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A CHARACTERIZATION OF LOCALLY HOMOGENEOUS RIEMANN MANIFOLDS OF DIMENSION 3

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Introduction

It is classical to characterize locally homogeneous Riemann manifolds by infinitesimal conditions. For example, [Si] asserts that the local-homogeneity is equivalent to the existence of linear isometries between tangent spaces which preserve the curvatures and their covariant derivatives up to certain orders. It is also known that the local homogeneity is equivalent to the existence of a certain tensor field of type (1, 2) (for this and a further study, see [TV]).

In connection with his characterization theorem, Singer raised the following questions:

(Q1) What are the Riemann manifolds which are completely determined by their curvatures only?

(Q2) Do there exist curvature homogeneous spaces which are not locally homogeneous?

The purpose of the present paper is to give, in the 3-dimensional case, an explicit characterization (i.e. in terms of Riemannian invariants) of locally homogeneous Riemann manifolds, and to give some answers to the questions of Singer.

Our characterization is as follows: Let M be a connected, compact Riemann manifold of dimension 3 and S the Ricci tensor. Assume that the eigenvalues ρ_1, ρ_2, ρ_3 of S are constant on M (in other words we assume that M is curvature homogeneous).

THEOREM A. *Suppose that* ρ_1, ρ_2, ρ_3 *are distinct. Then* M *is locally homogeneous if and only if the* 1*-form* $S \cdot \nabla S = \sum_{a,b} S^{ab} S_{ia;b}$ *vanishes. If that is the case, then* ρ_1, ρ_2, ρ_3 *give complete isometry invariants for the universal covering manifold of* M.

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THEOREM B. *Suppose that two of ρ_1, ρ_2, ρ_3 coincide. Then M is locally homogeneous if and only if the function* $\|\nabla S\|^2 = \sum_{a,b,c} S^{ab;c} S_{ab;c}$ *is constant. In that case, if* $\rho_i = \rho_j = 0 \neq \rho_k$ *for some distinct i, j, k, then* ρ_k, $\|\nabla S\|$ *give complete isometry invariants for the universal covering manifold, and otherwise* ρ_1, ρ_2, ρ_3 *give those.*

From the proof of Theorem B we obtain a sufficient condition for the local-homogeneity of M:

PROPOSITION 5.1. *Suppose that* $\rho_1 = \rho_2$. *If* $\rho_1 \geq 0$ *or* $\rho_3 \leq 0$, *then M is locally homogeneous.*

This proposition (with Theorem B) gives an answer in the 3-dimensional case to question (Q1). We also note that the assumption of compactness of M is essential in Proposition 5.1. Indeed, on $\mathbf{R}^3$, there exist non-homogeneous, complete metrics with $\rho_1 = \rho_2 = -1$, $\rho_3 = 0$ ([Se], [T], [KTV]).

We also give examples of non-homogeneous, complete metrics on $\mathbf{R}^3$ which have distinct, constant Ricci eivenvalues (§ 6). These give counter-examples in the non-degenerate case $\rho_i \neq \rho_j$ $(i \neq j)$ to question (Q2).

To obtain our result we proceed as follows. The proof of Theorem A is rather straightforward, because the assumption of the distinctness of the Ricci eigenvalues ensures the uniqueness of the Ricci eigenvector fields, and hence the isometry between neighborhoods of two points, if any, is also uniquely determined. On the other hand, if the Ricci eigenvalues are not distinct, say $\rho_1 = \rho_2 \neq \rho_3$, there are two possibilities: the one where the required isometry is uniquely determined and the other where the isometry is not uniquely determined, in other words, the isotropy group is nontrivial. In order to distinguish these cases, we introduce an isometry invariant function δ so that for a point p, the nontriviality of the isotropy group at p implies $\delta(p) = 0$, and δ being constant yields the local-homogeneity of M. The function δ is expressed by S and ∇S, and hence the condition that $\|\nabla S\|^2$ be constant as in Theorem B means that δ is constant. The main task in the proof of Proposition 5.1 is the proof of δ being constant, which is done by applying the geometric version of E. Hopf theorem to the function δ and some elliptic operator defined by means of Ricci eigenvector fields.

Throughout this paper, all manifolds and functions are assumed to be of class C^∞.

§ 1. Preliminary formulas

Let U be a 3-dimensional open ball with Riemann metric $\langle\, ,\rangle$. Assume that the eigenvalues ρ_1, ρ_2, ρ_3 of the Ricci tensor S (viewed as a tensor of type (1, 1)) are constant on U. Let (X_1, X_2, X_3) be the orthonormal frame field such that each X_i is the eigenvector field belonging to ρ_i (although such a frame is not unique in the degenerate cases $\rho_i = \rho_j$). Introduce the functions $d^k_{ij} = \langle \nabla_{X_i} X_j, X_k\rangle$ on U. Then $d^k_{ij} = -d^j_{ik}$, $\operatorname{div} X_i = -\sum_a d^i_{aa}$, and the following formulas are directly verified.

