Pressure live loads and the variational derivation of linear elasticity

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The rigorous derivation of linear elasticity from finite elasticity by means of Γ-convergence is a well-known result, which has been extended to different models also beyond the elastic regime. However, in these results the applied forces are usually assumed to be dead loads, that is, their density in the reference configuration is independent of the actual deformation. In this paper we begin a study of the variational derivation of linear elasticity in the presence of live loads. We consider a pure traction problem for a nonlinearly elastic body subject to a pressure live load and we compute its linearization for small pressure by Γ-convergence. We allow for a weakly coercive elastic energy density and we prove strong convergence of minimizers.

1. Introduction

Linear elasticity is a well-known and powerful mathematical approximation of the nonlinear theory of elasticity, with extensive application to the structural analysis and the numerical treatment of elastic bodies. In engineering textbooks its derivation is classical and is based on a formal linearization of finite elasticity about a reference configuration. A rigorous mathematical derivation via Γ-convergence was developed only rather recently in the pioneering work [8], where a Dirichlet boundary value problem was considered. A similar approach was then applied to different frameworks in elasticity, such as rubber-like materials [3], multiwell models [1, 2, 32], elasticity with residual stress [27, 28] and incompressible materials [21]. Beyond elasticity we also mention the papers [9, 10, 25, 26] for models in fracture mechanics, [12] for viscoelasticity, [23] for plasticity and the recent contribution [11] for materials with stress-driven rearrangement instabilities.

Linearization of pure traction problems has been recently studied in [14, 16, 17, 19], again in the context of elasticity. In this setting a full Γ-convergence result has been obtained in [22] and later extended to incompressible materials in [20]. As

observed in [**22**], in the Dirichlet case the boundary conditions prescribe the rigid motion to linearize about, whereas in the purely Neumann case the linearization process occurs around suitable rotations that are preferred by the applied forces.

In all this literature the main focus is on understanding the behaviour of the bulk elastic energy and the applied forces are usually assumed to be dead loads, namely their density in the reference configuration is independent of the actual deformation. This assumption is mathematically convenient, since the work done by the loadings turns out to be a continuous perturbation of the elastic energy, so that Γ-convergence of the total energy immediately follows from Γ-convergence of the elastic energy. However, restricting the analysis to dead loads is physically unsatisfactory, since the only realistic examples of dead loads are the gravitational body force and the zero surface load (see, e.g. [**29**, **30**] and [**4**, § 2.7]).

In this paper we begin a study of the derivation of linear elasticity in the presence of *live loads*. More precisely, we consider a pure traction problem for a hyperelastic body $\Omega \subset \mathbb{R}^n$ subject to a (small) *pressure load* on its boundary. In this setting the total energy of a deformation $y\colon \Omega \to \mathbb{R}^n$ is given by

$$\mathcal{T}_\varepsilon(y) := \int_\Omega W(x, \nabla y(x))\,\mathrm{d}x + \varepsilon \int_\Omega \pi(y(x)) \det \nabla y(x)\,\mathrm{d}x,$$

where the elastic energy density $W\colon \Omega \times \mathbb{R}^{n\times n} \to [0, +\infty]$ satisfies the usual assumptions of nonlinear elasticity (see (W1)–(W5)) and $\varepsilon\pi$ is the intensity of the applied pressure load, with $\varepsilon > 0$ a small parameter and $\pi\colon \mathbb{R}^n \to \mathbb{R}$ a given function. For simplicity in this Introduction we assume π to be continuous. As shown in [**30**, proposition 5.1] (see also [**15**, proposition 1.2.8]), the second term in the energy $\mathcal{T}_\varepsilon$ is the potential of the pressure load

$$-\varepsilon\pi(y(x))(\operatorname{cof} \nabla y(x))n_{\partial\Omega}(x) \quad \text{for } x \in \partial\Omega \tag{1.1}$$

acting on the whole boundary of Ω, where $\operatorname{cof} F$ denotes the cofactor of the matrix F and $n_{\partial\Omega}$ is the outward unit normal to $\partial\Omega$. In the deformed configuration $y(\Omega)$ the pressure load (1.1) corresponds to the surface force

$$-\varepsilon\pi(z)n_{\partial(y(\Omega))}(z) \qquad \text{for } z \in \partial(y(\Omega)).$$

Since $W(x, \cdot)$ is frame-indifferent and minimized at the identity, it is immediate to see that for $\varepsilon = 0$ the minimizers of $\mathcal{T}_\varepsilon$ are all the rigid motions of Ω. When ε is small, it is thus natural to expect minimizers to be close to rigid motions and their asymptotic behaviour to be described by a linearization of the energy. In pure traction problems, as mentioned before, the applied forces select the class of rigid motions around which the linearization takes place (see [**22**]). Indeed, if y_ε is a minimizer of $\mathcal{T}_\varepsilon$, then we have

$$\mathcal{T}_\varepsilon(y_\varepsilon) \leqslant \varepsilon \int_\Omega \pi(Rx)\,\mathrm{d}x \qquad \text{for every rotation } R \in SO(n).$$

If we assume y_ε to be of the form $y_\varepsilon(x) = R_0(x + \varepsilon u_0(x))$ with $R_0 \in SO(n)$, then by a formal expansion we obtain

$$\frac{\varepsilon^2}{2}\int_\Omega Q(x, e(u_0)(x))\,\mathrm{d}x + \varepsilon \int_\Omega \pi(R_0 x)\,\mathrm{d}x + O(\varepsilon^2) \leqslant \varepsilon \int_\Omega \pi(Rx)\,\mathrm{d}x$$

for every rotation $R \in SO(n)$. Here $Q(x, \cdot)$ is the quadratic form given by the Hessian of $W(x, \cdot)$ computed at the identity and $e(u_0)$ is the symmetric gradient of u_0. Dividing by ε and letting ε tend to zero, we deduce that R_0 is a so-called *optimal rotation*, that is, R_0 belongs to the set

$$\mathcal{R} := \operatorname*{argmin}_{R \in SO(n)} \left\{ \int_\Omega \pi(Rx) \, \mathrm{d}x \right\}.$$

Assume now for simplicity that the identity matrix belongs to $\mathcal{R}$ (one can always reduce to this case, up to rotating the whole system). The previous argument suggests that in order to identify the limiting behaviour of minimizers one needs to renormalize the energy as follows:

$$\frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y) := \frac{1}{\varepsilon^2} \left(\mathcal{T}_\varepsilon(y) - \varepsilon \int_\Omega \pi(x) \, \mathrm{d}x \right)$$

$$= \frac{1}{\varepsilon^2} \int_\Omega W(x, \nabla y(x)) \, \mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega \big(\pi(y(x)) \det \nabla y(x) - \pi(x) \big) \, \mathrm{d}x. \quad (1.2)$$

Under suitable assumptions for π and a weak p-coercivity condition on W with $1 < p \leqslant 2$ (see (W5)), we compute the Γ-limit of the rescaled energies $(1/\varepsilon^2)\mathcal{E}_\varepsilon$ and we establish a compactness result for deformations with equibounded energies. Here deformations are assumed to have zero average on Ω, as it is common in Neumann boundary value problems. More precisely, we prove the following results:

Compactness: If $\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant C\varepsilon^2$, then there exist rotations $R_\varepsilon \in SO(n)$ and displacements $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^n)$ such that

$$y_\varepsilon(x) = R_\varepsilon(x + \varepsilon u_\varepsilon(x)) \qquad \text{for } x \in \Omega \quad (1.3)$$

and, up to subsequences, there holds

- $u_\varepsilon \rightharpoonup u_0$ weakly in $W^{1,p}(\Omega; \mathbb{R}^n)$ with $u_0 \in H^1(\Omega; \mathbb{R}^n)$,

- $R_\varepsilon \to R_0$ with $R_0 \in \mathcal{R}$.

Γ-convergence: Under the above notion of convergence $y_\varepsilon \to (u_0, R_0)$, the rescaled energies $(1/\varepsilon^2)\mathcal{E}_\varepsilon$ Γ-converge to

$$\mathcal{E}_0(u_0, R_0) := \frac{1}{2} \int_\Omega Q(x, e(u_0)(x)) \, \mathrm{d}x + \int_{\partial\Omega} \pi(R_0 x) n_{\partial\Omega}(x) \cdot u_0(x) \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

We also deduce *strong convergence* of (almost) minimizers: if (y_ε) is a sequence of (almost) minimizers, then we have in addition that $u_\varepsilon \to u_0$ strongly in $W^{1,p}(\Omega; \mathbb{R}^n)$ and the pair (u_0, R_0) is a minimizer of $\mathcal{E}_0$.

We now comment on the expression of the limiting energy $\mathcal{E}_0$, which features two terms: the usual linear elastic energy and a potential term accounting for the surface load $-\pi(R_0 x) n_{\partial\Omega}(x)$ on $\partial\Omega$. The emergence of this boundary term can be explained by the following heuristic considerations: on the one hand, a formal

linearization of (1.1) leads to a pressure load of the above form; on the other hand, if y is smooth enough, the force term in (1.2) can be written as

$$\frac{1}{\varepsilon}\left(\int_{y(\Omega)} \pi(x)\,\mathrm{d}x - \int_{\Omega} \pi(x)\,\mathrm{d}x\right).$$

Hence, taking into account (1.3), computing the limit of the above expression on sequences (y_{ε}) with equibounded energies corresponds to a sort of shape derivative of the functional $\Omega \mapsto \int_{\Omega} \pi(x)\,\mathrm{d}x$ (see, e.g. [**18**, proposition 17.8]). However, we stress that in the present setting deformations y_{ε} are only of Sobolev regularity and are close to rigid motions only in the sense of $W^{1,p}(\Omega; \mathbb{R}^n)$, therefore the usual arguments in the context of shape derivatives do not apply.

From a mathematical viewpoint the main difference with respect to previous contributions dealing with dead loads, is that the force term in (1.2) is not a continuous perturbation of the elastic energy. Indeed, our assumptions on W imply that deformations are at most strongly convergent in $W^{1,p}(\Omega; \mathbb{R}^n)$ with $1 < p \leqslant 2$ and this is not enough to guarantee convergence of the determinants. Moreover, the crucial step in the proof of compactness is to show that deformations satisfying the bound $\mathcal{E}_{\varepsilon}(y_{\varepsilon}) \leqslant C\varepsilon^2$ have an elastic energy of order ε^2. Once this is established, one can apply the rigidity estimate by Friesecke *et al.* [**13**] and deduce (1.3), together with a uniform bound for (u_{ε}) in $W^{1,p}(\Omega; \mathbb{R}^n)$. In the case of dead loads deducing the ε^2-bound on the elastic energy is straightforward, since the force term is linear with respect to the deformation. In our setting, instead, this is one of the main difficulties. We show that the problem can be solved under two different sets of conditions:

- π Lipschitz continuous in a suitable neighbourhood of Ω and nonnegative;

- π Lipschitz continuous in a suitable neighbourhood of Ω, with a growth condition on its negative part (see (π3)) and an additional coercivity property for $W(\cdot, F)$ in terms of $\det F$ (see (W6)).

We note that this additional coercivity condition on W is satisfied by a large class of elastic materials (see, e.g. [**3**, remark 2.8]).

We also observe that both the nonlinear energy (1.2) and the Γ-limit $\mathcal{E}_0$ are well defined if π is merely a continuous function. However, because of the low regularity of deformations, in our proofs we need π to be Lipschitz continuous, at least in a suitable neighbourhood of Ω. How to extend our analysis to less regular pressure loads is an interesting question that we plan to consider in a future work.

In the case of dead loads the set $\mathcal{R}$ of optimal rotations is a submanifold of $SO(n)$ and, as a consequence, one can prove that the distance of the approximating rotations R_{ε} in (1.3) from $\mathcal{R}$ is at most of order $\sqrt{\varepsilon}$, see [**22**]. In the last part of the paper we show that neither of these properties is true, in general, in the present setting.

Finally, we mention that the displacement-traction problem, where a Dirichlet condition is prescribed on a part $\partial_D\Omega$ of $\partial\Omega$ and the remaining part of the boundary is subject to a pressure load, can be treated combining the techniques of this paper with the results in [**3**]. More precisely, assuming deformations y to satisfy a

boundary condition of the form $y(x) = x + \varepsilon w(x)$ for $x \in \partial_D\Omega$, the compactness and Γ-convergence results still hold with $R_0 = I$ and the boundary condition $u_0 = w$ on $\partial_D\Omega$. In particular, one can show that the approximating rotations R_ε in (1.3) are ε-close to the identity, see [3, lemma 3.3]. Therefore, there is no need to introduce the set of optimal rotations and the analysis turns out to be simpler than the one considered here.

Plan of the paper: In § 2 we set the problem and we state the main assumptions. In § 3 we discuss the case of a nonnegative pressure intensity π. We then extend our analysis to pressures with arbitrary sign in § 4. Finally, in § 5 we compute a refined Γ-limit, which takes into account how much deformations differ from being optimal rotations, and we make a comparison with the results proved in [22] in the case of dead loads.

2. Setting of the problem

2.1. Notation and preliminaries

Throughout the paper, the symbols C or c will be used to denote some positive constants not depending on ε, whose value may change from line to line.

Given two (extended) real numbers a and b the notation $a \vee b$ (respectively, $a \wedge b$) stands for the maximum (respectively, the minimum) between the two numbers. Given a scalar function f, we denote its positive and negative part by f^+ and f^-, respectively, so that $f = f^+ - f^-$. By $B_r \subset \mathbb{R}^n$ we mean the open ball with radius $r > 0$ centred at the origin.

Let Ω be an open set in $\mathbb{R}^n$. For $p \in [1, \infty]$ the norms in $L^p(\Omega)$ and $L^p(\Omega; \mathbb{R}^n)$ will be simply denoted by $\| \cdot \|_p$. The conjugate exponent of $p \in [1, \infty]$ will be denoted by p'. The notation $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ stands for the space of Sobolev functions $y \in W^{1,p}(\Omega; \mathbb{R}^n)$ with zero average; if $p = 2$, we shall write $\mathring{H}^1(\Omega; \mathbb{R}^n)$ instead of $\mathring{W}^{1,2}(\Omega; \mathbb{R}^n)$.

We denote by $\mathbb{R}^{n \times n}$, $\mathbb{R}^{n \times n}_{\text{sym}}$ and $\mathbb{R}^{n \times n}_{\text{skew}}$ the set of $(n \times n)$-matrices and the subsets of symmetric and skew-symmetric matrices, respectively. The set of rotations is denoted by $SO(n)$, namely

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : \ R^T R = I, \ \det R = 1\}.$$

Finally, we recall that for every $F \in \mathbb{R}^{n \times n}$ and $\varepsilon > 0$ there holds

$$\det(I + \varepsilon F) = 1 + \sum_{k=1}^{n} \varepsilon^k \iota_k(F), \tag{2.1}$$

where $\iota_k(F)$ is a homogeneous polynomial of degree k in the entries of F. In particular, there exists a constant $C > 0$, depending only on n, such that

$$|\iota_k(F)| \leqslant C|F|^k \qquad \text{for every } F \in \mathbb{R}^{n \times n}. \tag{2.2}$$

For $k = 1$ and $k = n$ we have that $\iota_1(F) = \operatorname{tr} F$ and $\iota_n(F) = \det F$.

