BULL. AUSTRAL. MATH. SOC. 52A20
VOL. 24 (1981), 321-325.

A GENERALIZATION OF RADON'S THEOREM II

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A new proof is given of the following result:

Let m and d be positive integers, and let a set of $md + m - d$ points be given in d-dimensional space. Then the set can be partitioned into m sets such that the m convex polytopes spanned by the sets have a non-empty intersection.

Let n-set mean a set of n points in R^d. We shall say that an n-set is m-divisible if it can be divided into m sets in such a way that the convex hulls of the m sets have a non-empty intersection. In 1964, [6], I proved the following:

THEOREM. *Any $(m(d+1)-d)$-set is m-divisible.*

The proof has been regarded as difficult and it is therefore a pleasure to be able to present a much simpler proof below. It is also a pleasure to acknowledge my debt to Imre Bárány, whose proof of Theorem 2.2 in [1] inspired the present one.

The proof is by induction on m. The case $m = 1$ is trivial and so, assuming that the theorem holds for $m = k > 0$, we are to prove that it holds for $m = k + 1$. Put $K = (k+1)(d+1) - d$ and let a K-set $\Omega_0 = \left\{p_1^0, \ldots, p_K^0\right\}$ be given. If the theorem is false for Ω_0, then there is an $\varepsilon > 0$ such that it is also false for any set $\Omega = \{p_1, \ldots, p_K\}$,

Received 28 April 1981. This research was done while the author was a Visiting Fellow in the Department of Mathematics, Research School of Physical Sciences, Institute of Advanced Studies, Australian National University. He was on sabbatical leave from the University of Bergen, supported by the Norwegian Research Council for Science and the Humanities.

https://doi.org/10.1017/S0004972700004858 Published online by Cambridge University Press

where $|p_i - p_i^0| < \varepsilon$, $i = 1, \ldots, K$. We choose Ω strongly independent, as defined by Reay [4], and prove the theorem for Ω , which suffices. Strong independence of Ω means that for each t and for any t affine subspaces $B_1, \ldots, B_t$ of R^d , spanned by pairwise disjoint subsets of Ω , $\dim(B_1 \cap \ldots \cap B_t) = \max(-1, \dim B_1 + \ldots + \dim B_t - (t-1)d)$.

Consider a partition in Ω , that is a partition of a subset of Ω , consisting of disjoint non-empty sets $\Omega_0, \ldots, \Omega_k$, having the property that $\operatorname{conv} \Omega_1 \cap \ldots \cap \operatorname{conv} \Omega_k \neq \emptyset$. Such partitions exist by the case $m = k$ of the theorem. Choose such a partition for which the distance from $\operatorname{conv} \Omega_0$ to $\operatorname{conv} \Omega_1 \cap \ldots \cap \operatorname{conv} \Omega_k$ is minimal. If this distance is zero, we are through, and so we assume it to be positive. A contradiction will now be obtained by showing that some other partition in Ω will make the distance considered smaller.

Let $q \in \operatorname{conv} \Omega_0$ and $r \in \operatorname{conv} \Omega_1 \cap \ldots \cap \operatorname{conv} \Omega_k$ be a pair of points realizing the distance. By Carathéodory's theorem there is a simplex, with vertex set $\Omega_0' \subseteq \Omega_0$, such that q is in the relative interior of $\operatorname{conv} \Omega_0'$. Replacing Ω_0 by Ω_0' , we may thus assume that q is in the relative interior of $\operatorname{conv} \Omega_0$. Similarly, we may assume that each $\operatorname{conv} \Omega_i$, $i = 1, \ldots, k$, is a simplex with r in its relative interior. Putting $A_i = \operatorname{aff} \Omega_i$, the affine hull of Ω_i , we then get $\dim A_i = |\Omega_i| - 1$, $i = 0, \ldots, k$. Furthermore, by the condition of strong independence,

$$\dim(A_1 \cap \ldots \cap A_k) = \dim A_1 + \ldots + \dim A_k - (k-1)d$$
$$= |\Omega_1 \cup \ldots \cup \Omega_d| - k + d - kd .$$

We now want to prove that $\Omega_0 \cup \ldots \cup \Omega_k$ is a proper subset of Ω , which will leave us some point p_j to add to a suitable Ω_i so as to lower the distance in question. Consider the parallel hyperplanes H_q through q and H_r through r , both orthogonal to $q - r$. The open