LEMMA 1.1. *For each $i, j, l = 1, 2, 3$, we have*

$$(i, i) \quad \rho_i = -X_i(\operatorname{div} X_i) + \sum_a X_a(d^a_{ii}) + \sum_a d^a_{ii} \operatorname{div} X_a - \sum_{a,b} d^a_{bi} d^b_{ai}\,,$$

$$(j, l) \quad 0 = -X_j(\operatorname{div} X_l) + \sum_a X_a(d^a_{jl}) + \sum_a d^a_{jl} \operatorname{div} X_a - \sum_{a,b} d^a_{bj} d^b_{al}\,, \quad j \neq l\,,$$

and as Bianchi's 2nd identity,

$$\sum_a (\rho_i - \rho_a) d^i_{aa} = 0\,.$$

In particular, if $\rho_1 = \rho_2 \neq \rho_3$, then

$$d^1_{33} = d^2_{33} = 0\,, \quad d^3_{11} + d^3_{22} = 0\,, \quad \rho_3 = 2(d^3_{12} d^1_{23} - (d^3_{11})^2)\,.$$

Hence, in the case $\rho_1 = \rho_2 \neq \rho_3$, every integral curve for X_3 is a geodesic, and the divergence $\operatorname{div} X_3$ *of X_3 vanishes.*

LEMMA 1.2. *Assume that $\rho_1 = \rho_2 \neq \rho_3$. Define $\delta : U \to \mathbf{R}$ by the formula*

$$\delta(x) = \det(\nabla X_3|P + (\nabla X_3|P)^*)\,, \quad x \in U\,,$$

where $\nabla X_3|P$ denotes the restriction to $P = (X_3)^{\perp}_x$ of the linear mapping $\nabla X_3 : T_x(U) \to T_x(U)$, and $(\nabla X_3|P)^*$ its adjoint. Then δ is independent of the choice (i.e. the orientation) of X_3, and satisfies $\delta \leq 0$. Furthermore we have $X_3(\delta) = 0$.

Proof. Note that

$$\delta = -4(d^3_{11})^2 - (d^3_{12} - d^1_{23})^2 = 2\rho_3 - (d^3_{12} + d^1_{23})^2\,.$$

Then the former parts of our lemma are obvious. By (1, 2), (2, 1) of Lemma 1.1 we have $X_3(d^3_{12} + d^1_{23}) = 0$, which implies the latter part. This expression for δ also yields

LEMMA 1.3. *Let the assumption and the notation be as in Lemma 1.2.*

If $\delta \equiv 0$, *then*

$$d_{11}^3 = d_{22}^3 = 0\,, \qquad d_{12}^3 = d_{23}^1\,, \qquad \rho_3 = 2(d_{12}^3)^2\,.$$

In the case where $\delta < 0$ on U, the following lemma gives us a "canonical" frame (X_1, X_2, X_3) for the Ricci tensor S.

LEMMA 1.4. *Let the assumption and the notation be as in Lemma* 1.2. *Suppose that* $\delta(x) < 0$ *for any* $x \in U$. *Then we can take* X_1, X_2 *so that* $d_{11}^3 = d_{22}^3 = d_{31}^2 = 0$, $d_{12}^3 > d_{23}^1$ *at every point of* U, *and so that* $d_{12}^3 + d_{23}^1 > 0$ *at some point of* U *or* $d_{12}^3 + d_{23}^1 = 0$ *at every point of* U.

Proof. Let (Y_1, Y_2) be the orthonormal frame field diagonalizing the symmetric operator $\nabla X_3 | P + (\nabla X_3 | P)^*$. (Since (Y_1, Y_2) is uniquely determined up to permutations, they are globally defined on U.) Rotate (Y_1, Y_2) by $\pi/4$ to get (X_1, X_2). Then (X_1, X_2, X_3) satisfies $d_{11}^3 = d_{22}^3 = 0$. By formulas (1, 1), (2, 2) in Lemma 1.1 we have also $d_{31}^2 = 0$. Since $d_{12}^3 - d_{23}^1 \neq 0$ on U, by reversing the orientation of X_1 if necessary, we may assume $d_{12}^3 - d_{23}^1 > 0$ on U. Moreover, if $d_{12}^3 + d_{23}^1$ is negative at some point of U, then we exchange X_1, X_2. Then the new vector fields satisfy the desired properties.

§ 2. Local theory for the case $\rho_1 = \rho_2 \neq \rho_3$

Let U be as in § 1 and we continue to consider the degenerate case $\rho_1 = \rho_2 \neq \rho_3$. Let δ be as in Lemma 1.2. The purpose of this section is to prove

PROPOSITION 2.1. *If* δ *is constant, then* U *is locally homogeneous, i.e. for any two points* p, q *of* U *there exists an isometry, taking* p *to* q, *of a neighborhood of* p *onto a neighborhood of* q.

We prepare some lemmas.