For the definition and the properties of Γ-convergence we refer to the monograph [7].

2.2. The main assumptions

Let $\Omega \subset \mathbb{R}^n$, with $n \geqslant 2$, be a bounded domain with Lipschitz boundary representing the reference configuration of a hyperelastic body. Up to a translation of the axes, we can assume without loss of generality that the origin is the barycentre of Ω, i.e.

$$\int_{\Omega} x \, \mathrm{d}x = 0. \tag{2.3}$$

For future use we also introduce the set

$$\mathcal{O} := \bigcup_{R \in SO(n)} R\Omega,$$

which is an open annulus (if $0 \notin \Omega$; an open ball if $0 \in \Omega$) centred at 0 and containing Ω.

The stored energy density of the body is assumed to be a Carathéodory function $W \colon \Omega \times \mathbb{R}^{n \times n} \to [0, +\infty]$ satisfying the following conditions for almost every $x \in \Omega$:

(W1) $W(x, F) = +\infty$ if $\det F \leqslant 0$ (orientation preserving condition);

(W2) $W(x, RF) = W(x, F)$ for every $F \in \mathbb{R}^{n \times n}$ and $R \in SO(n)$ (frame indifference);

(W3) $W(x, I) = 0$ (the reference configuration is stress-free);

(W4) $W(x, \cdot)$ is of class C^2 in a neighbourhood of $SO(n)$, independent of x, where the second derivatives of W are bounded, uniformly with respect to $x \in \Omega$;

(W5) $W(x, F) \geqslant c_1 g_p(\mathrm{dist}(F; SO(n)))$ for every $F \in \mathbb{R}^{n \times n}$ and for some $p \in (1, 2]$, where g_p is defined as

$$g_p(t) := \begin{cases} \dfrac{t^2}{2} & \text{if } t \in [0, 1], \\[2ex] \dfrac{t^p}{p} + \dfrac{1}{2} - \dfrac{1}{p} & \text{if } t > 1, \end{cases} \tag{2.4}$$

and $c_1 > 0$ is a constant independent of x (coercivity).

Assumptions (W1)–(W3) are natural conditions in elasticity theory (see, e.g. [4, 15]), assumption (W4) is the minimal regularity hypothesis needed to perform the linearization, while condition (W5) is satisfied by a large class of compressible rubber-like materials (see, e.g. [2, 3, 19–21]).

We note that

$$g_p(t) \geqslant \frac{1}{2}(t^2 \wedge t^p) \qquad \text{for every } t \geqslant 0. \tag{2.5}$$

Moreover, condition (W5) implies the following bound:

$$W(x, F) \geqslant c \, |\det F - 1|^2 \quad \text{for a.e. } x \in \Omega \text{ and for every } F \in \mathbb{R}^{n \times n}$$
$$\text{with } |\det F - 1| \leqslant 1, \tag{2.6}$$

for a suitable constant $c > 0$, independent of x. Indeed, by (W5) the bound (2.6) is satisfied for F outside a neighbourhood of $SO(n)$. If instead $\mathrm{dist}(F; SO(n))$ is small enough, then one has

$$c|\det F - 1|^2 \leqslant \mathrm{dist}^2(F; SO(n)),$$

which implies (2.6) by using again (W5).

We assume the body to be subjected to a *pressure load*, whose (unscaled) intensity is a Borel measurable function $\pi \colon \mathbb{R}^n \to \mathbb{R}$ such that

($\pi 1$) π is Lipschitz continuous in an open set containing $\overline{\mathcal{O}}$.

Since we do not prescribe any Dirichlet boundary condition, the linearization process will naturally select, as in [**22**] in the case of dead loads, a particular set of rotations that are 'preferred' by the force. This set is called the set of *optimal rotations* and in our framework it is defined as

$$\mathcal{R} := \operatorname*{argmin}_{R \in SO(n)} \left\{ \int_\Omega \pi(Rx) \, \mathrm{d}x \right\}. \tag{2.7}$$

Since the map $R \mapsto \int_\Omega \pi(Rx) \, \mathrm{d}x$ is continuous and $SO(n)$ is compact, the set of optimal rotations is not empty and is a compact subset of $SO(n)$. For simplicity we assume that

$$I \in \mathcal{R}, \tag{2.8}$$

where I is the identity matrix. Indeed, if this is not the case, we can always replace π by $\pi(R_0 \cdot)$ and deformations y by $R_0^T y$, where R_0 is a given optimal rotation.

Let $R_0 \in \mathcal{R}$. By computing the first variation of the functional in (2.7) along the curve $t \mapsto R_0 \mathrm{e}^{tA}$ with $A \in \mathbb{R}^{n \times n}_{\mathrm{skew}}$, we deduce that any optimal rotation R_0 satisfies the following Euler–Lagrange equation:

$$\int_\Omega \nabla\pi(R_0 x) \cdot R_0 A x \, \mathrm{d}x = 0 \quad \text{for every } A \in \mathbb{R}^{n \times n}_{\mathrm{skew}}. \tag{2.9}$$

Applying the divergence theorem, condition (2.9) can be rewritten as

$$\int_{\partial\Omega} \pi(R_0 x) n_{\partial\Omega}(x) \cdot A x \, \mathrm{d}\mathcal{H}^{n-1}(x) = 0 \qquad \text{for every } A \in \mathbb{R}^{n \times n}_{\mathrm{skew}}, \tag{2.10}$$

where $n_{\partial\Omega}$ is the outward unit normal to $\partial\Omega$.

3. Nonnegative pressure loads

We start our analysis by considering a pressure load with nonnegative intensity, that is,

($\pi 2$) $\pi(y) \geqslant 0$ for every $y \in \mathbb{R}^n$.

This includes, for instance, the relevant case of hydrostatic pressure $\pi(y) = g\rho y_3^-$, where g is the gravitational constant, ρ is the constant density of the fluid and y_3^- denotes the negative part of the third component of y.

For every $\varepsilon \in (0, 1)$ we consider the energy $\mathcal{E}_\varepsilon \colon \mathring{W}^{1,p}(\Omega; \mathbb{R}^n) \to (-\infty, +\infty]$ defined as

$$\mathcal{E}_\varepsilon(y) := \begin{cases} \displaystyle\int_\Omega W(x, \nabla y(x)) \, \mathrm{d}x + \varepsilon \int_\Omega (\pi(y(x)) \det \nabla y(x) - \pi(x)) \, \mathrm{d}x & \text{if } y \in Y^p, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.1}$$

where the set of admissible deformations is

$$Y^p := \left\{ y \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n) : \ \det \nabla y(x) > 0 \text{ for a.e. } x \in \Omega \right\}.$$

In other words, admissible deformations are orientation preserving and, as it is common in Neumann boundary value problems, have zero average on Ω.

By (2.3) any rigid motion of the form

$$y_R(x) := Rx \quad \text{with } R \in SO(n) \tag{3.2}$$

belongs to Y^p. Moreover, under the assumption $(\pi 2)$, the energy is well defined since the two integrands $W(\cdot, \nabla y)$ and $\pi(y) \det \nabla y$ are nonnegative for $y \in Y^p$.

REMARK 3.1. Here we do not assume deformations to be injective in any sense. However, one can easily include the requirement that admissible deformations are a.e. injective (see, e.g. [**4**]), without affecting the results of the paper, see also remark 3.13.

The key ingredient in the proof of compactness is the following variant of the celebrated *rigidity estimate* by Friesecke *et al.* [**13**], whose proof can be found, e.g. in [**3**, lemma 3.1]. Similar variants of the rigidity estimates with mixed growth condition have been proved in [**5**, **24**, **31**].

THEOREM 3.2. *There exists a positive constant $C = C(\Omega, p) > 0$ with the following property: for every $y \in W^{1,p}(\Omega; \mathbb{R}^n)$ there exists a constant rotation $R \in SO(n)$ such that*

$$\int_\Omega g_p(|\nabla y(x) - R|) \, \mathrm{d}x \leqslant C \int_\Omega g_p(\mathrm{dist}(\nabla y(x); SO(n))) \, \mathrm{d}x.$$

The following generalized rigidity estimate will be used in theorem 3.14 to infer strong convergence of almost minimizers. For a proof we refer to [**6**, theorem 1.1].

THEOREM 3.3. *Let $1 < p_1 < p_2 < \infty$. Then there exists a positive constant $C = C(\Omega, p_1, p_2) > 0$ with the following property: for every $y \in W^{1,1}(\Omega; \mathbb{R}^n)$ with*

$$\mathrm{dist}(\nabla y; SO(n)) = f_1 + f_2 \qquad \text{for some } f_i \in L^{p_i}(\Omega), \quad i = 1, 2,$$

there exist a constant rotation $\widetilde{R} \in SO(n)$ and two functions $g_i \in L^{p_i}(\Omega)$ such that

$$\nabla y = \widetilde{R} + g_1 + g_2 \quad \text{and} \quad \|g_i\|_{p_i} \leqslant C \|f_i\|_{p_i}, \quad i = 1, 2.$$

Our arguments will strongly rely on the Lipschitz continuity of the pressure function π. This, however, holds only in a suitable set Ω' containing Ω (see $(\pi 1)$).

Since deformations $y \in Y^p$ may be a priori valued outside Ω', it is convenient to introduce an auxiliary Lipschitz continuous function that coincides with π in Ω' and is bounded on the whole of $\mathbb{R}^n$. This is the content of the following lemma, which is clearly not necessary if π itself is Lipschitz continuous and bounded.

LEMMA 3.4. *Assume* $(\pi 1)$ *and* $(\pi 2)$. *Then there exists a Lipschitz continuous function* $\hat{\pi} \colon \mathbb{R}^n \to [0, +\infty)$, *with compact support, such that* $\hat{\pi}$ *coincides with* π *in an open neighbourhood of* $\overline{\mathcal{O}}$ *and* $\hat{\pi}(y) \leqslant \pi(y)$ *for all* $y \in \mathbb{R}^n$. *In particular,* $\hat{\pi}$ *is bounded.*

REMARK 3.5. Note that the set of optimal rotations (2.7) stays the same if π is replaced by $\hat{\pi}$, since π and $\hat{\pi}$ coincide in a neighbourhood of $\overline{\mathcal{O}}$.

Proof of lemma 3.4. Assume $0 \notin \Omega$, so that $\mathcal{O}$ is an open annulus centred at 0 (the case where $0 \in \Omega$ can be treated similarly). Let $0 < r_1 < r_2$ and $0 < \delta < r_1$ be such that $\overline{\mathcal{O}} \subset B_{r_2} \setminus \overline{B_{r_1}}$ and π is Lipschitz continuous in $\overline{B_{r_2+\delta}} \setminus B_{r_1-\delta}$ with Lipschitz constant L. Let $M \geqslant 0$ be the maximum of π on $\partial B_{r_1} \cup \partial B_{r_2}$. We first define $\widetilde{\pi} \colon \overline{B_{r_2+\delta}} \setminus B_{r_1-\delta} \to \mathbb{R}$ as

$$
\widetilde{\pi}(y) := \begin{cases} \pi\left(r_1 \frac{y}{|y|}\right) - K(r_1 - |y|) & \text{if } r_1 - \delta \leqslant |y| < r_1, \\ \pi(y) & \text{if } r_1 \leqslant |y| < r_2, \\ \pi\left(r_2 \frac{y}{|y|}\right) - K(|y| - r_2) & \text{if } r_2 \leqslant |y| \leqslant r_2 + \delta, \end{cases}
$$

where $K := L \vee M/\delta$. It is easy to see that $\widetilde{\pi}$ is Lipschitz continuous in its domain; moreover, by construction it coincides with π in an open neighbourhood of $\overline{\mathcal{O}}$. We now show that $\widetilde{\pi} \leqslant \pi$ on $\overline{B_{r_2+\delta}} \setminus B_{r_1-\delta}$. Indeed, if $r_1 - \delta \leqslant |y| < r_1$, by the Lipschitz continuity of π we have

$$
\widetilde{\pi}(y) \leqslant \pi(y) + L\left|r_1 \frac{y}{|y|} - y\right| - K(r_1 - |y|) = \pi(y) - (K - L)(r_1 - |y|) \leqslant \pi(y),
$$

and similarly if $r_2 \leqslant |y| \leqslant r_2 + \delta$. Finally, we note that, if $y \in \partial B_{r_1-\delta} \cup \partial B_{r_2+\delta}$, then

$$
\widetilde{\pi}(y) \leqslant M - K\delta \leqslant 0.
$$

We now conclude by considering $\hat{\pi}(y) := \widetilde{\pi}(y) \vee 0$ for $y \in \overline{B_{r_2+\delta}} \setminus B_{r_1-\delta}$, $\hat{\pi}(y) := 0$ otherwise in $\mathbb{R}^n$. $\qquad\square$

We are now in a position to state and prove some estimates which will be crucial to infer compactness of deformations with equibounded (rescaled) energy.

LEMMA 3.6. *Assume* (W1)–(W5), (π1), *and* (π2). *If* $\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant C\varepsilon^2$ *for every* $\varepsilon \in (0, 1)$, *then there holds*

$$\int_{\{|\det \nabla y_\varepsilon - 1| \leqslant 1\}} |\det \nabla y_\varepsilon(x) - 1|^2 \, \mathrm{d}x \leqslant C\varepsilon^2 \quad \text{for every } \varepsilon \in (0,1). \tag{3.3}$$

Furthermore, there exist constant rotations $R_\varepsilon \in SO(n)$ *such that the rescaled displacements* $u_\varepsilon \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ *defined by*

$$u_\varepsilon := \frac{1}{\varepsilon} R_\varepsilon^T (y_\varepsilon - y_{R_\varepsilon}) \tag{3.4}$$

(see (3.2) *for the definition of* y_{R_ε}*) satisfy*

$$\int_\Omega g_p(\varepsilon |\nabla u_\varepsilon(x)|) \, \mathrm{d}x \leqslant C\varepsilon^2 \quad \text{for every } \varepsilon \in (0,1) \tag{3.5}$$

and are uniformly bounded in $W^{1,p}(\Omega; \mathbb{R}^n)$.

If, moreover, $(R'_\varepsilon) \subset SO(n)$ *is another sequence for which the rescaled displacements, defined as in* (3.4), *satisfy* (3.5), *then*

$$|R_\varepsilon - R'_\varepsilon| \leqslant C\varepsilon \tag{3.6}$$

for every $\varepsilon \in (0, 1)$.

