B are again arbitrary functions in one variable. We conclude that the general solution to the given Frobenius ideal F equals

$$\phi(z_1, z_2, z_3) = A(z_1) + z_1 \cdot \log(z_2) \cdot B'(z_1) + \log(z_3) \cdot B(z_1).$$

This method can also be applied for $k \geq 2$, enabling us to study solutions for any D -module.

7.3. Socle solutions

The solution space $\text{Sol}(M)$ to a system M of linear PDE is a complex vector space, typically infinite-dimensional. Our algorithm in Section 5 decomposes that space into finitely many natural pieces, one for each of the integrals in (2.6). The number $\text{amult}(M)$ of pieces is a meaningful invariant from commutative algebra. Each piece is labeled by a polynomial $B_{ij}(\mathbf{x}, \mathbf{z})$ in $2n$ variables, and it is parametrized by measures μ_{ij} on the irreducible variety V_i .

This approach does not take full advantage of the fact that $\text{Sol}(M)$ is an R -module where $R = \mathbb{C}[\partial_1, \dots, \partial_n]$. Indeed, if $\psi(\mathbf{z})$ is any solution to M then so is $(\partial_i \bullet \psi)(\mathbf{z})$. So, if we list all solutions then $\partial_i \bullet \psi$ is redundant provided ψ is already listed. More precisely, we consider

$$\text{Sol}(M) / \langle \partial_1, \dots, \partial_n \rangle \text{Sol}(M). \tag{7.2}$$

This quotient space is still infinite-dimensional over \mathbb{C} , but it often has a much smaller description than $\text{Sol}(M)$. A solution to M is called a *socle solution* if it is nonzero in (7.2). We pose the problem of

modifying `solvePDE` so that the output is a minimal subset of Noetherian multipliers which represent all the socle solutions. The solution will require the prior development of additional theory in commutative algebra, along the lines of [9, 11, 12].

The situation is straightforward in the case of Theorem 3.8 when the support $V(M)$ is finite. Here the space $\text{Sol}(M)$ is finite-dimensional, and it is canonically isomorphic to the vector space dual of R^k/M , as shown in [30]. Finding the socle solutions is a computation using linear algebra over $K = \mathbb{C}$, similar to the three steps after Proposition 3.7. For instance, let $k = 1$ and suppose that I is a homogeneous ideal in R . The socle solutions are sometimes called *volume polynomials* [33, Lemma 3.6.20]. The most desirable case arises when I is *Gorenstein*. Here the socle solution is unique up to scaling, and it fully characterizes I . For instance, consider the power sum ideal $(\sum_{i=1}^n \partial_i^s : s = 1, \dots, n)$. This is Gorenstein with volume polynomial $\Delta = \prod_{1 \leq i < j \leq n} (z_i - z_j)$. For $n = 3$, the ideal I is $J_{1,2,3}$ in Subsection 7.1. Here $\text{Sol}(I)$ is a \mathbb{C} -vector space of dimension $n!$. However, as an R -module, it is generated by a single polynomial Δ . A future version of `solvePDE` should simply output $\text{Sol}(I) = R\Delta$.

It is instructive to revisit the general solutions to PDE we presented in this paper and to highlight the socle solutions for each of them. For instance, in Example 2.3, we have $\text{amult}(I) = 4$ but only one of the four Noetherian multipliers B_i gives a socle solution. The last summand in (2.9) gives the socle solutions. The first three summands can be obtained from the last summand by taking derivatives. What are the socle solutions in Example 2.4?

7.4. From calculus to analysis

The storyline of this paper is meant to be accessible for students of multivariable calculus. These students know how to check that (1.9) is a solution to (1.8). The derivations in Examples 2.3, 2.4, 5.1, 5.2, 6.1, 6.3, and 6.8 are understandable as well. No prior exposure to abstract algebra is needed to follow these examples, or to download `Macaulay2` and run `solvePDE`.

The objective of this subsection is to move beyond calculus and to build a bridge to advanced themes and current research in analysis. First of all, we ought to consider inhomogeneous systems of linear PDE with constant coefficients. Such a system has the form

$$A(\partial) \bullet \psi(\mathbf{z}) = f(\mathbf{z}), \quad (7.3)$$

where A is a $k \times l$ matrix as before and f is a vector in \mathcal{F}^l , where \mathcal{F} is a space of functions or distributions. Writing a_i for the i th column of A , the system (7.3) describes vectors $\psi = (\psi_1, \dots, \psi_k)$ with $a_i \bullet \psi = f_i$ for $i = 1, \dots, l$. The study of the inhomogeneous system (7.3) is a major application of Theorem 2.2. We see this in Palamodov's book [32, Chapter VII], but also in the work of Oberst who addresses the "canonical Cauchy problem" in [28, Section 5]. An important role is played by the syzygy module $\ker_R(A) \subset R^l$, whose elements are the R -linear relations on the columns a_1, \dots, a_l . A necessary condition for solvability of (7.3) is that the Fourier transform of the right-hand side $f = (f_1, \dots, f_l)$ satisfies the same syzygies. Hörmander shows in [23, Theorem 7.6.13] that the converse is also true, under certain regularity hypotheses on f . Thus, the computation of syzygies and other homological methods (cf. [18, Part III]) are useful for solving (7.3). Treves calls this *Simple algebra in the general case* [35, Section 3.1]. We point to his exact sequence in [35, equation (3.5)]. Syzygies can be lifted to D-modules [33, Section 2.4] via the Gröbner deformations in Subsection 7.2.