LEMMA 2.2. *Assume that* $\delta \equiv 0$. *Then* $\rho_3 \geq 0$, *and the eigenvector field* X_3 *belonging to* ρ_3 *is a Killing vector field. Moreover, for any point* p *of* U *there exist orthonormal vector fields* Y_1, Y_2 *on a neighborhood of* p *such that*

(1) $\langle Y_i, X_3\rangle = 0$, $[Y_i, X_3] = 0$ *for* $i = 1, 2$,

(2) $\langle [Y_1, Y_2], X_3\rangle = \sqrt{2\rho_3}$, *and hence*

(3) $\rho_1 + \rho_3 = Y_1(c_{12}^2) - Y_2(c_{12}^1) - (c_{12}^2)^2 - (c_{12}^1)^2$, *where* $c_{ij}^k = \langle [Y_i, Y_j], Y_k\rangle$.

Hence, the neighborhood of p *is isometric to an open subset of the total* *space of a Riemann submersion over a surface B with curvature $\rho_1 + \rho_3$, whose fibers (integral curves for X_3) are geodesics, and whose integrability tensor is equal to $\sqrt{2\rho_3}$ times the area form of B.*

Proof. Lemma 1.3 implies $\rho_3 \geq 0$. For any orthonormal vector fields X_1, X_2 orthogonal to X_3, by Lemmas 1.1 and 1.3 we have $d^3_{12} = d^1_{23}$ and $d^3_{ii} = 0$, $d^i_{33} = 0$ for any i. These show especially that ∇X_3 is skew-symmetric, and therefore X_3 is Killing. Since the problem is local, it is easy to take orthonormal vector fields Y_1, Y_2 orthogonal to X_3, defined in a neighborhood of p, so that $[Y_1, X_3] = [Y_2, X_3] = 0$. To apply Lemmas in § 1 to the orthonormal frame fields (Y_1, Y_2, X_3), we may use the same notations d^k_{ij} for the functions $\langle \nabla_{Y_i} Y_j, Y_k \rangle$, where $Y_3 = X_3$. Then by Lemma 1.3, we obtain $\langle [Y_1, Y_2], X_3 \rangle^2 = 2\rho_3$. By replacing Y_1 by $-Y_1$, if necessary, we may assume that $\langle [Y_1, Y_2], X_3 \rangle \geq 0$, and hence (2) is satisfied. Condition (3) follows from (1, 1), (3, 3) of Lemma 1.1 and the fact $d^3_{12} + d^2_{31} = 0$. The latter part is an immediate consequence of (1), (2), (3) and the fact that for a surface B with orthonormal vector fields Y_1, Y_2, the curvature of B is given by $\langle R(Y_1, Y_2)Y_2, Y_1 \rangle = Y_1(c^2_{12}) - Y_2(c^1_{12}) - (c^2_{12})^2 - (c^1_{12})^2$, where $c^k_{ij} = \langle [Y_i, Y_j], Y_k \rangle$. This completes the proof of Lemma 2.2.

LEMMA 2.3. *Assume that δ is negative constant. Then the vector fields X_1, X_2, X_3 taken as in Lemma* 1.4 *satisfy the conditions $d^k_{ij} =$ constant. In fact, d^3_{12} and d^1_{23} are the constants determined by ρ_3 and δ, and other d^k_{ij}'s vanish. Moreover we have $\rho_1 = \rho_2 = 0$.*

Proof. We know already that all the d^k_{ij}'s except d^1_{22}, d^2_{11}, d^3_{12}, d^1_{23} are identically zero. As for d^3_{12}, d^1_{23}, by the expression for δ in the proof of Lemma 1.2, we have $d^3_{12} - d^1_{23} = \sqrt{-\delta}$, $d^3_{12} + d^1_{23} = \sqrt{2\rho_3 - \delta}$. Hence d^3_{12}, d^1_{23} are the constants determined by ρ_3, δ. Furthermore, by formulas (2, 3), (1, 3) of Lemma 1.1 we observe that $d^1_{22} = 0$, $d^2_{11} = 0$, respectively. Hence (1, 1) yields $\rho_1 = 0$. Lemma 2.3 is proved.

We need two more general lemmas.

LEMMA 2.4. *Let π be a submersion of an n-dimensional Riemann manifold M onto an $(n-1)$-dimensional Riemann manifold B with structure tensors T, A (cf.* [O]). *Suppose that the submersion π is induced by a Killing vector field, that is, there exists a non-singular Killing vector field f on M such that f is vertical at every point of M. Put $\nu = f/\|f\|$, $N = T_\nu \nu$. Consider the 2-form on M defined by $(X, Y) \mapsto \langle A_X Y, \nu \rangle$ and denote it by*

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the same letter A. (Then we can regard N, A as the vector field, 2-form on B, respectively.) Let $\pi' : M' \to B'$ be another such submersion with T', A', f', N'. Let $p \in M$, $p' \in M'$. Suppose that there exists an isometry ϕ of B onto B', $\phi(\pi(p)) = \pi'(p')$, such that $\langle X, N\rangle = \langle \phi_* X, N'\rangle$ and $A_X Y = A'_{\phi_* X}\phi_* Y$ for any tangent vectors X, Y of B. Then the isometry ϕ can be lifted to an isometry $\bar{\phi}$ of a neighborhood of p onto a neighborhood of $p' = \bar{\phi}(p)$.