Proof. Let $\widehat{\mathcal{E}}_\varepsilon$ be the auxiliary energy defined as in (3.1) with π replaced by the function $\hat{\pi}$ given by lemma 3.4. Since $\pi \geqslant \hat{\pi}$ everywhere and $\pi \equiv \hat{\pi}$ on Ω, we have that

$$\widehat{\mathcal{E}}_\varepsilon(y) \leqslant \mathcal{E}_\varepsilon(y) \qquad \text{for every } y \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n). \tag{3.7}$$

For the sake of brevity we introduce the notation

$$\Omega_\varepsilon^- := \{x \in \Omega : |\det \nabla y_\varepsilon(x) - 1| \leqslant 1\}, \qquad \Omega_\varepsilon^+ := \Omega \setminus \Omega_\varepsilon^-. \tag{3.8}$$

Since $\widehat{\mathcal{E}}_\varepsilon(y_\varepsilon) \leqslant \mathcal{E}_\varepsilon(y_\varepsilon) \leqslant C\varepsilon^2$, we have that y_ε belongs to Y^p, hence in particular $\det \nabla y_\varepsilon > 0$ a.e. in Ω. By (W5) and theorem 3.2 we infer the existence of $R_\varepsilon \in SO(n)$ such that

$$\int_\Omega g_p(|\nabla y_\varepsilon(x) - R_\varepsilon|) \, \mathrm{d}x \leqslant C \int_\Omega W(x, \nabla y_\varepsilon(x)) \, \mathrm{d}x. \tag{3.9}$$

By (2.8) we deduce that

$$\int_\Omega W(x, \nabla y_\varepsilon) \, \mathrm{d}x = \widehat{\mathcal{E}}_\varepsilon(y_\varepsilon) + \varepsilon \int_\Omega (\hat{\pi}(x) - \hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon) \, \mathrm{d}x$$

$$\leqslant C\varepsilon^2 + \varepsilon \int_\Omega (\hat{\pi}(y_{R_\varepsilon}) - \hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon) \, \mathrm{d}x$$

$$\leqslant C\varepsilon^2 + \varepsilon \int_\Omega |\hat{\pi}(y_\varepsilon) - \hat{\pi}(y_{R_\varepsilon})| \, \mathrm{d}x + \varepsilon \int_\Omega \hat{\pi}(y_\varepsilon)(1 - \det \nabla y_\varepsilon) \, \mathrm{d}x.$$

Since $\hat{\pi} \geqslant 0$ by construction, the integrand in the last integral above is nonpositive on Ω_ε^+. Thus, using the fact that $\hat{\pi}$ is Lipschitz continuous and bounded, and

applying Hölder's inequality we deduce that

$$\int_\Omega W(x, \nabla y_\varepsilon)\, \mathrm{d}x \leqslant C\varepsilon^2 + C\varepsilon\|y_\varepsilon - y_{R_\varepsilon}\|_1 + C\varepsilon\|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)}. \tag{3.10}$$

By (2.6) this implies

$$\|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)}^2 \leqslant C\varepsilon^2 + C\varepsilon\|y_\varepsilon - y_{R_\varepsilon}\|_1 + C\varepsilon\|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)},$$

which, in turn, by Young's inequality yields

$$\|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)}^2 \leqslant C\varepsilon^2 + C\varepsilon\|y_\varepsilon - y_{R_\varepsilon}\|_1. \tag{3.11}$$

By combining (3.10) and (3.11) we deduce

$$\int_\Omega W(x, \nabla y_\varepsilon)\, \mathrm{d}x \leqslant C\varepsilon^2 + C\varepsilon\|y_\varepsilon - y_{R_\varepsilon}\|_1, \tag{3.12}$$

and, as a consequence of (3.4), (3.9) and (3.12), we obtain

$$\int_\Omega g_p(|\varepsilon\nabla u_\varepsilon|)\, \mathrm{d}x = \int_\Omega g_p(|\nabla y_\varepsilon - R_\varepsilon|)\, \mathrm{d}x \leqslant C\varepsilon^2 + C\varepsilon\|y_\varepsilon - y_{R_\varepsilon}\|_1 = C\varepsilon^2(1 + \|u_\varepsilon\|_1). \tag{3.13}$$

Using the definition (2.4) of g_p this implies that

$$\int_{\{|\varepsilon\nabla u_\varepsilon|\leqslant 1\}} |\varepsilon\nabla u_\varepsilon|^2\, \mathrm{d}x \leqslant 2\int_\Omega g_p(|\varepsilon\nabla u_\varepsilon|)\, \mathrm{d}x \leqslant C\varepsilon^2(1 + \|u_\varepsilon\|_1). \tag{3.14}$$

By Hölder's inequality we obtain

$$\int_{\{|\varepsilon\nabla u_\varepsilon|\leqslant 1\}} |\varepsilon\nabla u_\varepsilon|^p\, \mathrm{d}x \leqslant C\left(\int_{\{|\varepsilon\nabla u_\varepsilon|\leqslant 1\}} |\varepsilon\nabla u_\varepsilon|^2\, \mathrm{d}x\right)^{p/2}$$

$$\leqslant C\varepsilon^p(1 + \|u_\varepsilon\|_1)^{p/2} \leqslant C\varepsilon^p(1 + \|u_\varepsilon\|_1), \tag{3.15}$$

where the last inequality follows from the fact that $t^{p/2} \leqslant 1 + t$ for $t \geqslant 0$.

Again from (2.4) and recalling that $p \leqslant 2$ we also have that

$$\int_{\{|\varepsilon\nabla u_\varepsilon|>1\}} |\varepsilon\nabla u_\varepsilon|^p\, \mathrm{d}x \leqslant \int_\Omega g_p(|\varepsilon\nabla u_\varepsilon|)\, \mathrm{d}x \leqslant C\varepsilon^2(1 + \|u_\varepsilon\|_1) \leqslant C\varepsilon^p(1 + \|u_\varepsilon\|_1). \tag{3.16}$$

By (3.15), (3.16) and the continuous embedding of $W^{1,p}(\Omega; \mathbb{R}^n)$ into $L^1(\Omega; \mathbb{R}^n)$ we deduce that

$$\|\nabla u_\varepsilon\|_p^p \leqslant C + C\|u_\varepsilon\|_{W^{1,p}}.$$

Since u_ε has zero average, Poincaré–Wirtinger inequality finally yields

$$\|u_\varepsilon\|_{W^{1,p}}^p \leqslant C + C\|u_\varepsilon\|_{W^{1,p}},$$

which implies $\|u_\varepsilon\|_{W^{1,p}} \leqslant C$. This inequality, combined with (3.11) and (3.13), provides (3.3) and (3.5).

Finally, if (R'_ε) is a sequence of rotations whose corresponding rescaled displacements satisfy (3.5), then

$$|R_\varepsilon - R'_\varepsilon| \leqslant C(\|\nabla y_\varepsilon - R_\varepsilon\|_p + \|\nabla y_\varepsilon - R'_\varepsilon\|_p) \leqslant C\varepsilon.$$

This concludes the proof.   □

As an immediate corollary we obtain that the infimum of the energy $\mathcal{E}_\varepsilon$ is of order ε^2.

COROLLARY 3.7. *Assume (W1)–(W5), (π1) and (π2). Then*

$$-C\varepsilon^2 \leqslant \inf_{\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon \leqslant 0 \quad \text{for every } \varepsilon \in (0,1). \tag{3.17}$$

Proof. Let (y_ε) be a minimizing sequence satisfying

$$\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant \inf_{\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon + \varepsilon^2.$$

Using the fact that W is nonnegative and arguing as in the proof of lemma 3.6, we deduce that

$$\begin{aligned}
\mathcal{E}_\varepsilon(y_\varepsilon) &\geqslant \varepsilon \int_\Omega (\hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon - \hat{\pi}(y_{R_\varepsilon})) \, \mathrm{d}x \\
&\geqslant -C\varepsilon\|y_\varepsilon - y_{R_\varepsilon}\|_1 - C\varepsilon\|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)} \\
&\geqslant -C\varepsilon^2,
\end{aligned}$$

where the last inequality follows from lemma 3.6. This proves the first inequality in (3.17).

The other inequality in (3.17) follows trivially by the fact that the energy $\mathcal{E}_\varepsilon$ is zero on the identity map.   □

We now have all the main ingredients to prove compactness of deformations with equibounded rescaled energies.

PROPOSITION 3.8 (**Compactness**). *Assume (W1)–(W5), (π1) and (π2). If $\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant C\varepsilon^2$ for every $\varepsilon \in (0,1)$, then for any R_ε, u_ε given by lemma 3.6 we have that, up to subsequences,*

- *$u_\varepsilon \rightharpoonup u_0$ weakly in $\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$ with $u_0 \in \mathring{H}^1(\Omega;\mathbb{R}^n)$,*

- *$R_\varepsilon \to R_0$ with $R_0 \in \mathcal{R}$,*

as $\varepsilon \to 0$. Moreover, R_0 is independent of the choice of R_ε and u_0 is independent up to infinitesimal rigid motions of the form Ax, with $A \in \mathbb{R}^{n\times n}_{\text{skew}}$.

Proof. By lemma 3.6 the sequence (u_ε) is uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^n)$. Hence, up to subsequences, $u_\varepsilon \rightharpoonup u_0$ weakly in $\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$. We now show that

u_0 belongs to $\mathring{H}^1(\Omega; \mathbb{R}^n)$. We first introduce the set

$$G_\varepsilon := \left\{ x \in \Omega : \ \varepsilon^{1/2}|\nabla u_\varepsilon(x)| \leqslant 1 \right\}, \tag{3.18}$$

and we observe that by Tchebichev inequality

$$|\Omega \setminus G_\varepsilon| \leqslant C\varepsilon^{p/2}. \tag{3.19}$$

We claim that

  (i) $\chi_{G_\varepsilon} \nabla u_\varepsilon$ is bounded in $L^2(\Omega; \mathbb{R}^{n \times n})$;

  (ii) $\nabla u_0 \in L^2(\Omega; \mathbb{R}^{n \times n})$ and, up to subsequences, $\chi_{G_\varepsilon} \nabla u_\varepsilon \rightharpoonup \nabla u_0$ weakly in $L^2(\Omega; \mathbb{R}^{n \times n})$.

Assertion (i) easily follows from (3.5) arguing as in (3.14) and using that $G_\varepsilon \subset \{|\varepsilon \nabla u_\varepsilon| \leqslant 1\}$. To prove (ii) we first note that (i) ensures that $\chi_{G_\varepsilon} \nabla u_\varepsilon \rightharpoonup v$ weakly in $L^2(\Omega; \mathbb{R}^{n \times n})$, up to subsequences, for some $v \in L^2(\Omega; \mathbb{R}^{n \times n})$. On the other hand, by (3.19) we have that χ_{G_ε} converges to 1 boundedly in measure. Since $\nabla u_\varepsilon \rightharpoonup \nabla u_0$ in $L^p(\Omega; \mathbb{R}^{n \times n})$, we conclude that $\chi_{G_\varepsilon} \nabla u_\varepsilon \rightharpoonup \nabla u_0$ in $L^p(\Omega; \mathbb{R}^{n \times n})$. Hence, $v = \nabla u_0$ and (ii) is proved. By Sobolev embedding we have that $u_0 \in \mathring{H}^1(\Omega; \mathbb{R}^n)$.

Since $SO(n)$ is a compact set, there exists $R_0 \in SO(n)$ such that $R_\varepsilon \to R_0$, up to subsequences. To prove that $R_0 \in \mathcal{R}$ we argue as in the proof of lemma 3.6 and deduce

$$C \geqslant \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon)$$

$$\geqslant \frac{1}{\varepsilon^2} \widehat{\mathcal{E}}_\varepsilon(y_\varepsilon) \geqslant \frac{1}{\varepsilon} \int_\Omega (\hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon - \hat{\pi}(x)) \, \mathrm{d}x \geqslant -c + \frac{1}{\varepsilon} \int_\Omega (\pi(R_\varepsilon x) - \pi(x)) \, \mathrm{d}x.$$

Note that in the last integral we used that $\hat{\pi} \equiv \pi$ on $\mathcal{O}$.

Multiplying by ε and then letting $\varepsilon \to 0$ we infer that

$$\int_\Omega (\pi(R_0 x) - \pi(x)) \, \mathrm{d}x \leqslant 0.$$

This implies that $R_0 \in \mathcal{R}$ since by assumption the identity matrix is an optimal rotation.

Uniqueness of R_0 is a straightforward consequence of (3.6). Uniqueness (up to an infinitesimal rigid motion) of u_0 follows by arguing as in [22, theorem 5.1], recalling that displacements have zero average in our setting. □

The following proposition will be useful in both the liminf and the limsup inequalities to characterize the asymptotic behaviour of the rescaled pressure potential. Note that, besides the presence of $\hat{\pi}$ in place of π, the integral at the left-hand side of (3.20) differs from the rescaled pressure potential in the total energy whenever R_ε is not an optimal rotation.

PROPOSITION 3.9. *Let $\hat{\pi}$ be a function as in lemma 3.4 and let $y_\varepsilon \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ satisfy (3.3). Assume there exist $R_\varepsilon \in SO(n)$ converging to $R_0 \in \mathcal{R}$ such that the*

corresponding displacements u_ε, defined as in (3.4), weakly converge in $\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$ to $u_0 \in \mathring{H}^1(\Omega;\mathbb{R}^n)$. Then,

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} &\int_\Omega (\hat\pi(y_\varepsilon)\det\nabla y_\varepsilon - \hat\pi(y_{R_\varepsilon}))\, \mathrm{d}x \\
&\geqslant \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{|\det\nabla y_\varepsilon - 1|\leqslant 1\}} (\hat\pi(y_\varepsilon)\det\nabla y_\varepsilon - \hat\pi(y_{R_\varepsilon}))\, \mathrm{d}x \\
&= \int_{\partial\Omega} \pi(R_0 x) n_{\partial\Omega}(x)\cdot u_0(x)\, \mathrm{d}\mathcal{H}^{n-1}(x).
\end{aligned}
\tag{3.20}
$$

Proof. We write

$$
\begin{aligned}
\frac{1}{\varepsilon} &\int_\Omega (\hat\pi(y_\varepsilon)\det\nabla y_\varepsilon - \hat\pi(y_{R_\varepsilon}))\, \mathrm{d}x \\
&= \frac{1}{\varepsilon} \int_\Omega \hat\pi(y_\varepsilon)(\det\nabla y_\varepsilon - 1)\, \mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega (\hat\pi(y_\varepsilon) - \hat\pi(y_{R_\varepsilon}))\, \mathrm{d}x \\
&=: I_\varepsilon + II_\varepsilon.
\end{aligned}
\tag{3.21}
$$

We start by considering I_ε. Let Ω_ε^- and G_ε be defined as in (3.8) and (3.18). Since (u_ε) is bounded in $W^{1,p}(\Omega;\mathbb{R}^n)$, property (3.19) still holds. Moreover, by (2.1)

$$
\det\nabla y_\varepsilon(x) = \det(I + \varepsilon\nabla u_\varepsilon(x)) = 1 + \sum_{k=1}^n \varepsilon^k \iota_k(\nabla u_\varepsilon(x)) \quad \text{for a.e. } x\in\Omega.
$$