Another issue is to better understand which collections of vectors B_{ij} arise as Noetherian multipliers for some PDE. The analogous question for Noetherian operators of ideals is addressed in [11, Theorem 3.1]. That result is essentially equivalent to the characterization in [23, Theorem 7.7.7] of spaces \mathcal{A} of Noetherian operators for a primary module as being closed under the Lie bracket. More work on this topic is needed. This is related to the issue of primary fusion, discussed at the end of [12, Section 5], which concerns the compatibility of minimal sets of Noetherian operators for associated primes that are contained in one another.

We end with a pointer to current research in calculus of variations by De Phillippis and collaborators in [2, 16]. Each solution μ to the PDE $A \bullet \mu = 0$ is a Radon measure on an open set in \mathbb{R}^n with values

in \mathbb{R}^k . Such a measure μ is called *A-free*, and one is interested in the singular part μ^s of μ . Analysts view solutions among smooth functions as classical and well-understood, and they care primarily about irregularities and their rectifiability. One studies μ^s via the polar vector $\frac{d\mu}{d|\mu|}$ in \mathbb{R}^k . The main result in [16] states that this vector always lies in the wave cone Λ_A . This is a real algebraic variety in \mathbb{R}^k which is an invariant of our module $M = \text{image}_R(A)$. When $A = \text{curl}$ as in (7.1), the wave cone is a Veronese variety, and the result is Alberti's Rank-One Theorem. The article [2] proves the same conclusion for more refined wave cones, and it offers a conjecture relating the geometry of wave cones to the singular supports of solutions [2, Conjecture 1.6]. It would be interesting to compute these real varieties in practice and to learn about *A-free* measures from the output of `solvePDE`.

7.5. Numerical algebraic geometry

In applications, one often does not have access to an exact representation of a problem, but rather some approximation with possible errors introduced by measurements or finite-precision arithmetic. The last decade of developments in *numerical algebraic geometry* [4] provides tools for the numerical treatment of such polynomial models. In that paradigm, a prime ideal $P \subset \mathbb{C}[\mathbf{x}]$ is represented by a *witness set*, i.e. a set of $\deg(P)$ points approximately on $V(P) \cap L$, where L is a generic affine-linear space of dimension $c = \text{codim}(P)$. Similarly, radical ideals are collections of witness sets corresponding to irreducible components. Dealing with general ideals and modules is much more subtle, since these have embedded primes. One idea, pioneered by Leykin [24], is to consider *deflations* of ideals. Modules were not considered in [24]. Deflation has the effect of exposing embedded and nonreduced components as isolated components, which can subsequently be represented using witness sets. One drawback is that the deflated ideal lies in a polynomial ring with many new variables.

We advocate the systematic development of numerical methods for linear PDE with constant coefficients. Noetherian operators and multipliers can be used to represent arbitrary ideals and modules. For each prime P , both the field $\mathbb{K} = \text{Frac}(R/P)$ and the spaces in (4.8) should be represented purely numerically. Along the way, one would extend the current repertoire of numerical algebraic geometry to modules and their coherent sheaves.

First steps toward the numerical encoding of affine schemes were taken in [10], for ideals I with no embedded primes. The key observation is that the coefficients of the Noetherian operators for the P -primary component of I can be evaluated at a point $\mathbf{u} \in V(P)$ using only linear algebra over \mathbb{C} . This linear algebra step can be carried out purely numerically.

Inspired by this, we propose a numerical representation of an arbitrary module $M \subseteq R^k$. Let $(P_i, \mathcal{S}_i, \mathcal{A}_i)$ be a differential primary decomposition as in Theorem 4.3. Assuming the ability to sample generic points $\mathbf{u}_i \in V(P_i)$, we encode the sets \mathcal{A}_i by their point evaluations $\mathcal{A}_i(\mathbf{u}_i) = \{A(\mathbf{u}_i, \partial_{\mathbf{x}}) : A(\mathbf{x}, \partial_{\mathbf{x}}) \in \mathcal{A}_i\}$. Each evaluated operator $A(\mathbf{u}_i, \partial_{\mathbf{x}})$ gives an exponential solution $B(\mathbf{u}_i, \mathbf{z}) \exp(\mathbf{u}_i^T \mathbf{z})$ to the PDE given by M via the correspondence in Proposition 4.8. We obtain a numerical module membership test: a polynomial vector $m \in R^k$ belongs to M with high probability if $A(\mathbf{u}_i, \partial_{\mathbf{x}}) \bullet m$ vanishes at the point \mathbf{u}_i for all $A \in \mathcal{A}_i(\mathbf{u}_i)$ and $i = 1, \dots, s$. The exponential functions $\mathbf{z} \rightarrow B(\mathbf{u}_i, \mathbf{z}) \exp(\mathbf{u}_i^T \mathbf{z})$, which depend on numerical parameters \mathbf{u}_i , serve as an encoding of the infinite-dimensional \mathbb{C} -vector space $\text{Sol}(M)$.

Another potential research direction is the development of hybrid algorithms, where numerical information is used to speed up symbolic computations. Assuming the numerical approximations to be accurate enough, the output of a hybrid algorithm is exact. For Noetherian operators of ideals with no embedded components, this is explored in [10], and it is already implemented in the `Macaulay2` package `NoetherianOperators` [8] using the command `noetherianOperators(I, Strategy => "Hybrid")`. It will be desirable to extend this hybrid method to the command `solvePDE`, in the full generality seen in Algorithm 1.

In conclusion, the numerical solution of partial differential equations is the key to computational science. The case of linear PDE with constant coefficients serves as a base case. We hope that the techniques described in this article will be useful for the future applications.

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