Proof. Let $p \in M$, and $\gamma(t)$ a geodesic in B with $\gamma(0) = \pi(p)$. Let $\{\gamma_\varepsilon(t)\}$ be a family of geodesics such that $\gamma_0(t) = \gamma(t)$, and let $J(t)$ be the Jacobi field along γ determined by $\{\gamma_\varepsilon(t)\}$. Let $\bar{\gamma}(t)$ be the horizontal lift of $\gamma(t)$ with $\bar{\gamma}(0) = p$, and let $\{\bar{\gamma}_\varepsilon(t)\}$ be any horizontal lift of $\{\gamma_\varepsilon(t)\}$ such that $\bar{\gamma}_0(t) = \bar{\gamma}(t)$. Let $\tilde{J}(t)$ be the Jacobi field along $\bar{\gamma}$ determined by $\{\bar{\gamma}_\varepsilon(t)\}$. Then we have

$$\frac{d}{dt}\langle \tilde{J}(t), \nu\rangle = -\langle \dot{\gamma}(t), N\rangle\langle \tilde{J}(t), \nu\rangle + 2\langle A_{J(t)}\dot{\gamma}(t), \nu\rangle .$$

By this formula and the usual argument using geodesics, we can verify that the lift $\bar{\phi}$ is an isometry.

LEMMA 2.5. *Let $X_1, X_2, \cdots, X_n$ be orthonormal vector fields on a connected, simply connected Riemann manifold U of dimension n. Let p be a point of U. Suppose that $c_{ij}^k = \langle [X_i, X_j], X_k\rangle$ is constant on U for each i, j, k. Then there exists a connected, simply connected Lie group G of dimension n, which is uniquely determined by constants c_{ij}^k, such that $U \subset G$ ($U = G$ if U is complete) and p is the unit element of G, and such that X_i are the restrictions to U of the left invariant vector fields $\tilde{X}_i$, $\tilde{X}_i(p) = X_i(p)$, on G. The metric on G defined so that $\tilde{X}_i$ are orthonormal is left invariant.*

Proof. This is a direct consequence of the Lie's fundamental theorem on local groups of transformations.

Proof of Proposition 2.1. First consider the case $\delta \equiv 0$. Then by Lemma 2.2 we see that every point of U has a neighborhood which is a total space of a submersion. Applying Lemma 2.4, since $N = 0$ in this case, we conclude that the neighborhood is locally homogeneous, and hence so is U. Next, consider the case $\delta < 0$. Then by Lemma 2.3 we can apply Lemma 2.5 to identify U with an open subset of a Lie group with left invariant metric. Hence U is in particular locally homogeneous.

§ 3. Proof of Theorem A

Let M be a 3-dimensional connected Riemann manifold, and let S, ∇S be the Ricci tensor, its covariant differential. Let $S\cdot\nabla S$ be the 1-form defined by $(S\cdot\nabla S)_i = \sum_{a,b} S^{ab} S_{ia;b}$.

We have to prove a lemma.

LEMMA 3.1. *Assume that the Ricci eigenvalues ρ_1, ρ_2, ρ_3 are constant and distinct. Moreover, assume that the* 1*-form $S\cdot\nabla S$ vanishes. Let (X_1, X_2, X_3) be the orthonormal frame field on the universal covering manifold $\tilde{M}$ of M such that each X_i is the eigenvector field belonging to ρ_i. Then the bracket products $[X_i, X_j]$ are expressed as linear combinations of X_1, X_2, X_3, whose coefficients are constants determined by ρ_1, ρ_2, ρ_3.*

Proof. Since $(\nabla S)(X_i, X_j; X_k) = (p_i - p_j)d^j_{ki}$, we note that the 1-form $S\cdot\nabla S$ satisfies

$$(S\cdot\nabla S)(X_i) = \sum_a \rho_a(\rho_a - \rho_i)d^i_{aa}\,,$$

where $d^k_{ij} = \langle\nabla_{X_i}X_j, X_k\rangle$. Then by the assumption $S\cdot\nabla S = 0$ and Bianchi 2nd identity, we have $d^i_{jj} = 0$ for each i, j. Hence, from formula (i, i) of Lemma 1.1 we get $\rho_i = 2d^i_{jk}d^j_{ki}$ for any even permutation i, j, k of 1, 2, 3. Using the distinctness of ρ_i we conclude that d^3_{12}, d^1_{23}, d^2_{31} are non-zero constants, and can be expressed as functions of ρ_1, ρ_2, ρ_3. Therefore the coefficients $d^k_{ij} - d^k_{ji}$ of $[X_i, X_j]$ with respect to X_k are the constants determined by ρ_1, ρ_2, ρ_3.