Since by (2.2) we have that for $k = 1,\, \ldots,\, n$

$$
|\varepsilon^k \iota_k(\nabla u_\varepsilon(x))| \leqslant C\varepsilon^k|\nabla u_\varepsilon(x)|^k \leqslant C\varepsilon^{k/2} \leqslant C\varepsilon^{1/2} \quad \text{for a.e. } x\in G_\varepsilon,
$$

we deduce that $G_\varepsilon \subset \Omega_\varepsilon^-$ for ε small enough. Therefore, using the nonnegativity of $\hat\pi$ and (2.1) again, we obtain

$$
\begin{aligned}
I_\varepsilon &\geqslant \frac{1}{\varepsilon} \int_{\Omega_\varepsilon^-} \hat\pi(y_\varepsilon)(\det\nabla y_\varepsilon - 1)\, \mathrm{d}x \\
&= \int_{G_\varepsilon} \hat\pi(y_\varepsilon)\mathrm{div}\, u_\varepsilon\, \mathrm{d}x \\
&\quad + \sum_{k=2}^n \varepsilon^{k-1} \int_{G_\varepsilon} \hat\pi(y_\varepsilon)\iota_k(\nabla u_\varepsilon)\, \mathrm{d}x + \frac{1}{\varepsilon} \int_{\Omega_\varepsilon^- \setminus G_\varepsilon} \hat\pi(y_\varepsilon)(\det\nabla y_\varepsilon - 1)\, \mathrm{d}x \\
&=: J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3.
\end{aligned}
$$

We first show that

$$
\lim_{\varepsilon\to 0} J_\varepsilon^1 = \int_\Omega \pi(R_0 x)\mathrm{div}\, u_0(x)\, \mathrm{d}x.
\tag{3.22}
$$

Indeed, since $\hat{\pi} = \pi$ on $\mathcal{O}$, we may write

$$\left| J_\varepsilon^1 - \int_\Omega \pi(R_0 x) \operatorname{div} u_0(x) \, \mathrm{d}x \right| \leqslant \int_{G_\varepsilon} |\hat{\pi}(R_\varepsilon x + \varepsilon R_\varepsilon u_\varepsilon(x)) - \hat{\pi}(R_0 x)| |\operatorname{div} u_\varepsilon| \, \mathrm{d}x$$
$$+ \left| \int_{G_\varepsilon} \pi(R_0 x) \operatorname{div} u_\varepsilon \, \mathrm{d}x - \int_\Omega \pi(R_0 x) \operatorname{div} u_0 \, \mathrm{d}x \right|.$$
$$(3.23)$$

Using the Lipschitz continuity of $\hat{\pi}$ and the definition of G_ε, the first integral at the right-hand side can be bounded as follows:

$$\int_{G_\varepsilon} |\hat{\pi}(R_\varepsilon x + \varepsilon R_\varepsilon u_\varepsilon(x)) - \hat{\pi}(R_0 x)| |\operatorname{div} u_\varepsilon| \, \mathrm{d}x$$
$$\leqslant C |R_\varepsilon - R_0| \|\nabla u_\varepsilon\|_p + C\varepsilon \int_{G_\varepsilon} |u_\varepsilon| |\nabla u_\varepsilon| \, \mathrm{d}x$$
$$\leqslant C |R_\varepsilon - R_0| + C\varepsilon^{1/2} \|u_\varepsilon\|_p$$
$$\leqslant C |R_\varepsilon - R_0| + C\varepsilon^{1/2},$$

where the last term goes to zero, as $\varepsilon \to 0$. Since χ_{G_ε} converges to 1 boundedly in measure, we have that $\chi_{G_\varepsilon} \operatorname{div} u_\varepsilon \rightharpoonup \operatorname{div} u_0$ weakly in $L^p(\Omega)$, hence the second term in (3.23) goes to zero, as well. This proves (3.22).

We now prove that both J_ε^2 and J_ε^3 converge to 0, as $\varepsilon \to 0$. By (2.2) and (3.4), since $\hat{\pi}$ is bounded, we obtain

$$|J_\varepsilon^2| \leqslant C \sum_{k=2}^n \varepsilon^{k-1} \int_{G_\varepsilon} |\nabla u_\varepsilon|^k \, \mathrm{d}x = C \sum_{k=2}^n \varepsilon^{k-1} \int_{G_\varepsilon} |\nabla u_\varepsilon|^{k-p} |\nabla u_\varepsilon|^p \, \mathrm{d}x.$$

Since $|\nabla u_\varepsilon| \leqslant \varepsilon^{-1/2}$ on G_ε, we have that

$$|J_\varepsilon^2| \leqslant C \sum_{k=2}^n \varepsilon^{(k+p-2)/2} \|\nabla u_\varepsilon\|_p^p \leqslant C\varepsilon^{p/2} \to 0.$$

To bound J_ε^3 we use (3.3) and deduce

$$|J_\varepsilon^3| \leqslant \frac{C}{\varepsilon} \| \det \nabla y_\varepsilon - 1 \|_{L^2(\Omega_\varepsilon^-)} |\Omega \setminus G_\varepsilon|^{1/2} \leqslant C |\Omega \setminus G_\varepsilon|^{1/2},$$

which vanishes by (3.19).

By combining the previous inequalities we conclude that

$$\liminf_{\varepsilon \to 0} I_\varepsilon \geqslant \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon^-} \hat{\pi}(y_\varepsilon)(\det \nabla y_\varepsilon - 1) \, \mathrm{d}x = \int_\Omega \pi(R_0 x) \operatorname{div} u_0(x) \, \mathrm{d}x. \quad (3.24)$$

We now claim that

$$\lim_{\varepsilon \to 0} II_\varepsilon = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon^-} (\hat{\pi}(y_\varepsilon) - \hat{\pi}(y_{R_\varepsilon})) \, \mathrm{d}x = \int_\Omega \nabla \pi(R_0 x) \cdot R_0 u_0(x) \, \mathrm{d}x. \quad (3.25)$$

Assuming this is true, the thesis follows by (3.21), (3.24), (3.25) and the divergence theorem, since

$$\pi(R_0 x)\operatorname{div} u_0(x) + \nabla\pi(R_0 x) \cdot R_0 u_0(x) = \operatorname{div}(\pi(R_0 x)u_0(x)).$$

To conclude we only need to prove (3.25). We can write the integrand in II_ε as

$$\frac{1}{\varepsilon}(\hat{\pi}(y_\varepsilon) - \hat{\pi}(y_{R_\varepsilon})) = \frac{1}{\varepsilon}(\hat{\pi}(R_\varepsilon x + \varepsilon R_\varepsilon u_\varepsilon(x)) - \hat{\pi}(R_\varepsilon x)) \qquad (3.26)$$

and owing to the Lipschitz continuity of $\hat{\pi}$ we have

$$\frac{1}{\varepsilon}|\hat{\pi}(y_\varepsilon) - \hat{\pi}(y_{R_\varepsilon})| \leqslant C|u_\varepsilon(x)| \qquad \text{for a.e. } x \in \Omega. \qquad (3.27)$$

Since (u_ε) is bounded in $L^p(\Omega; \mathbb{R}^n)$ and $|\Omega \setminus \Omega_\varepsilon^-| \leqslant |\Omega \setminus G_\varepsilon| \to 0$ by (3.19), we deduce that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega \setminus \Omega_\varepsilon^-} (\hat{\pi}(y_\varepsilon) - \hat{\pi}(y_{R_\varepsilon})) \, \mathrm{d}x = 0.$$

Hence, proving (3.25) is equivalent to show that

$$\lim_{\varepsilon \to 0} II_\varepsilon = \int_\Omega \nabla\pi(R_0 x) \cdot R_0 u_0(x) \, \mathrm{d}x. \qquad (3.28)$$

On the other hand, $u_\varepsilon \to u_0$ strongly in $L^1(\Omega; \mathbb{R}^n)$ by compact embedding. Thus, by (3.27) and the generalized dominated convergence theorem, (3.28) is proved if we show that the integrand (3.26) converges a.e. to $\nabla\pi(R_0 x) \cdot R_0 u_0(x)$.

To this aim, we first note that, up to subsequences,

$$\lim_{\varepsilon \to 0} \nabla\hat{\pi}(R_\varepsilon x) = \nabla\hat{\pi}(R_0 x) \quad \text{for a.e. } x \in \Omega.$$

Indeed, the convergence is actually in $L^1(\mathcal{O}; \mathbb{R}^n)$. This can be easily proved by approximating $\nabla\hat{\pi}$ with functions in $C^0(\overline{\mathcal{O}}; \mathbb{R}^n)$. Now, by Rademacher theorem (we point out that we are working with a countable sequence of rotations R_ε) for almost every $x \in \Omega$ we have

$$\frac{\hat{\pi}(R_\varepsilon x + \varepsilon R_\varepsilon u_\varepsilon(x)) - \hat{\pi}(R_\varepsilon x)}{\varepsilon} = \nabla\hat{\pi}(R_\varepsilon x) \cdot R_\varepsilon u_\varepsilon(x) + \frac{1}{\varepsilon}o(\varepsilon|u_\varepsilon(x)|).$$

Since $u_\varepsilon \to u_0$ a.e., up to subsequences, and $\hat{\pi} \equiv \pi$ on $\mathcal{O}$, we deduce the desired convergence. This concludes the proof. $\qquad \square$

With the result of proposition 3.9 at hand, we are now in a position to state and prove the liminf and the limsup inequalities for the energy functionals $(1/\varepsilon^2)\mathcal{E}_\varepsilon$.

PROPOSITION 3.10 (**Liminf inequality**). *Assume (W1)–(W5), (π1) and (π2). For every $\varepsilon \in (0, 1)$ let $y_\varepsilon \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ be such that there exist $R_\varepsilon \in SO(n)$ converging to $R_0 \in \mathcal{R}$ and the corresponding displacements u_ε, defined as in (3.4), weakly*

converge in $\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$ to $u_0 \in \mathring{H}^1(\Omega;\mathbb{R}^n)$. Then

$$\mathcal{E}_0(u_0, R_0) \leqslant \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon),$$

where $\mathcal{E}_0 \colon \mathring{H}^1(\Omega;\mathbb{R}^n) \times \mathcal{R} \to \mathbb{R}$ is defined by

$$\mathcal{E}_0(u_0, R_0) := \frac{1}{2} \int_\Omega Q(x, e(u_0)(x)) \,\mathrm{d}x + \int_{\partial\Omega} \pi(R_0 x) n_{\partial\Omega}(x) \cdot u_0(x) \,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{3.29}$$

The density $Q(x, \cdot)$ is the quadratic form given by

$$Q(x, F) = D_F^2 W(x, I) F : F \quad \text{for } F \in \mathbb{R}^{n \times n},$$

and $e(u_0)$ denotes the symmetric gradient of u_0.

REMARK 3.11. For any optimal rotation $R_0 \in \mathcal{R}$ the functional $\mathcal{E}_0(\cdot, R_0)$ is invariant under perturbations by infinitesimal rigid motions. Indeed, if $u_0'(x) = u_0(x) + Ax$ with $A \in \mathbb{R}_{\mathrm{skew}}^{n \times n}$, then clearly $e(u_0') = e(u_0)$ and by (2.10)

$$\int_{\partial\Omega} \pi(R_0 x) n_{\partial\Omega}(x) \cdot u_0'(x) \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\partial\Omega} \pi(R_0 x) n_{\partial\Omega}(x) \cdot u_0(x) \,\mathrm{d}\mathcal{H}^{n-1}(x).$$

Proof of proposition 3.10. Without loss of generality we can assume

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) < +\infty,$$

so that $y_\varepsilon \in Y^p$ and, up to subsequence, $\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant C\varepsilon^2$. By lemma 3.6 and proposition 3.8 there exist a (possibly different) sequence $(R_\varepsilon') \subset SO(n)$ such that, up to subsequences, $R_\varepsilon' \to R_0$, the corresponding displacements u_ε' satisfy (3.5) and, up to subsequences, weakly converge to $u_0 + Ax$ for some $A \in \mathbb{R}_{\mathrm{skew}}^{n \times n}$. However, by remark 3.11 we can assume, without loss of generality, that $R_\varepsilon = R_\varepsilon'$ and so, $u_\varepsilon = u_\varepsilon'$ and $A = 0$.

Let $\widehat{\mathcal{E}}_\varepsilon$ be the auxiliary energy defined as in (3.1) with π replaced by the function $\hat{\pi}$ given by lemma 3.4. By the properties of $\hat{\pi}$ we have

$$\frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) \geqslant \frac{1}{\varepsilon^2} \widehat{\mathcal{E}}_\varepsilon(y_\varepsilon) = \frac{1}{\varepsilon^2} \int_\Omega W(x, \nabla y_\varepsilon) \,\mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega (\hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon - \hat{\pi}(y_{R_\varepsilon})) \,\mathrm{d}x$$

$$+ \frac{1}{\varepsilon} \int_\Omega (\hat{\pi}(y_{R_\varepsilon}) - \hat{\pi}(x)) \,\mathrm{d}x. \tag{3.30}$$

Arguing as in [3, proof of theorem 2.4] one can prove that

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\Omega W(x, \nabla y_\varepsilon) \,\mathrm{d}x \geqslant \frac{1}{2} \int_\Omega Q(x, e(u_0)(x)) \,\mathrm{d}x.$$

Since condition (3.3) is satisfied by lemma 3.6, we can apply proposition 3.9 and we obtain

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega (\hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon - \hat{\pi}(y_{R_\varepsilon})) \,\mathrm{d}x \geqslant \int_{\partial\Omega} \pi(R_0 x) n_{\partial\Omega}(x) \cdot u_0(x) \,\mathrm{d}\mathcal{H}^{n-1}(x).$$

Finally, assumption (2.8) guarantees that the last term in (3.30) is nonnegative. This proves the desired inequality. □

PROPOSITION 3.12 (**Limsup inequality**). *Assume (W1)–(W5), (π1) and (π2).* *For every $(u_0, R_0) \in \mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R}$ there exist $(u_\varepsilon, R_\varepsilon) \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n) \times SO(n)$ such that $u_\varepsilon \rightharpoonup u_0$ weakly in $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$, $R_\varepsilon \to R_0$ and, setting $y_\varepsilon(x) := R_\varepsilon(x + \varepsilon u_\varepsilon(x))$, there holds*

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) \leqslant \mathcal{E}_0(u_0, R_0),$$

where $\mathcal{E}_0$ is the functional defined in (3.29).

Proof. Let $(u_0, R_0) \in \mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R}$. By mollification there exists $(u_\varepsilon) \subset \mathring{W}^{1,\infty}(\Omega; \mathbb{R}^n)$ such that

$$u_\varepsilon \to u_0 \quad \text{strongly in } H^1(\Omega; \mathbb{R}^n) \quad \text{and} \quad \varepsilon^{1/2} \|u_\varepsilon\|_{W^{1,\infty}} \leqslant 1. \qquad (3.31)$$

We define $R_\varepsilon := R_0$, so that $y_\varepsilon(x) = R_0(x + \varepsilon u_\varepsilon(x))$.