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slab between them clearly separates conv Ω_0 from conv $\Omega_1 \cap \ldots \cap \text{conv } \Omega_k$. Furthermore $\Omega_0 \subset H_q$, while conv $\Omega_1 \cap \ldots \cap \text{conv } \Omega_k \subset H_r$. The first of these inclusions holds true because H_q is a supporting hyperplane of conv Ω_0 in q and q is in the relative interior of the simplex conv Ω_0. The second one holds because r is in the relative interior of each simplex conv $\Omega_1, \ldots,$ conv Ω_k, so that some neighbourhood of r in $A_1 \cap \ldots \cap A_k$ is contained in conv $\Omega_1 \cap \ldots \cap \text{conv } \Omega_k$, and is thus supported by H_r in R. (If it was not, it would meet the open slab mentioned above.) These two inclusions show that $A_0 \subset H_q$ and $A_1 \cap \ldots \cap A_k \subset H_r$, which, by the strong independence in Ω can happen only if

$$\dim A_0 + \dim\left(A_1 \cap \ldots \cap A_k\right) < d .$$

Thus

$$|\Omega_0| - 1 + |\Omega_1 \cup \ldots \cup \Omega_k| - k + d - kd < d ,$$

so that

$$|\Omega_0 \cup \ldots \cup \Omega_k| < k(d+1) + 1 = |\Omega| .$$

It is no restriction to assume that $p_1 \notin \Omega_0 \cup \ldots \cup \Omega_k$.

The easier case is when p_1 is in that open halfspace, bounded by H_q, in which H_r lies. The segment qp_1 will then be in $\text{conv}\left(\Omega_0 \cup \{p_1\}\right)$ and for any point q' on it sufficiently near q, but not equal to q, we will have $|q'-r| < |q-r|$. Thus the distance from $\text{conv}\left(\Omega_0 \cup \{p_1\}\right)$ to conv $\Omega_1 \cap \ldots \cap \text{conv } \Omega_k$ will be smaller than that from conv Ω_0.

In the second, more difficult, case, p_1 is separated (weakly) from H_r by H_q. We shall see that for some $i \in \{1, \ldots, k\}$ there is a ray from r, contained in $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k$, which passes through the simplex $\text{conv}\left(\Omega_i \cup \{p_1\}\right)$ and also lies in the halfspace bounded by H_r

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and containing H_q . Any point r' $(\neq r)$ on this ray, sufficiently near r , will then satisfy $|q-r'| < |q-r|$ and also be in $\operatorname{conv} \Omega_1 \cap \ldots \cap \operatorname{conv}(\Omega_i \cup \{p_1\}) \cap \ldots \cap \operatorname{conv} \Omega_k$, so that the desired contradiction will be obtained once again.

For the proof of the existence of the ray just mentioned, we introduce an affine coordinate system in which $r = (0, \ldots, 0)$, $p_1 = (1, \ldots, 1)$, A_1 is given by $x_j = 0$, $j \in C_1, \ldots, A_k$ by $x_j = 0$, $j \in C_k$. Here $C_1, \ldots, C_k$, form a partition in $\{1, \ldots, d\}$ (with possibly some empty parts). The disjointness of the C_i is a consequence of the strong independence in Ω . Furthermore H_r is given by $a_1x_1 + \ldots + a_dx_d = 0$, where $a_j = 0$ for $j \notin (C_1 \cup \ldots \cup C_k)$, as we have seen that $A_1 \cap \ldots \cap A_k \subset H_r$. The a_j's are normalized so that p_1 (which is not in H_r) lies in the hyperplane $a_1x_1 + \ldots + a_dx_d = 1$. Thus H_q will have the equation $a_1x_1 + \ldots + a_dx_d = a$, with $0 < a \leq 1$.

Now consider the flat $\operatorname{aff}(\Omega_1 \cup \{p_1\}) \cap A_2 \cap \ldots \cap A_s$. It is no restriction to assume that $C_1 = \{1, \ldots, |C_1|\}$, $C_2 = \{|C_1|+1, \ldots, |C_1|+|C_2|\}$ and so on. Then the point $p_1' = (1, \ldots, 1, 0, \ldots, 0)$ ($|C_1|$ 1's) is in the flat, as it equals $p_1 + (0, \ldots, 0, -1, \ldots, -1)$. Furthermore p_1' and p_1 are in the same open halfspace of $\operatorname{aff}(\Omega_1 \cup \{p_1\})$, bounded by A_1 , as $(0, \ldots, -1, \ldots, -1)$ is in A_1 . Hence the ray from r through p_1' passes through the simplex $\operatorname{conv}(\Omega_1 \cup \{p_1\})$. It will also lie in the halfspace bounded by H_r and containing H_q , provided $(a_1, \ldots, a_d) \cdot (1, \ldots, 1, 0, \ldots, 0) = a_1 + \ldots + a_{|C_1|} > 0$. Thus, if $a_1 + \ldots + a_{|C_1|} > 0$, we have what we want. Similarly, we shall be satisfied with $a_{|C_1|+1} + \ldots + a_{|C_1|+|C_2|} > 0$, and so on. But one of these equalities must hold, as $a_1 + \ldots + a_{|C_1|+\ldots+|C_k|} = 1$.

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Doignon and Valette [2] have proved that our theorem remains valid in any affine space over an ordered division ring. The proof given above can be modified so as to show this. Finally I would like to call the reader's attention to the recent survey papers by Eckhoff [3] and Reay [5], which give a lot of information on Radon's theorem and related matters.

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https://doi.org/10.1017/S0004972700004858 Published online by Cambridge University Press