Proof of Theorem A. Suppose that $S\cdot\nabla S = 0$. Then by Lemmas 2.5 and 3.1, we can regard the universal covering manifold $\tilde{M}$ as a Lie group with left invariant metric. Hence M is itself locally homogeneous. The latter part of Theorem A also follows from Lemmas 2.5, 3.1. In order to prove the necessity of the condition $S\cdot\nabla S = 0$, suppose that M is locally homogeneous. By considering a suitable finite covering manifold, we may assume that the Ricci eigenvector fields X_1, X_2, X_3 are defined globally on M. Then clearly $\operatorname{div} X_i$ are constant. By the compactness of M we get $\operatorname{div} X_i = 0$. These and Bianchi 2nd identity yield $d^i_{jj} = 0$ for each i, j, and hence $S\cdot\nabla S = 0$. This completes the proof of Theorem A.

In the noncompact case, we have the following criterion for the homogeneity of M.

THEOREM 3.2. *Assume that the Ricci eigenvalues ρ_1, ρ_2, ρ_3 are constant* *and distinct. Then M is locally homogeneous if and only if the symmetric tensor* $T = (S\cdot\nabla S)\otimes(S\cdot\nabla S)$ *satisfies the conditions* $\operatorname{tr} T =$ *constant and* $[S, T] = 0$, *where* $[S, T]$ *denotes the 2-form* $\sum_a (S_i^a T_{aj} - T_i^a S_{aj})$.

Proof. Note that the condition $[S, T] = 0$ means that at least two of $\operatorname{div} X_i$ vanish, and recall the following fact. For any 3-dimensional non-unimodular Lie group G with left invariant metric, if the Ricci eigenvalues are distinct, then two of $\operatorname{div} e_i$'s of the eigenvector fields e_i vanish ([M, p. 321]). Then the similar argument as before proves our theorem.

§ 4. Proof of Theorem B

We now discuss the degenerate case $\rho_1 = \rho_2 \neq \rho_3$. Theorem B is an immediate consequence of the following assertion (If $\rho_1 = \rho_2 = \rho_3$, then Theorem B holds clearly, because in that case M is a space of constant curvature).

THEOREM 4.1. *Let M be a 3-dimensional, connected, complete Riemann manifold. Assume that ρ_1, ρ_2, ρ_3 are constant, and now assume that $\rho_1 = \rho_2 \neq \rho_3$. Define $\delta: M \to \mathbf{R}$ by the formula*

$$\delta = \rho_3 - \frac{\|\nabla S\|^2}{2(\rho_1 - \rho_3)^2},$$

where $\|\nabla S\|^2 = \sum_{a,b,c} S^{ab;c} S_{ab;c}$. Then $\delta \leq 0$, and the necessary and sufficient condition for M to be locally homogeneous is that δ is constant. If $\delta = 0$, then $\rho_3 \geq 0$, and the isometry class of the universal covering manifold $\tilde{M}$ of M is determined by ρ_1, ρ_3. If δ is negative constant, then $\rho_1 = \rho_2 = 0$, the manifold $\tilde{M}$ is a Lie group with left invariant metric, and the isometry class of $\tilde{M}$ is determined by ρ_3, δ.

Proof. Clearly, if M is locally homogeneous, then δ is constant. To prove the converse, note that the function δ is nothing but the function defined in Lemma 1.2. Then by Proposition 2.1 we see that the constancy of δ ensures the local homogeneity of M. To prove the latter parts, suppose that $\delta = 0$. Then Lemmas 2.2 and 2.4 imply that the isometry class of $\tilde{M}$ is determined by ρ_1, ρ_3. In the case $\delta \equiv$ negative constant, Lemmas 2.3 and 2.5 prove our assertion.

§ 5. Global theory for the case $\rho_1 = \rho_2 \neq \rho_3$

There are some cases in which only the constancy of ρ_1, ρ_2, ρ_3 ensures the local homogeneity of M. Indeed, we shall prove

PROPOSITION 5.1. *Let M be a 3-dimensional, connected, compact Riemann manifold with constant Ricci eigenvalues ρ_1, ρ_2, ρ_3. If $\rho_1 = \rho_2$, and if $\rho_1 \geq 0$ or $\rho_3 \leq 0$, then M is locally homogeneous.*

In order to prove this proposition we may assume that $\rho_1 = \rho_2 \neq \rho_3$. Then we have the function $\delta : M \to \mathbf{R}$ defined in Theorem 4.1, which coincides locally with the function δ in Lemma 1.2. Thus, to prove the local homogeneity of M it suffices to show the constancy of δ. This comes from

LEMMA 5.2. *Let M be as in Proposition 5.1, and assume that $\rho_1 = \rho_2 \neq \rho_3$.*

(1) *If $\rho_3 < 0$, then $\delta =$ negative constant, and $\rho_1 = \rho_2 = 0$.*
(2) *If $\rho_3 = 0$, then $\delta \equiv 0$.*
(3) *If $\rho_1 > 0$, then $\delta \equiv 0$.*
(4) *If $\rho_1 = 0$, then $\delta =$ constant.*

To prove Lemma 5.2 we prepare a lemma.