We first observe that $y_\varepsilon \in Y^p$ for ε small enough. Indeed, by (2.3) it has zero-average and by (2.1) it satisfies

$$\det \nabla y_\varepsilon(x) = \det(I + \varepsilon \nabla u_\varepsilon(x)) = 1 + \sum_{k=1}^n \varepsilon^k \iota_k(\nabla u_\varepsilon(x)) \quad \text{for a.e. } x \in \Omega. \quad (3.32)$$

Since by (2.2) we have that for $k = 1, \ldots, n$

$$|\varepsilon^k \iota_k(\nabla u_\varepsilon(x))| \leqslant C\varepsilon^k |\nabla u_\varepsilon(x)|^k \leqslant C\varepsilon^{k/2} \leqslant C\varepsilon^{1/2} \qquad \text{for a.e. } x \in \Omega,$$

for ε small enough we obtain

$$|\det \nabla y_\varepsilon(x) - 1| \leqslant C\varepsilon^{1/2} \leqslant \frac{1}{2} \text{ for a.e. } x \in \Omega, \qquad (3.33)$$

and thus, $\det \nabla y_\varepsilon > 0$ a.e. in Ω.

By (3.31) we have that for ε small enough the set $y_\varepsilon(\Omega)$ is contained in the neighbourhood of $\mathcal{O}$ where π and $\hat{\pi}$ coincide. Therefore, using also that $R_0 \in \mathcal{R}$, we can write

$$\frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) = \frac{1}{\varepsilon^2} \int_\Omega W(x, I + \varepsilon \nabla u_\varepsilon(x)) \, \mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega (\hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon - \hat{\pi}(y_{R_0})) \, \mathrm{d}x.$$

Arguing as in [**3**, proof of theorem 2.4], one can show that

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\Omega W(x, I + \varepsilon \nabla u_\varepsilon(x)) \, \mathrm{d}x \leqslant \frac{1}{2} \int_\Omega Q(x, e(u_0)(x)) \, \mathrm{d}x.$$

On the other hand, by (3.32) we have that

$$|\det \nabla y_\varepsilon(x) - 1| \leqslant \varepsilon |\nabla u_\varepsilon(x)| + C\varepsilon \quad \text{for a.e. } x \in \Omega,$$

hence condition (3.3) is satisfied and we can apply proposition 3.9. By (3.33) we deduce

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega (\hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon - \hat{\pi}(y_{R_0})) \, \mathrm{d}x = \int_{\partial \Omega} \pi(R_0 x) n_{\partial \Omega}(x) \cdot u_0(x) \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

This concludes the proof. $\qquad \square$

REMARK 3.13. If we include a.e. injectivity in the definition of the space Y^p of admissible deformations (see remark 3.1), the limsup inequality can be proved by means of the same recovery sequence. Indeed, by [4, theorem 5.5-1(b)] the deformations y_ε are a.e. injective owing to (3.31).

Combining together the previous propositions, we can prove the main result of this section. It ensures that almost minimizers of the nonlinear energy strongly converge to minimizers of the limiting energy.

THEOREM 3.14 (**Convergence of almost minimizers**). *Assume (W1)–(W5), (π1) and (π2). If (y_ε) is a sequence of almost minimizers for the energies $\mathcal{E}_\varepsilon$, that is,*

$$\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant \inf_{\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon + o(\varepsilon^2), \tag{3.34}$$

then there exist $R_\varepsilon \in SO(n)$ such that, up to passing to a subsequence, we have

- $u_\varepsilon \to u_0$ *strongly in $\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)$ with $u_0 \in \mathring{H}^1(\Omega;\mathbb{R}^n)$,*

- $R_\varepsilon \to R_0$ *with $R_0 \in \mathcal{R}$,*

as $\varepsilon \to 0$. Furthermore, the pair (u_0, R_0) is a minimizer of $\mathcal{E}_0$ on $\mathring{H}^1(\Omega;\mathbb{R}^n) \times \mathcal{R}$ and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left(\inf_{\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon \right) = \min \left\{ \mathcal{E}_0(u, R) : (u, R) \in \mathring{H}^1(\Omega;\mathbb{R}^n) \times \mathcal{R} \right\}. \tag{3.35}$$

Proof. Let (y_ε) be a sequence of almost minimizers. By corollary 3.7 we have that

$$\inf_{\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)} \mathcal{E}_\varepsilon \leqslant 0,$$

hence by proposition 3.8 there exist $u_0 \in \mathring{H}^1(\Omega;\mathbb{R}^n)$ and $R_0 \in \mathcal{R}$ such that, up to a subsequence,

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } \mathring{W}^{1,p}(\Omega;\mathbb{R}^n) \qquad \text{and} \qquad R_\varepsilon \to R_0.$$

We now show that (u_0, R_0) is a minimizer of $\mathcal{E}_0$. To this aim let $(v, S) \in \mathring{H}^1(\Omega;\mathbb{R}^n) \times \mathcal{R}$ and let $(v_\varepsilon, S_\varepsilon)$ be a recovery sequence for (v, S), as in proposition 3.12. Let $z_\varepsilon(x) := S_\varepsilon(x + \varepsilon v_\varepsilon(x))$. By proposition 3.10 we have

$$\mathcal{E}_0(u_0, R_0) \leqslant \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) \leqslant \liminf_{\varepsilon \to 0} \left(\inf_{\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon \right)$$

$$\leqslant \limsup_{\varepsilon \to 0} \left(\inf_{\mathring{W}^{1,p}(\Omega;\mathbb{R}^n)} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon \right)$$

$$\leqslant \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(z_\varepsilon) \leqslant \mathcal{E}_0(v, S). \tag{3.36}$$

This implies that $\mathcal{E}_0$ is minimized at (u_0, R_0) and, as a consequence, (3.35) holds.

To conclude, it remains to prove that $u_\varepsilon \to u_0$ strongly in $W^{1,p}(\Omega; \mathbb{R}^n)$. We adapt the argument in [**3**, theorem 2.5] to our framework. We claim that the following properties hold:

(a) $\chi_{G_\varepsilon} e(u_\varepsilon) \to e(u_0)$ strongly in $L^2(\Omega; \mathbb{R}^{n \times n}_{\mathrm{sym}})$, where the set G_ε is defined as in (3.18);

(b) the sequence $((1/\varepsilon^p) \, \mathrm{dist}^p(\nabla y_\varepsilon; SO(n)))$ is equi-integrable;

(c) the sequence $(|\nabla u_\varepsilon|^p)$ is equi-integrable.

The thesis follows from (a) and (b), by using Vitali's convergence theorem together with Korn's second inequality, see [**3**, proof of theorem 2.5] for more details.

We now prove (a). By choosing $(v, S) = (u_0, R_0)$ in (3.36) we deduce

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) = \mathcal{E}_0(u_0, R_0).$$

By (3.30) and assumption (2.8) we have

$$\frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) \geqslant \frac{1}{\varepsilon^2} \int_\Omega W(x, \nabla y_\varepsilon) \, \mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega (\hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon - \hat{\pi}(y_{R_\varepsilon})) \, \mathrm{d}x.$$

Therefore, letting $\varepsilon \to 0$ and applying proposition 3.9 yield

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\Omega W(x, \nabla y_\varepsilon) \, \mathrm{d}x \leqslant \mathcal{E}_0(u_0, R_0) - \int_{\partial \Omega} \pi(R_0 x) n_{\partial \Omega}(x) \cdot u_0(x) \, \mathrm{d}\mathcal{H}^{n-1}(x)$$

$$= \frac{1}{2} \int_\Omega Q(x, e(u_0)) \, \mathrm{d}x.$$

On the other hand, by Taylor expansion of W around I and by the weak convergence of $\chi_{G_\varepsilon} e(u_\varepsilon)$ to $e(u_0)$ in $L^2(\Omega; \mathbb{R}^{n \times n}_{\mathrm{sym}})$ (see property (ii) in the proof of proposition 3.8) we obtain

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\Omega W(x, \nabla y_\varepsilon) \, \mathrm{d}x \geqslant \limsup_{\varepsilon \to 0} \frac{1}{2} \int_\Omega Q(x, \chi_{G_\varepsilon} e(u_\varepsilon)) \, \mathrm{d}x$$

$$\geqslant \liminf_{\varepsilon \to 0} \frac{1}{2} \int_\Omega Q(x, \chi_{G_\varepsilon} e(u_\varepsilon)) \, \mathrm{d}x$$

$$\geqslant \frac{1}{2} \int_\Omega Q(x, e(u_0)) \, \mathrm{d}x,$$

see, e.g. [**3**, **8**, **22**]. Combining the previous inequalities yields

$$\lim_{\varepsilon \to 0} \frac{1}{2} \int_\Omega Q(x, \chi_{G_\varepsilon} e(u_\varepsilon)) \, \mathrm{d}x = \frac{1}{2} \int_\Omega Q(x, e(u_0)) \, \mathrm{d}x. \tag{3.37}$$

Since $\chi_{G_\varepsilon} e(u_\varepsilon) \rightharpoonup e(u_0)$ weakly in $L^2(\Omega; \mathbb{R}^{n \times n}_{\mathrm{sym}})$ and the quadratic form $Q(x, \cdot)$ is coercive on $\mathbb{R}^{n \times n}_{\mathrm{sym}}$ by (W2), (W4) and (W5), equation (3.37) proves claim (a).

To show claim (b) one can repeat verbatim the proof in [**3**, theorem 2.5].

We now prove claim (c). Given $\alpha > p$ and $\eta > 0$, we deduce by (b) that there exists $M_\eta > 0$ such that, setting

$$f_1^{\varepsilon,\eta}(x) := \text{dist}(\nabla y_\varepsilon(x); SO(n))\chi_{E^{\varepsilon,\eta}}(x), \quad f_2^{\varepsilon,\eta}(x) := \text{dist}(\nabla y_\varepsilon(x); SO(n))\chi_{\Omega\setminus E^{\varepsilon,\eta}}(x),$$

where

$$E^{\varepsilon,\eta} := \left\{ x \in \Omega : \frac{1}{\varepsilon^p} \text{dist}^p(\nabla y_\varepsilon(x); SO(n)) \geqslant M_\eta \right\},$$

we have that

$$\left\| \frac{f_1^{\varepsilon,\eta}}{\varepsilon} \right\|_p^p \leqslant \eta \text{ and } \left\| \frac{f_2^{\varepsilon,\eta}}{\varepsilon} \right\|_\alpha^\alpha \leqslant |\Omega| M_\eta^{\alpha/p}. \tag{3.38}$$

Theorem 3.3 now ensures the existence of $\widetilde{R}_{\varepsilon,\eta} \in SO(n)$ and of $g_1^{\varepsilon,\eta}$, $g_2^{\varepsilon,\eta}$ such that

$$\nabla y_\varepsilon = \widetilde{R}_{\varepsilon,\eta} + g_1^{\varepsilon,\eta} + g_2^{\varepsilon,\eta} \text{ and } \|g_1^{\varepsilon,\eta}\|_p \leqslant C\|f_1^{\varepsilon,\eta}\|_p, \quad \|g_2^{\varepsilon,\eta}\|_\alpha \leqslant C\|f_2^{\varepsilon,\eta}\|_\alpha. \tag{3.39}$$

Since $\nabla y_\varepsilon = R_\varepsilon + \varepsilon R_\varepsilon \nabla u_\varepsilon$, we deduce that

$$\frac{\widetilde{R}_{\varepsilon,\eta} - R_\varepsilon}{\varepsilon} = R_\varepsilon \nabla u_\varepsilon - \frac{g_1^{\varepsilon,\eta}}{\varepsilon} - \frac{g_2^{\varepsilon,\eta}}{\varepsilon}, \tag{3.40}$$

hence

$$\frac{|\widetilde{R}_{\varepsilon,\eta} - R_\varepsilon|^p}{\varepsilon^p} \leqslant C\left(\|\nabla u_\varepsilon\|_p^p + \left\| \frac{g_1^{\varepsilon,\eta}}{\varepsilon} \right\|_p^p + \left\| \frac{g_2^{\varepsilon,\eta}}{\varepsilon} \right\|_p^p \right) \leqslant C(1 + \eta + M_\eta), \tag{3.41}$$

where the last inequality follows from Hölder's inequality, (3.38), and (3.39).

On the other hand, by (3.40) we can write

$$\nabla u_\varepsilon = R_\varepsilon^T \left(\frac{\widetilde{R}_{\varepsilon,\eta} - R_\varepsilon}{\varepsilon} + \frac{g_1^{\varepsilon,\eta}}{\varepsilon} + \frac{g_2^{\varepsilon,\eta}}{\varepsilon} \right).$$

Thus, by (3.38), (3.39) and (3.41) we have that for every measurable set $A \subset \Omega$

$$\int_A |\nabla u_\varepsilon|^p \, dx \leqslant C\left(\frac{|\widetilde{R}_{\varepsilon,\eta} - R_\varepsilon|^p}{\varepsilon^p}|A| + \int_A \left| \frac{g_1^{\varepsilon,\eta}}{\varepsilon} \right|^p dx + \int_A \left| \frac{g_2^{\varepsilon,\eta}}{\varepsilon} \right|^p dx \right)$$

$$\leqslant C\left((1 + \eta + M_\eta)|A| + \eta + \left\| \frac{f_2^{\varepsilon,\eta}}{\varepsilon} \right\|_\alpha^p |A|^{1-p/\alpha} \right)$$

$$\leqslant C\left((1 + \eta + M_\eta)|A| + \eta + M_\eta|A|^{1-p/\alpha} \right).$$

Now, for every $\delta > 0$ we can choose first $\eta = \eta(\delta)$ and then $\omega = \omega(\delta, \eta)$ in such a way that the right-hand side above is less than δ for every measurable set $A \subset \Omega$ with $|A| < \omega$. This proves claim (c) and concludes the proof of the theorem. $\quad\square$

4. Pressure loads of arbitrary sign

Here we extend the results of the previous section to pressure loads whose intensity π is not necessarily nonnegative (and still satisfies (π1)). To deal with the negative part of π we need to assume an additional bound from below for $W(\cdot, F)$ in terms of $\det F$:

(W6) $W(x, F) \geqslant c_2 g_q(|\det F - 1|)$ for a.e. $x \in \Omega$ and for every $F \in \mathbb{R}^{n \times n}$, for some $q \in [1, 2]$,

where g_q is defined as in (2.4) and $c_2 > 0$ is a constant independent of x. According to the value of q in (W6), we assume π to satisfy the following condition:

(π3) if $q = 1$, π^- is bounded; if $q \in (1, 2]$, $\pi^-(y) \leqslant C(1 + |y|^{p/q'})$ for every $y \in \mathbb{R}^n$.

We note that the growth condition in (π3) is at most linear, since $p, q \in (1, 2]$ implies $p/q' \in (0, 1]$.

In the current framework the energy $\mathcal{E}_\varepsilon$ is defined as in (3.1) with the set of admissible deformations Y^p replaced by

$$Y_q^p := \{y \in Y^p : \det \nabla y \in L^q(\Omega)\}. \tag{4.1}$$

Owing to (π3) the energy is well defined on Y_q^p: indeed, if $y \in Y_q^p$, then the composition $\pi^- \circ y$ belongs to $L^{q'}(\Omega)$ and thus, $\pi(y) \det \nabla y$ is integrable. Clearly, all rigid motions y_R with $R \in SO(n)$ are still admissible deformations. As observed in remark 3.1, also in this setting the a.e. injectivity condition can be included in the definition of Y_q^p without altering the results of this section.

As in the previous section we need a Lipschitz continuous function that extends π outside a neighbourhood of $\overline{\mathcal{O}}$, is below π everywhere, and satisfies the same growth condition (π3) as π.

LEMMA 4.1. *Assume conditions (π1) and (π3). Then, there exists a Lipschitz continuous function $\hat{\pi} \colon \mathbb{R}^n \to \mathbb{R}$ such that $\hat{\pi}$ coincides with π in a neighbourhood of $\overline{\mathcal{O}}$, $\hat{\pi}(y) \leqslant \pi(y)$ for all $y \in \mathbb{R}^n$, and $\hat{\pi}$ has the following property: if $q = 1$, $\hat{\pi}$ is bounded; if $q \in (1, 2]$, $|\hat{\pi}(y)| \leqslant C(1 + |y|^{p/q'})$ for every $y \in \mathbb{R}^n$.*

Proof. We consider only the case $q \in (1, 2]$, being the case $q = 1$ analogous and even simpler. Let $\overline{C} > 0$ be a constant for which (π3) is satisfied and let

$$h(y) := \left(\overline{C}(1 + |y|^{p/q'})\right) \vee (1 + \overline{C}).$$

Since $p/q' \in (0, 1]$, the function h is Lipschitz continuous in the whole of $\mathbb{R}^n$ and $\pi \geqslant -h$ by (π3). Hence we can apply lemma 3.4 to the function $\Pi := \pi + h$. This provides us with a function $\widehat{\Pi}$. It is now easy to check that the function $\hat{\pi} := \widehat{\Pi} - h$ has all the required properties. $\qquad \square$

Under this new set of assumptions, the results of § 3 can be modified as follows.

LEMMA 4.2. *Assume (W1)–(W6), (π1) and (π3). Then there exists $\varepsilon_0 \in (0, 1)$ such that, if $\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant C\varepsilon^2$ for every $\varepsilon \in (0, \varepsilon_0)$, then there holds*

$$\int_\Omega g_q(|\det \nabla y_\varepsilon(x) - 1|) \, \mathrm{d}x \leqslant C\varepsilon^2 \quad \text{for every } \varepsilon \in (0, \varepsilon_0). \tag{4.2}$$

Furthermore, there exist constant rotations $R_\varepsilon \in SO(n)$ such that the rescaled displacements u_ε defined by (3.4) satisfy

$$\int_\Omega g_p(\varepsilon|\nabla u_\varepsilon(x)|) \, \mathrm{d}x \leqslant C\varepsilon^2 \quad \text{for every } \varepsilon \in (0, \varepsilon_0) \tag{4.3}$$

and are uniformly bounded in $W^{1,p}(\Omega; \mathbb{R}^n)$.

If, moreover, $(R'_\varepsilon) \subset SO(n)$ is another sequence for which the rescaled displacements, defined as in (3.4), satisfy (4.3), then

$$|R_\varepsilon - R'_\varepsilon| \leqslant C\varepsilon$$

for every $\varepsilon \in (0, \varepsilon_0)$. Finally, one has

$$-C\varepsilon^2 \leqslant \inf_{\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)} \mathcal{E}_\varepsilon \leqslant 0 \quad \text{for every } \varepsilon \in (0, \varepsilon_0).$$

Proof. The choice of ε_0 will be made throughout the proof. We follow the lines of the proof of lemma 3.6. By lemma 4.1 inequality (3.7) still holds. By (W5) and theorem 3.2 there exists a sequence $(R_\varepsilon) \subset SO(n)$ such that

$$\int_\Omega g_p(|\nabla y_\varepsilon(x) - R_\varepsilon|) \, \mathrm{d}x \leqslant C \int_\Omega W(x, \nabla y_\varepsilon(x)) \, \mathrm{d}x,$$

and

$$\int_\Omega W(x, \nabla y_\varepsilon) \, \mathrm{d}x \leqslant C\varepsilon^2 + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_1 + \varepsilon \int_\Omega |\hat{\pi}(y_\varepsilon)| |\det \nabla y_\varepsilon - 1| \, \mathrm{d}x. \tag{4.4}$$

We denote by P_ε the last term in the above inequality. In the following, c_2 is the constant in condition (W6). If $q = 1$, the function $\hat{\pi}$ is bounded and thus, recalling the definition (3.8) of the sets $\Omega_\varepsilon^\pm$, we have

$$P_\varepsilon \leqslant C\varepsilon \|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)} + C\varepsilon \|\det \nabla y_\varepsilon - 1\|_{L^1(\Omega_\varepsilon^+)}$$

$$\leqslant C\varepsilon^2 + \frac{c_2}{4} \|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)}^2 + C\varepsilon \|\det \nabla y_\varepsilon - 1\|_{L^1(\Omega_\varepsilon^+)}$$

$$\leqslant C\varepsilon^2 + \left(\frac{c_2}{2} + C\varepsilon\right) \int_\Omega g_1(|\det \nabla y_\varepsilon - 1|) \, \mathrm{d}x,$$

where we used Cauchy's inequality and (2.5). If instead $q \in (1, 2]$, the function $\hat{\pi}$ is Lipschitz continuous and satisfies a p/q'-growth condition with $p/q' \leqslant 1$, so that

we obtain

$$P_\varepsilon \leqslant C\varepsilon \int_{\Omega_\varepsilon^-} (1 + |y_\varepsilon - y_{R_\varepsilon}|)|\det \nabla y_\varepsilon - 1|\, \mathrm{d}x$$

$$+ C\varepsilon \int_{\Omega_\varepsilon^+} (1 + |y_\varepsilon - y_{R_\varepsilon}|^{p/q'})|\det \nabla y_\varepsilon - 1|\, \mathrm{d}x.$$

Using that $|\det \nabla y_\varepsilon - 1| \leqslant 1$ on Ω_ε^- and applying Hölder's inequality, from the previous equation we deduce

$$P_\varepsilon \leqslant C\varepsilon \|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)} + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}} + C\varepsilon \|\det \nabla y_\varepsilon - 1\|_{L^q(\Omega_\varepsilon^+)}$$

$$+ C\varepsilon \int_{\Omega_\varepsilon^+} |y_\varepsilon - y_{R_\varepsilon}|^p \, \mathrm{d}x + C\varepsilon \int_{\Omega_\varepsilon^+} |\det \nabla y_\varepsilon - 1|^q \, \mathrm{d}x$$

$$\leqslant C\varepsilon^2 + \frac{c_2}{4} \|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)}^2 + C\varepsilon^{q'} + \left(\frac{c_2}{4} + C\varepsilon\right) \|\det \nabla y_\varepsilon - 1\|_{L^q(\Omega_\varepsilon^+)}^q$$

$$+ C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}} + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}}^p$$

$$\leqslant C\varepsilon^2 + \left(\frac{c_2}{2} + C\varepsilon\right) \int_\Omega g_q(|\det \nabla y_\varepsilon - 1|)\, \mathrm{d}x$$

$$+ C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}} + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}}^p,$$

where we used Young's inequality, (2.5) and the fact that $q' \geqslant 2$. Combining (4.4) with the previous bounds on P_ε, we obtain that

$$\int_\Omega W(x, \nabla y_\varepsilon)\, \mathrm{d}x \leqslant C\varepsilon^2 + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}} + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}}^p$$

$$+ \left(\frac{c_2}{2} + C\varepsilon\right) \int_\Omega g_q(|\det \nabla y_\varepsilon - 1|)\, \mathrm{d}x \tag{4.5}$$

in both cases $q = 1$ and $q \in (1, 2]$.

Now, if $\varepsilon_0 \leqslant c_2/(4\overline{C})$, where $\overline{C}$ is a constant for which (4.5) is true, then by (W6) we deduce that

$$\int_\Omega g_q(|\det \nabla y_\varepsilon - 1|)\, \mathrm{d}x \leqslant C\varepsilon^2 + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}} + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}}^p \tag{4.6}$$

for every $\varepsilon \in (0, \varepsilon_0)$. Combining (4.6) and (4.5) yields

$$\int_\Omega W(x, \nabla y_\varepsilon)\, \mathrm{d}x \leqslant C\varepsilon^2 + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}} + C\varepsilon \|y_\varepsilon - y_{R_\varepsilon}\|_{W^{1,p}}^p$$

$$= C\varepsilon^2 (1 + \|u_\varepsilon\|_{W^{1,p}} + \varepsilon^{p-1} \|u_\varepsilon\|_{W^{1,p}}^p). \tag{4.7}$$

Arguing as in (3.14)–(3.16) and using Poincaré–Wirtinger inequality we deduce that

$$\|u_\varepsilon\|_{W^{1,p}}^p \leqslant C(1 + \|u_\varepsilon\|_{W^{1,p}} + \varepsilon^{p-1} \|u_\varepsilon\|_{W^{1,p}}^p).$$

Up to choosing ε_0 smaller, if needed, we obtain

$$\|u_\varepsilon\|_{W^{1,p}}^p \leqslant C(1 + \|u_\varepsilon\|_{W^{1,p}}),$$

which implies $\|u_\varepsilon\|_{W^{1,p}} \leqslant C$.

Inequality (4.2) now follows easily from (4.6), while (4.3) is a consequence of (4.7) and (W5). The last two statements of the lemma can be proved arguing exactly as in the proofs of lemma 3.6 and corollary 3.7. □

The proof of the following compactness result is completely analogous to that of proposition 3.8.

PROPOSITION 4.3 (**Compactness**). *Assume (W1)–(W6), (π1) and (π3). If $\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant C\varepsilon^2$ for $\varepsilon \in (0, \varepsilon_0)$, then for any R_ε, u_ε given by lemma 4.2, we have that, up to subsequences,*

- *$u_\varepsilon \rightharpoonup u_0$ weakly in $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ with $u_0 \in \mathring{H}^1(\Omega; \mathbb{R}^n)$,*

- *$R_\varepsilon \to R_0$ with $R_0 \in \mathcal{R}$,*

as $\varepsilon \to 0$. Moreover, R_0 is independent of the choice of R_ε and u_0 is independent up to infinitesimal rigid motions of the form Ax, with $A \in \mathbb{R}^{n \times n}_{\text{skew}}$.

The next proposition is the analogue of proposition 3.9. However, in the present setting, owing to the assumptions (W6) and (π3), we can improve the result and show convergence on the whole of Ω.

PROPOSITION 4.4. *Let $\hat{\pi}$ be a function as in lemma 4.1 and let $y_\varepsilon \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ satisfy (4.2). Assume there exist $R_\varepsilon \in SO(n)$ converging to $R_0 \in \mathcal{R}$ such that the corresponding displacements u_ε, defined as in (3.4), weakly converge in $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ to $u_0 \in \mathring{H}^1(\Omega; \mathbb{R}^n)$. Then,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega (\hat{\pi}(y_\varepsilon) \det \nabla y_\varepsilon - \hat{\pi}(y_{R_\varepsilon})) \, \mathrm{d}x = \int_{\partial\Omega} \pi(R_0 x) n_{\partial\Omega}(x) \cdot u_0(x) \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

Proof. We follow the lines of the proof of proposition 3.9. The only difference is in the analysis of the term I_ε in (3.21), which now can be written as

$$I_\varepsilon = \int_{G_\varepsilon} \hat{\pi}(y_\varepsilon) \mathrm{div}\, u_\varepsilon \, \mathrm{d}x + \sum_{k=2}^n \varepsilon^{k-1} \int_{G_\varepsilon} \hat{\pi}(y_\varepsilon) \iota_k(\nabla u_\varepsilon) \, \mathrm{d}x$$

$$+ \frac{1}{\varepsilon} \int_{\Omega_\varepsilon^- \setminus G_\varepsilon} \hat{\pi}(y_\varepsilon)(\det \nabla y_\varepsilon - 1) \, \mathrm{d}x + \frac{1}{\varepsilon} \int_{\Omega_\varepsilon^+} \hat{\pi}(y_\varepsilon)(\det \nabla y_\varepsilon - 1) \, \mathrm{d}x$$

$$=: \tilde{J}_\varepsilon^1 + \tilde{J}_\varepsilon^2 + \tilde{J}_\varepsilon^3 + \tilde{J}_\varepsilon^4,$$

where the sets $\Omega_\varepsilon^\pm$ and G_ε are defined as in (3.8) and (3.18). Here we recall that $G_\varepsilon \subset \Omega_\varepsilon^-$ for ε small enough. The first integral $\tilde{J}_\varepsilon^1$ can be handled exactly as in the proof of proposition 3.9. To conclude it is enough to show that the remaining terms are infinitesimal, as $\varepsilon \to 0$.

By (2.2) and the Lipschitz continuity of $\hat{\pi}$ we obtain

$$|\tilde{J}_\varepsilon^2| \leqslant C \sum_{k=2}^n \varepsilon^{k-1} \int_{G_\varepsilon} (1 + \varepsilon|u_\varepsilon|)|\nabla u_\varepsilon|^k \, dx$$

$$= C \sum_{k=2}^n \left(\varepsilon^{k-1} \int_{G_\varepsilon} |\nabla u_\varepsilon|^{k-p} |\nabla u_\varepsilon|^p \, dx + \varepsilon^k \int_{G_\varepsilon} |u_\varepsilon||\nabla u_\varepsilon|^k \, dx \right).$$

Since $|\nabla u_\varepsilon| \leqslant \varepsilon^{-1/2}$ on G_ε, we have that

$$|\tilde{J}_\varepsilon^2| \leqslant C \sum_{k=2}^n \left(\varepsilon^{(k+p-2)/2} \|\nabla u_\varepsilon\|_p^p + \varepsilon^{k/2} \|u_\varepsilon\|_p \right)$$

$$\leqslant C \sum_{k=2}^n \left(\varepsilon^{(k+p-2)/2} + \varepsilon^{k/2} \right) \leqslant C\varepsilon^{p/2} \to 0.$$

Using (4.2), the boundedness of (u_ε) in $W^{1,p}(\Omega; \mathbb{R}^n)$, and recalling that $|\det \nabla y_\varepsilon - 1| \leqslant 1$ on Ω_ε^-, we deduce that

$$|\tilde{J}_\varepsilon^3| \leqslant \frac{C}{\varepsilon} \int_{\Omega_\varepsilon^- \backslash G_\varepsilon} (1 + \varepsilon|u_\varepsilon|)|\det \nabla y_\varepsilon - 1| \, dx$$

$$\leqslant \frac{C}{\varepsilon} \|\det \nabla y_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^-)} |\Omega \backslash G_\varepsilon|^{1/2} + C \int_{\Omega_\varepsilon^- \backslash G_\varepsilon} |u_\varepsilon| \, dx$$

$$\leqslant C|\Omega \backslash G_\varepsilon|^{1/2} + C\|u_\varepsilon\|_p |\Omega \backslash G_\varepsilon|^{1/p'} \leqslant C|\Omega \backslash G_\varepsilon|^{1/2} + C|\Omega \backslash G_\varepsilon|^{1/p'},$$

where the last term goes to zero, as $\varepsilon \to 0$, by (3.19).