LEMMA 5.3. *Under the same hypothesis as in Lemma 5.2, if $\delta(x) < 0$ for all $x \in M$, then we have $\rho_1 = \rho_2 = 0$.*

Proof. Let X_1, X_2, X_3 be the orthonormal vector fields on some finite covering manifold $\tilde{M}$ of M such that X_3 is the eigenvector field belonging to ρ_3 and X_1, X_2 are locally as in Lemma 1.4. Then by the choice of X_1, X_2 the functions $d_{ij}^k = \langle \nabla_{X_i} X_j, X_k \rangle : \tilde{M} \to \mathbf{R}$ satisfy

$$d_{33}^1 = d_{33}^2 = d_{11}^3 = d_{22}^3 = d_{31}^2 = 0\,, \quad \rho_3 = 2 d_{12}^3 d_{23}^1\,,$$
$$\delta = -(d_{12}^3 - d_{23}^1)^2\,, \quad d_{12}^3 > d_{23}^1\,,$$

and satisfy

$$d_{12}^3 + d_{23}^1 > 0 \text{ at some point of } \tilde{M} \quad \text{or} \quad d_{12}^3 + d_{23}^1 \equiv 0\,.$$

Hence (1, 1) in Lemma 1.1 becomes $\rho_1 = X_1(d_{22}^1) - (d_{22}^1)^2 + X_2(d_{11}^2) - (d_{11}^2)^2$. By integrating this formula over $\tilde{M}$ we obtain $\rho_1 = 0$, because the right hand side can be written as $\operatorname{div}(d_{22}^1 X_1 + d_{11}^2 X_2)$.

Proof of (1), *Lemma* 5.2. Assume $\rho_3 < 0$. From the expression for δ in the proof of Lemma 1.2, we see that $\delta < 0$ on M. Let X_1, X_2, X_3, d_{ij}^k be as in the proof of Lemma 5.3. Then the functions d_{ij}^k satisfy the same properties as before. We contend that d^3_{12}, d^1_{23} are constant. This and Lemma 5.3 will prove (1). Now, since $\rho_3 = 2d^3_{12}d^1_{23}$ and $X_3(\delta) = 0$, we see that d^3_{12}, d^1_{23} are X_3-invariant. Hence by (3, 1), (3, 2) we see that the function d^1_{22} satisfies the differential equation $(X_3)^2(d^1_{22}) + (\rho_3/2)d^1_{22} = 0$. Hence, for each integral curve $c(t)$ of X_3, the function $d^1_{22}(c(t))$ is of the form constant $e^{\sqrt{-\rho_3/2}\,t}$ + constant $e^{-\sqrt{-\rho_3/2}\,t}$. Therefore, since M is assumed to be compact, the bounded function d^1_{22} has to be identically zero. Similarly, $d^2_{11} \equiv 0$. Using (1, 3), (2, 3) we conclude that d^3_{12}, d^1_{23} are constant, as desired.

Proof of (2), *Lemma* 5.2. Assume that $\rho_3 = 0$, and assume on the contrary that $\delta \not\equiv 0$. Now, consider the open submanifold $M_0 = \{p \in M \,|\, \delta(p) < 0\}$ and the vector fields X_1, X_2, X_3 on some finite covering manifold $\tilde{M}_0$ of M_0 as in the proof of Lemma 5.3. Then we have the functions $d^k_{ij} = \langle \nabla_{X_i}X_j, X_k\rangle : \tilde{M}_0 \to \mathbf{R}$ satisfying the same properties as before. By the assumption $\rho_3 = 0$, using the latter part of the properties in Lemma 1.4, we get $d^3_{12} > 0$ and $d^1_{23} = 0$. Then (2, 3) of Lemma 1.1 yields $d^1_{22} = 0$, and hence by (1, 1) we get $\rho_1 = X_2(d^2_{11}) - (d^2_{11})^2$. Furthermore, by (1, 3) we have $X_2(d^3_{12}) - d^2_{11}d^3_{12} = 0$. We contend that $\rho_1 < 0$. In fact, let p be the point where δ takes its minimum. Then d^3_{12} is maximal at p, and hence $X_2(d^3_{12})(p) = 0$, and $(X_2)^2(d^3_{12})(p) \leq 0$. Therefore $d^2_{11}(p) = 0$, $X_2(d^2_{11})(p) \leq 0$, and hence $\rho_1 \leq 0$. Recalling the assumption $\rho_1 \neq \rho_3$, we get $\rho_1 < 0$. Thus we can consider the non-empty open set $V = \{p \in \tilde{M}_0 \,|\, (d^2_{11})^2(p) < -\rho_1\}$. We shall show that the volume of V is infinite. This contradiction will prove our assertion (2). To estimate the volume of V, we denote by $p(t)$ the integral curve for X_2 through a point p. We contend that for any point $p \in V$, the curve $p(t)$ is defined for all $t \in \mathbf{R}$, and that $p(t) \in V$ for all t. In fact, from the differential equation $\rho_1 = X_2(d^2_{11}) - (d^2_{11})^2$ and the initial condition $|d^2_{11}(p)| < \sqrt{-\rho_1}$, we observe that $|d^2_{11}(p(t))| < \sqrt{-\rho_1}$ as far as $p(t)$ is defined. Hence, from the differential equation $X_2(d^3_{12}) - d^2_{11}d^3_{12} = 0$ we see that the function $d^3_{12}(p(t)) = \sqrt{-\delta(p(t))}$ does not accumulate to zero in finite t. Consequently, the curve $p(t)$ is defined for all t and lies in V. The above argument also shows that X_2 is transversal to the 2-dimensional submanifold $V_0 = \{p \in \tilde{M}_0 \,|\, d^2_{11}(p) = 0\}$, and each curve $p(t)$ in V intersects with V_0 once and only once. Furthermore, for any curve $p(t)$ starting at $p \in V_0$, we have $(\operatorname{div} X_2)(p(t)) = -d^2_{11}(p(t)) > 0$ for any $t > 0$. Thus we have to conclude that the volume of V is infinite, as desired. Assertion (2) is proved.