To deal with $\tilde{J}_\varepsilon^4$ we consider the two cases $q = 1$ and $q \in (1, 2]$, separately. If $q = 1$, the function $\hat{\pi}$ is bounded and so, we have

$$|\tilde{J}_\varepsilon^4| \leqslant \frac{C}{\varepsilon} \|\det \nabla y_\varepsilon - 1\|_{L^1(\Omega_\varepsilon^+)} \leqslant C\varepsilon,$$

where in the last inequality we used (4.2).

If instead $q \in (1, 2]$, by using the p/q'-growth of $\hat{\pi}$ we have

$$|\tilde{J}_\varepsilon^4| \leqslant \frac{C}{\varepsilon} \int_{\Omega_\varepsilon^+} (1 + (\varepsilon|u_\varepsilon|)^{p/q'})|\det \nabla y_\varepsilon - 1| \, dx$$

$$\leqslant \frac{C}{\varepsilon} \|\det \nabla y_\varepsilon - 1\|_{L^q(\Omega_\varepsilon^+)} |\Omega_\varepsilon^+|^{1/q'} + C\varepsilon^{p/q'-1} \|\det \nabla y_\varepsilon - 1\|_{L^q(\Omega_\varepsilon^+)} \|u_\varepsilon\|_p^{p/q'}.$$

By (4.2) and the boundedness of (u_ε) in $W^{1,p}(\Omega; \mathbb{R}^n)$ we deduce

$$|\tilde{J}_\varepsilon^4| \leqslant C\varepsilon^{2/q-1} |\Omega_\varepsilon^+|^{1/q'} + C\varepsilon^{2/q+p/q'-1} \|u_\varepsilon\|_p^{p/q'} \leqslant C|\Omega_\varepsilon^+|^{1/q'} + C\varepsilon^{2/q+p/q'-1}.$$

It is easy to verify that $2/q + p/q' - 1 > 0$. Moreover, since $G_\varepsilon \subset \Omega_\varepsilon^-$ for ε small enough, we have that $|\Omega_\varepsilon^+| \to 0$ by (3.19). Therefore, we conclude that in both cases $\tilde{J}_\varepsilon^4$ is infinitesimal, as $\varepsilon \to 0$. □

The next proposition collects both the liminf and the limsup inequalities, which thus provide a full Γ-convergence result.

PROPOSITION 4.5 (**Liminf and limsup inequalities**). *Assume (W1)–(W6), (π1) and (π3). For every $\varepsilon \in (0, \varepsilon_0)$ let $y_\varepsilon \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ be such that there exist $R_\varepsilon \in SO(n)$ converging to $R_0 \in \mathcal{R}$ and the corresponding displacements u_ε, defined as in (3.4), weakly converge in $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ to $u_0 \in \mathring{H}^1(\Omega; \mathbb{R}^n)$. Then*

$$\mathcal{E}_0(u_0, R_0) \leqslant \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon),$$

where $\mathcal{E}_0$ is the functional defined in (3.29).

On the other hand, for every $(u_0, R_0) \in \mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R}$ there exist $(u_\varepsilon, R_\varepsilon) \in \mathring{W}^{1,p}(\Omega; \mathbb{R}^n) \times SO(n)$ such that $u_\varepsilon \rightharpoonup u_0$ weakly in $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$, $R_\varepsilon \to R_0$ and, setting $y_\varepsilon(x) := R_\varepsilon(x + \varepsilon u_\varepsilon(x))$, there holds

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) \leqslant \mathcal{E}_0(u_0, R_0).$$

Proof. The proof is the same of propositions 3.10 and 3.12, once we have at our disposal proposition 4.4. The only additional remark is that the deformations y_ε in the recovery sequence are admissible since $\det \nabla y_\varepsilon \in L^\infty(\Omega)$ by construction (see (4.1) for the definition of Y_q^p). □

Combining the previous results and arguing as in theorem 3.14, one can infer the following convergence result for almost minimizers.

THEOREM 4.6 (**Convergence of almost minimizers**). *Assume (W1)–(W6), (π1) and (π3). If (y_ε) is a sequence of almost minimizers for the energies $\mathcal{E}_\varepsilon$, that is,*

$$\mathcal{E}_\varepsilon(y_\varepsilon) \leqslant \inf_{\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)} \mathcal{E}_\varepsilon + o(\varepsilon^2),$$

then there exist $R_\varepsilon \in SO(n)$ such that, up to passing to a subsequence, we have

- $u_\varepsilon \to u_0$ *strongly in $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ with $u_0 \in \mathring{H}^1(\Omega; \mathbb{R}^n)$;*

- $R_\varepsilon \to R_0$ *with $R_0 \in \mathcal{R}$.*

Furthermore, the pair (u_0, R_0) is a minimizer of $\mathcal{E}_0$ on $\mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R}$ and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left(\inf_{\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)} \mathcal{E}_\varepsilon \right) = \min \left\{ \mathcal{E}_0(u, R) : \ (u, R) \in \mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R} \right\}.$$

5. A refined Γ-limit and a comparison with dead loads

In this section we make a comparison with the results obtained in [**22**], in the case of dead loads. In particular, we compute a refined version of the Γ-limit of the rescaled energies $(1/\varepsilon^2)\mathcal{E}_\varepsilon$.

We assume (W1)–(W5) and (π1), together with either (π2) or (π3) and (W6). In addition, we require

(π4) π is of class C^2 in the open set given by (π1).

Under this assumption any optimal rotation $R_0 \in \mathcal{R}$ satisfies, in addition to (2.9), the following condition:

$$\int_{\Omega} \left(\nabla\pi(R_0 x) \cdot R_0 A^2 x + D^2\pi(R_0 x) R_0 A x \cdot R_0 A x \right) \mathrm{d}x \geqslant 0 \qquad \text{for every } A \in \mathbb{R}^{n\times n}_{\text{skew}}. \tag{5.1}$$

This is obtained by imposing that the second variation of the functional in (2.7) is positive semidefinite along the curve $t \mapsto R_0 e^{tA}$. By the divergence theorem, condition (5.1) can be rewritten as

$$\int_{\partial\Omega} \left(\nabla\pi(R_0 x) \cdot R_0 A x \right) n_{\partial\Omega}(x) \cdot A x \, \mathrm{d}\mathcal{H}^{n-1}(x) \geqslant 0 \qquad \text{for every } A \in \mathbb{R}^{n\times n}_{\text{skew}}. \tag{5.2}$$

In [22] the applied body force is assumed to be a dead load of the form

$$-\int_{\Omega} g(x) \cdot y(x) \, \mathrm{d}x,$$

where $g \in L^2(\Omega; \mathbb{R}^n)$ is given. In this setting the authors proved that the set of optimal rotations, which is defined as

$$\mathcal{R}_g := \operatorname*{argmin}_{R \in SO(n)} \left\{ -\int_{\Omega} g(x) \cdot R x \, \mathrm{d}x \right\}, \tag{5.3}$$

is a submanifold of $SO(n)$ (see [22, proposition 4.1]). Moreover, if (y_ε) is a sequence of deformations with total energy of order ε^2, then any sequence of rotations (R_ε) provided by the rigidity estimate converges to an optimal rotation R_0 (as in propositions 3.8 and 4.3) and, in addition, satisfies

$$\operatorname{dist}_{SO(n)}(R_\varepsilon; \mathcal{R}_g) \leqslant C\sqrt{\varepsilon}, \tag{5.4}$$

where $\operatorname{dist}_{SO(n)}$ is the intrinsic distance in $SO(n)$, that is,

$$\operatorname{dist}_{SO(n)}(R, S) := \min \left\{ |A| : A \in \mathbb{R}^{n\times n}_{\text{skew}}, \ R = S e^A \right\}.$$

Finally, the Γ-limit of the rescaled energies can be expressed as

$$\frac{1}{2} \int_{\Omega} Q(x, e(u_0)(x)) \, \mathrm{d}x - \int_{\Omega} g(x) \cdot R_0 u_0(x) \, \mathrm{d}x - \frac{1}{2} \int_{\Omega} g(x) \cdot R_0 A_0^2 x \, \mathrm{d}x, \tag{5.5}$$

where u_0 is the limit displacement, $A_0 \in \mathbb{R}^{n\times n}_{\text{skew}}$ is the limit of

$$\frac{1}{\sqrt{\varepsilon}} R_0^T (R_\varepsilon - \mathcal{P}_g(R_\varepsilon))$$

(which exists up to subsequences), and $\mathcal{P}_g$ is the projection operator on $\mathcal{R}_g$ (see [22, § 5]). We note that the last term in (5.5) is the second variation of the functional in (5.3) at R_0 computed in the direction A_0. In [22] it is then proved that $A_0 = 0$ for sequences (y_ε) of almost minimizers, so that the last term in (5.5) is identically equal to 0 on minimizers.
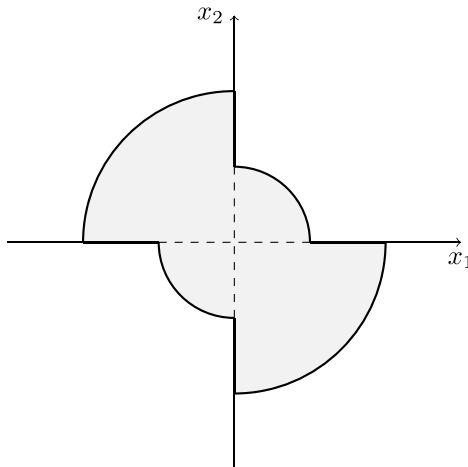
Figure 1. Set Ω in example 5.1.

In our setting of a pressure live load, a first difference with [**22**] is that the set of optimal rotations may not be a manifold, as the following example shows.

EXAMPLE 5.1. Let $n = 2$ and let Ω be the set given in polar coordinates by

$$\Omega = \{(\rho, \theta) : \ \rho < 1 \text{ if } \theta \in [0, \pi/2] \cup [\pi, 3\pi/2], \text{ and } \rho < 2 \text{ otherwise}\}$$

(see figure 1). We note that (2.3) is satisfied.

Let $\varphi \in C^3([0, \pi/2])$ be a nonnegative function satisfying $\varphi(0) = 0$ and attaining its maximum at $\alpha = \pi/2$. Suppose, in addition, that the first, second and third derivatives of φ vanish at $\alpha = 0$ and at $\alpha = \pi/2$. Let $\psi \in C^2([1, +\infty))$ be a bounded function with bounded first and second derivatives satisfying $\psi(1) = \psi'(1) = \psi''(1) = 0$ and $\int_1^2 \rho\psi(\rho)\,d\rho = 1$. A simple choice of ψ could be $\psi(\rho) = (20/9)(\rho - 1)^3$ for $\rho \in [1, 2]$, suitably extended to $[2, +\infty)$.

We consider the following pressure function:

$$\pi(x_1, x_2) = \begin{cases} \psi(\sqrt{x_1^2 + x_2^2})\varphi'(\arctan \frac{x_2}{x_1}) & \text{if } x_1 > 0, x_2 > 0, \text{ and } x_1^2 + x_2^2 > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Elementary computations show that π is Lipschitz continuous and of class $C^2(\mathbb{R}^n)$, so that (π1) and (π4) are satisfied; furthermore, π is bounded, so also (π3) holds true.

We recall that $SO(2)$ can be identified with the unit sphere $\mathbb{S}^1$ via the map

$$[0, 2\pi) \ni \alpha \mapsto R^\alpha := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

so that we can write

$$\mathcal{R} = \operatorname*{argmin}_{\alpha \in [0, 2\pi)} \left\{ \int_{R^\alpha\Omega} \pi(x_1, x_2)\,dx_1\,dx_2 \right\}.$$

For $\alpha \in [0, \pi/2)$ we can compute

$$\int_{R^\alpha\Omega} \pi(x_1, x_2)\, \mathrm{d}x_1 \,\mathrm{d}x_2 = \int_1^2 \int_0^\alpha \psi(\rho)\varphi'(\theta)\rho \,\mathrm{d}\theta \,\mathrm{d}\rho$$

$$= \int_1^2 \rho\psi(\rho)\,\mathrm{d}\rho \int_0^\alpha \varphi'(\theta)\,\mathrm{d}\theta = \varphi(\alpha).$$

Similarly, we obtain

$$\int_{R^\alpha\Omega} \pi(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \phi(\alpha) := \begin{cases} \varphi(\alpha) & \text{if } \alpha \in [0, \pi/2), \\ \varphi(\pi/2) - \varphi(\alpha - \pi/2) & \text{if } \alpha \in [\pi/2, \pi), \\ \varphi(\alpha - \pi) & \text{if } \alpha \in [\pi, 3\pi/2), \\ \varphi(\pi/2) - \varphi(\alpha - 3\pi/2) & \text{if } \alpha \in [3\pi/2, 2\pi). \end{cases}$$

$$(5.6)$$

Since φ is nonnegative and maximized at $\pi/2$, the set of optimal rotations corresponds to the zero-level set of the function ϕ. In particular, in $[0, \pi/2)$ this is given by the zero-level set of φ, which can be any closed set at positive distance from $\pi/2$.

This example shows that, in general, we cannot expect $\mathcal{R}$ to be a manifold.

Since $\mathcal{R}$ is not in general a manifold, the projection operator on $\mathcal{R}$ is not well defined. However, in the limiting process we can keep track of the distance of the approximating rotations R_ε from $\mathcal{R}$ through a suitable sequence of skew-symmetric matrices A_ε. In contrast with (5.4), the scaling of this distance may be larger than $\sqrt{\varepsilon}$ (actually, larger than $\sqrt[k]{\varepsilon}$ for any given $k > 2$), see example 5.3. To recover compactness of (A_ε) we rescale it by $|A_\varepsilon| \vee \sqrt{\varepsilon}$ and we denote by A_0 its limit. The Γ-limit of the rescaled energies can be then expressed as

$$\mathcal{E}_0(u_0, R_0) + \frac{1}{2}\mathcal{F}(R_0, A_0),$$

where $\mathcal{F}: \mathcal{R} \times \mathbb{R}^{n\times n}_{\mathrm{skew}} \to [0, +\infty)$ is the second variation of the functional in (2.7). This additional term measures the cost due to the fluctuations of the approximating rotations from the set $\mathcal{R}$. Arguing as in (5.1) and (5.2), the functional $\mathcal{F}$ takes the form

$$\mathcal{F}(R_0, A_0) = \int_{\partial\Omega} \left(\nabla\pi(R_0 x) \cdot R_0 A_0 x\right) A_0 x \cdot n_{\partial\Omega}(x) \,\mathrm{d}\mathcal{H}^{n-1}(x).$$

For sequences (y_ε) of almost minimizers the limit A_0 may be different from 0; however, we have $\mathcal{F}(R_0, A_0) = 0$. More precisely, we have the following result.