Proof of (3) *and* (4), *Lemma* 5.2. Assume that $\rho_1 \geq 0$. By (1), (2), in order to prove (3), (4), we may assume that $\rho_3 > 0$. Now, suppose that $\delta \not\equiv 0$. Then we can consider the submanifold $M_- = \{p \in M \mid \delta(p) \leq -\varepsilon\}$ with some small $\varepsilon > 0$ and the vector fields X_1, X_2, X_3 on some finite covering manifolds $\tilde{M}_-$ as in the proof of Lemma 5.3. Noting that the functions $d_{ij}^k = \langle \nabla_{X_i} X_j, X_k \rangle : \tilde{M}_- \to \mathbf{R}$ satisfy the same properties as in the proof of Lemma 5.3, we see that the assumption $\rho_3 > 0$ yields $d_{12}^3, d_{23}^1 > 0$, and from (2, 3), (1, 3) of Lemma 1.1 we have $X_1(d_{23}^1) = -d_{22}^1(d_{12}^3 - d_{23}^1)$, $X_2(d_{12}^3) = d_{11}^2(d_{12}^3 - d_{23}^1)$. We contend that the inner product of the vector field $d_{22}^1 X_1 + d_{11}^2 X_2$ and the vector field grad δ is given by

$$\langle d_{22}^1 X_1 + d_{11}^2 X_2, \operatorname{grad} \delta \rangle = 4(d_{12}^3 + d_{23}^1)\left(\frac{d_{12}^3 (d_{22}^1)^2 + d_{23}^1 (d_{11}^2)^2}{\rho_3}\right)\delta\,,$$

which is clearly nonpositive at every point, and which is zero if and only if $d_{22}^1 = d_{11}^2 = 0$. In fact, applying X_1, X_2 to $\rho_3 = 2d_{12}^3 d_{23}^1$, we have $X_1(d_{12}^3) = (d_{12}^3/d_{23}^1) d_{22}^1 (d_{12}^3 - d_{23}^1)$, $X_2(d_{23}^1) = -(d_{23}^1/d_{12}^3) d_{11}^2 (d_{12}^3 - d_{23}^1)$. Then our contention is easily verified. Now we can prove assertion (3). Since $\rho_1 = \operatorname{div}(d_{22}^1 X_1 + d_{11}^2 X_2)$, integrating ρ_1 over $\tilde{M}_-$ and using the integral formula, we get

$$\rho_1 \operatorname{vol}(\tilde{M}_-) = \int_{\partial \tilde{M}_-} \langle d_{22}^1 X_1 + d_{11}^2 X_2, \operatorname{grad} \delta \rangle \frac{1}{\|\operatorname{grad} \delta\|}\,.$$

Since the integrand of the right hand side is nonpositive as proved above, we obtain $\rho_1 \leq 0$. This contradiction proves (3). In order to prove (4), it suffices to verify that d_{12}^3, d_{23}^1 are constant and hence that $d_{22}^1 = d_{11}^2 = 0$. This also follows from the above integral expression. Indeed, if $\rho_1 = 0$, then the integrand has to be zero identically on the boundary, and hence $d_{22}^1 = d_{11}^2 = 0$. This proves (4) and completes the proof of Lemma 5.2.

Lemma 5.2, (1) has the following corollary:

PROPOSITION 5.4. *Let M be a compact Riemann manifold with constant Ricci eigenvalues ρ_1, ρ_2, ρ_3. If $\rho_1 = \rho_2 \neq \rho_3$ and $\rho_3 < 0$, then $\rho_1 = \rho_2 = 0$.*

Remark. Proposition 5.1 still holds for non-compact but comlete M with $\rho_1 = \rho_2 > 0$, $\rho_3 = 0$. This fact is essentially proved in [Se]. In our notation in the proof of Lemma 5.2, (2), this is verified as follows: It suffices to prove $\delta \equiv 0$. Let $p \in M_0$, if any. By (1, 1), (1, 3) we observe that the function $d_{12}^3(p(t))$ is of the form

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$$d^3_{12}(p(t)) = \frac{\text{const}}{\cos(\sqrt{\rho_1}\, t + \text{const})},$$

where $p(t)$ is the integral curve for X_2 through p. This expression shows, on the one hand, that the curve $p(t)$ does not approach the boundary of $\tilde{M}_0$, and hence that $p(t)$ is defined for all t, and on the other hand, that the function $d^3_{12}(p(t))$ is not defined for some t. This contradicts to the fact that d^3_{12} is defined on the whole $\tilde{M}_0$. Consequently $M_0 = \emptyset$, i.e. $\delta \equiv 0$, as desired.

Note that Proposition 5.1 does not hold generally in the noncompact case. In fact, there exist examples of non-homogeneous, complete metric on $\mathbf{R}^3$ with $\rho_1 = \rho_2 = -1$, $\rho_3 = 0$ ([Se], also [T]). K. Sekigawa found these examples in his study of Nomizu conjecture in dimension 3. Moreover, the recent result by O. Kowalski, F. Tricerri and L. Vanhecke ([KTV]) asserts that the set $\mathscr{M}(\mathbf{R}^3; -1, -1, 0)$ of isometry classes of complete metrics on $\mathbf{R}^3$ with Ricci eigenvalues $\rho_1 = \rho_2 = -1$, $\rho_3 = 0$ is infinite dimensional. On the other hand, Proposition 5.1 (with Theorem B) implies that the set $\mathscr{M}(S^3; \rho_1, \rho_2, 1)$, $0 < \rho_1 = \rho_2 \neq 1$, for example, is just one point (the isometry class of some left invariant metric on $SU(2)$). We will also see that the set $\mathscr{M}(M; \rho_1, \rho_2, -1)$ for compact M is empty if $0 \neq \rho_1 = \rho_2 \neq -1$ (Proposition 5.4). Much yet remains to be studied about the set $\mathscr{M}(M; \rho_1, \rho_2, \rho_3)$.

§ 6. Some curvature homogeneous metrics on $\mathbf{R}^3$

We give 3-dimensional, complete Riemann manifolds (diffeomorphic to $\mathbf{R}^3$) which are not homogeneous, but have distinct, constant Ricci eigenvalues. To give such examples, let ρ_1, ρ_2, ρ_3 be three distinct real numbers such that the numbers

$$A = \frac{\rho_1 + \rho_2 - \rho_3}{2}, \quad B = \frac{\rho_1 - \rho_3}{\rho_3 - \rho_2}, \quad C = -\frac{(\rho_1 + \rho_2)(\rho_3 - \rho_2)^2}{(\rho_2 - \rho_1)^2}.$$

satisfy the inequalities $A > 0$, $C > 0$, $A + BC > 0$. Then there exists a complete, non-homogeneous Riemann metric g on $\mathbf{R}^3$ with constant Ricci eigenvalues ρ_1, ρ_2, ρ_3. In fact, consider the vector fields X_1, X_2, X_3 on $\mathbf{R}^3$ defined by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y},$$

$$X_3 = \frac{\partial}{\partial z} + (x\phi(z) + yf(z))\frac{\partial}{\partial x} + (xg(z) + By\phi(z))\frac{\partial}{\partial y},$$

where $\phi(z)$ is a solution of the differential equation

$$\frac{d\phi}{dz} + (1 + B)(\phi^2 - C) = 0 \quad \text{satisfying } \phi^2 < C,$$

and f, g are the functions chosen so that

$$f^2 - g^2 = 2(\rho_1 + (1 + B)C), \quad (f + g)^2 = 4(A + B\phi^2), \quad f + g > 0.$$

Specifically, if $\rho_1 = -5/6$, $\rho_2 = 7/12$, $\rho_3 = -9/4$, then

$$\phi = \frac{1 - e^{-z}}{1 + e^{-z}}, \quad f = \sqrt{1 - \tfrac{1}{2}\phi^2} - \frac{1}{6\sqrt{1 - \tfrac{1}{2}\phi^2}}, \quad g = \sqrt{1 - \tfrac{1}{2}\phi^2} + \frac{1}{6\sqrt{1 - \tfrac{1}{2}\phi^2}}.$$

Let $\mathbf{g}$ be the Riemann metric on $\mathbf{R}^3$ such that X_1, X_2, X_3 are orthonormal. Then the Riemann manifold $(\mathbf{R}^3, \mathbf{g})$ is complete and non-homogeneous, but has constant Ricci eigenvalues ρ_1, ρ_2, ρ_3. The completeness follows from the fact that there exist constants K, L such that

$$u^2 + v^2 + w^2 \leq K(x^2 + y^2 + z^2) + L$$

for any tangent vector $Y = u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$ at (x, y, z) satisfying $\mathbf{g}(Y, Y) = 1$. Since div $X_3 = (1 + B)\phi$ is nonconstant, the non-homogeneity is obvious.

Finally, we should mention the following

PROBLEM. *Give a compact connected Riemann manifold of dimension* 3 *which is not locally homogeneous, but has constant Ricci eigenvalues.*

This seems to be difficult, but is interesting as a problem of global analysis on manifolds.

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