THEOREM 5.2. *Under the assumptions of propositions* 3.8 *or* 4.3, *we have in addition that there exist* $A_\varepsilon \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ *such that* $R_\varepsilon = S_\varepsilon \mathrm{e}^{A_\varepsilon}$ *for some* $S_\varepsilon \in \mathcal{R}$ *and, up to subsequences,* $A_\varepsilon \to 0$ *and* $A_\varepsilon/(|A_\varepsilon| \vee \sqrt{\varepsilon}) \to A_0$ *for some* $A_0 \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ *with* $|A_0| \leqslant 1$.

If also (π_4) is in force, then

$$\Gamma - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) = \mathcal{E}_0(u_0, R_0) + \frac{1}{2} \mathcal{F}(R_0, A_0)$$

with respect to the following convergences: $u_\varepsilon \rightharpoonup u_0$ weakly in $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$, $R_\varepsilon \to R_0$, $A_\varepsilon \to 0$, and $A_\varepsilon/(|A_\varepsilon| \vee \sqrt{\varepsilon}) \to A_0$.

Finally, if (y_ε) is a sequence of almost minimizers for the energies $\mathcal{E}_\varepsilon$, that is, (3.34) holds, then there exist $R_\varepsilon \in SO(n)$ and $A_\varepsilon \in \mathbb{R}^{n \times n}_{\text{skew}}$ as above such that, up to a subsequence, we have

- *$u_\varepsilon \to u_0$ strongly in $\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)$ with $u_0 \in \mathring{H}^1(\Omega; \mathbb{R}^n)$,*

- *$R_\varepsilon \to R_0$ with $R_0 \in \mathcal{R}$,*

- *$A_\varepsilon \to 0$ and $A_\varepsilon/(|A_\varepsilon| \vee \sqrt{\varepsilon}) \to A_0$ with $A_0 \in \mathbb{R}^{n \times n}_{\text{skew}}$, $|A_0| \leqslant 1$.*

Furthermore, the triplet (u_0, R_0, A_0) is a minimizer of $\mathcal{E}_0 + (1/2)\mathcal{F}$ on $\mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R} \times \mathbb{R}^{n \times n}_{\text{skew}}$, $\mathcal{F}(R_0, A_0) = 0$, and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left(\inf_{\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)} \mathcal{E}_\varepsilon \right)$$

$$= \min \left\{ \mathcal{E}_0(u, R) + \frac{1}{2} \mathcal{F}(R, A) : \ (u, R, A) \in \mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R} \times \mathbb{R}^{n \times n}_{\text{skew}} \right\}$$

$$= \min \left\{ \mathcal{E}_0(u, R) : \ (u, R) \in \mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R} \right\}.$$

Proof. As for the compactness statement, since $\mathcal{R}$ is a closed set, there exist $S_\varepsilon \in \mathcal{R}$ and $A_\varepsilon \in \mathbb{R}^{n \times n}_{\text{skew}}$ such that $R_\varepsilon = S_\varepsilon e^{A_\varepsilon}$ and

$$\text{dist}_{SO(n)}(R_\varepsilon; \mathcal{R}) = \text{dist}_{SO(n)}(R_\varepsilon, S_\varepsilon) = |A_\varepsilon|.$$

Since, up to subsequences, $R_\varepsilon \to R_0 \in \mathcal{R}$, we have that $A_\varepsilon \to 0$. The remaining properties follow from the fact the sequence $(A_\varepsilon/(|A_\varepsilon| \vee \sqrt{\varepsilon}))$ is bounded by 1.

We now give a sketch of the proof of the liminf inequality. By proposition 3.10 or 4.5 we have that

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) \geqslant \mathcal{E}_0(u_0, R_0) + \liminf_{\varepsilon \to 0} \int_\Omega \frac{\pi(R_\varepsilon x) - \pi(x)}{\varepsilon} \, \mathrm{d}x.$$

In fact, the last term above was always neglected in the previous computations, since it is nonnegative by (2.8). Using that $S_\varepsilon \in \mathcal{R}$, a Taylor expansion of π and of

the exponential map yields

$$\int_\Omega \frac{\pi(R_\varepsilon x) - \pi(x)}{\varepsilon} \, \mathrm{d}x \tag{5.1}$$

$$= \frac{1}{\varepsilon} \int_\Omega \nabla\pi(S_\varepsilon x) \cdot S_\varepsilon A_\varepsilon x \, \mathrm{d}x$$

$$+ \frac{1}{2\varepsilon} \int_\Omega \left(\nabla\pi(S_\varepsilon x) \cdot S_\varepsilon A_\varepsilon^2 x + D^2\pi(S_\varepsilon(I + t_\varepsilon(x)K_\varepsilon)x)S_\varepsilon K_\varepsilon x \cdot S_\varepsilon K_\varepsilon x \right) \mathrm{d}x$$

$$+ \frac{1}{\varepsilon} \int_\Omega |A_\varepsilon|^3 \nabla\pi(S_\varepsilon x) \cdot S_\varepsilon H_\varepsilon x \, \mathrm{d}x, \tag{5.7}$$

where $t_\varepsilon(x) \in [0, 1]$, H_ε is a uniformly bounded matrix, and

$$K_\varepsilon := A_\varepsilon + \frac{1}{2}A_\varepsilon^2 + |A_\varepsilon|^3 H_\varepsilon.$$

We now note that the first integral in (5.7) is equal to 0 by (2.9); thus, by multiplying and dividing by $\alpha_\varepsilon := (|A_\varepsilon| \vee \sqrt{\varepsilon})^2$ we obtain

$$\int_\Omega \frac{\pi(R_\varepsilon x) - \pi(x)}{\varepsilon} \, \mathrm{d}x$$

$$= \frac{\alpha_\varepsilon}{2\varepsilon} \left[\int_\Omega \left(\nabla\pi(S_\varepsilon x) \cdot S_\varepsilon \frac{A_\varepsilon^2 x}{\alpha_\varepsilon} + D^2\pi(S_\varepsilon(I + t_\varepsilon(x)K_\varepsilon)x)S_\varepsilon \frac{K_\varepsilon x}{\sqrt{\alpha_\varepsilon}} \cdot S_\varepsilon \frac{K_\varepsilon x}{\sqrt{\alpha_\varepsilon}} \right) \mathrm{d}x \right.$$

$$\left. + \frac{2|A_\varepsilon|^3}{\alpha_\varepsilon} \int_\Omega \nabla\pi(S_\varepsilon x) \cdot S_\varepsilon H_\varepsilon x \, \mathrm{d}x \right].$$

We observe that the left-hand side is nonnegative, hence the term within square brackets is nonnegative, as well. Since $\sqrt{\alpha_\varepsilon} = |A_\varepsilon| \vee \sqrt{\varepsilon} \geqslant \sqrt{\varepsilon}$, $|A_\varepsilon| \to 0$, $S_\varepsilon \to R_0$, $A_\varepsilon/\sqrt{\alpha_\varepsilon} \to A_0$, letting $\varepsilon \to 0$ and using the dominated convergence theorem yield

$$\liminf_{\varepsilon \to 0} \int_\Omega \frac{\pi(R_\varepsilon x) - \pi(x)}{\varepsilon} \, \mathrm{d}x$$

$$\geqslant \frac{1}{2} \int_\Omega \left(\nabla\pi(R_0 x) \cdot R_0 A_0^2 x + D^2\pi(R_0 x)R_0 A_0 x \cdot R_0 A_0 x \right) \mathrm{d}x.$$

The liminf inequality follows now from the divergence theorem.

For the construction of the recovery sequence we proceed as in proposition 3.12 or 4.5, but choosing $R_\varepsilon := R_0 \mathrm{e}^{A_\varepsilon}$ with $A_\varepsilon := \sqrt{\varepsilon}A_0$. Since R_0 is in $\mathcal{R}$, we can write

$$\frac{1}{\varepsilon^2}\mathcal{E}_\varepsilon(y_\varepsilon) = \frac{1}{\varepsilon^2} \int_\Omega W(x, \nabla y_\varepsilon) \, \mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega \left(\pi(y_\varepsilon(x)) \det \nabla y_\varepsilon - \pi(R_\varepsilon x) \right) \mathrm{d}x$$

$$+ \int_\Omega \frac{\pi(R_\varepsilon x) - \pi(R_0 x)}{\varepsilon} \, \mathrm{d}x. \tag{5.8}$$

The first two integrals at the right-hand side can be bounded by $\mathcal{E}_0(u_0, R_0)$, arguing as in proposition 3.12 or 4.5. Repeating the same computations as for the liminf inequality, one can show that the last integral converges to $(1/2)\mathcal{F}(R_0, A_0)$.

Convergence of almost minimizers and of infima can be proved exactly as in theorems 3.14 and 4.6. Finally, we observe that the functional $\mathcal{F}$ is always nonnegative by (5.2) and $\mathcal{F}(R, 0) = 0$ for every $R \in \mathcal{R}$. Hence, by minimality we deduce that $\mathcal{F}(R_0, A_0) = 0$. This concludes the proof. □

We conclude the paper with an example showing that, given any sequence $\lambda_\varepsilon \to 0$ such that

$$\lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon^2}{\varepsilon} = +\infty \qquad \text{and} \qquad \lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon^k}{\varepsilon} = 0 \tag{5.9}$$

for some $k > 2$, there may exist sequences of almost minimizers of $\mathcal{E}_\varepsilon$ for which any approximating sequence of rotations has a distance from $\mathcal{R}$ of order λ_ε. This provides a further difference with the case of dead loads [**22**].

EXAMPLE 5.3. We start by considering a sequence (λ_ε) satisfying (5.9) with $k = 3$. We assume the pressure intensity π to be of class C^3. Moreover, we assume that the set of optimal rotations $\mathcal{R}$ is finite and that for every $R_0 \in \mathcal{R}$ there exists $A_0 \in \mathbb{R}^{n \times n}_{\text{skew}}$ such that

$$|A_0| = 1 \quad \text{and} \quad \mathcal{F}(R_0, A_0) = 0. \tag{5.10}$$

These properties are satisfied, for instance, in example 5.1 if the function φ is strictly increasing. Indeed, in this case the function ϕ in (5.6) attains its minimum only at $\alpha = 0$ and $\alpha = \pi$ and thus, $\mathcal{R} = \{\pm I\}$. Moreover, in this example $\nabla\pi(x) = \nabla\pi(-x) = 0$ for every $x \in \partial\Omega$, so that $\mathcal{F}(R_0, A_0) = 0$ for every $R_0 \in \mathcal{R}$ and every $A_0 \in \mathbb{R}^{n \times n}_{\text{skew}}$. Finally, π is of class C^3 if we assume, in addition, $\varphi \in C^4([0, \pi/2])$ with $\varphi^{(\text{iv})}(0) = \varphi^{(\text{iv})}(\pi/2) = 0$ and $\psi \in C^3([1, +\infty))$ with bounded third derivative and $\psi'''(1) = 0$.

Now let (u_0, R_0) be a minimizer of $\mathcal{E}_0$ on $\mathring{H}^1(\Omega; \mathbb{R}^n) \times \mathcal{R}$ and let $A_0 \in \mathbb{R}^{n \times n}_{\text{skew}}$ satisfy (5.10). Let $R_\varepsilon := R_0 e^{\lambda_\varepsilon A_0}$ and let (u_ε) be an approximating sequence for u_0 as in (3.31). We claim that the deformations $y_\varepsilon(x) := R_\varepsilon(x + \varepsilon u_\varepsilon(x))$ are a sequence of almost minimizers of $\mathcal{E}_\varepsilon$. Indeed, arguing as in the proof of the limsup inequality in theorem 5.2, we have by (5.8) that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{E}_\varepsilon(y_\varepsilon) = \mathcal{E}_0(u_0, R_0) + \lim_{\varepsilon \to 0} \int_\Omega \frac{\pi(R_\varepsilon x) - \pi(R_0 x)}{\varepsilon} \, dx$$

$$= \min_{\mathring{H}^1 \times \mathcal{R}} \mathcal{E}_0 + \lim_{\varepsilon \to 0} \int_\Omega \frac{\pi(R_\varepsilon x) - \pi(R_0 x)}{\varepsilon} \, dx$$

$$= \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon^2} \inf_{\mathring{W}^{1,p}(\Omega; \mathbb{R}^n)} \mathcal{E}_\varepsilon + \int_\Omega \frac{\pi(R_\varepsilon x) - \pi(R_0 x)}{\varepsilon} \, dx \right).$$

Therefore, the claim is proved if we show that the last term above vanishes, as $\varepsilon \to 0$. To prove it we argue as in the proof of the liminf inequality in theorem 5.2,

now expanding up to the third order. By (2.9) and (5.10) we obtain

$$\int_\Omega \frac{\pi(R_\varepsilon x) - \pi(R_0 x)}{\varepsilon}\, \mathrm{d}x$$
$$= \frac{\lambda_\varepsilon}{\varepsilon}\int_\Omega \nabla\pi(R_0 x)\cdot R_0 A_0 x\, \mathrm{d}x + \frac{\lambda_\varepsilon^2}{2\varepsilon}\mathcal{F}(R_0, A_0) + O\left(\frac{\lambda_\varepsilon^3}{\varepsilon}\right) = O\left(\frac{\lambda_\varepsilon^3}{\varepsilon}\right), \quad (5.11)$$

which proves the claim owing to (5.9) with $k = 3$.

We note that the intrinsic distance of $R_\varepsilon = R_0 \mathrm{e}^{\lambda_\varepsilon A_0}$ from $\mathcal{R}$ in $SO(n)$ is of order λ_ε. Indeed, since $\mathcal{R}$ is finite by assumption, we have that $\mathrm{d}_{SO(n)}(R_\varepsilon; \mathcal{R}) = \mathrm{d}_{SO(n)}(R_\varepsilon, R_0)$ for ε small enough. By definition we clearly have $\mathrm{d}_{SO(n)}(R_\varepsilon, R_0) \leqslant \lambda_\varepsilon$. On the other hand, since the intrinsic distance in $SO(n)$ is equivalent to the Euclidean distance, we obtain

$$\mathrm{d}_{SO(n)}(R_\varepsilon, R_0) \geqslant c|\mathrm{e}^{\lambda_\varepsilon A_0} - I| \geqslant c\lambda_\varepsilon.$$

We now prove that for the sequence (y_ε) constructed above, any sequence (R'_ε) of approximating rotations satisfies

$$\mathrm{d}_{SO(n)}(R'_\varepsilon; \mathcal{R}) \geqslant c\lambda_\varepsilon. \tag{5.12}$$

By lemma 3.6 or lemma 4.2 we deduce that $|R_\varepsilon - R'_\varepsilon| \leqslant C\varepsilon$, hence $R'_\varepsilon \to R_0$. Since $\mathcal{R}$ is finite, we have that $\mathrm{d}_{SO(n)}(R'_\varepsilon; \mathcal{R}) = \mathrm{d}_{SO(n)}(R'_\varepsilon, R_0)$ for ε small enough. Let $A'_\varepsilon \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ be such that $R'_\varepsilon = R_0 \mathrm{e}^{A'_\varepsilon}$ and $\mathrm{d}_{SO(n)}(R'_\varepsilon, R_0) = |A'_\varepsilon|$. Assume by contradiction that (5.12) does not hold, that is, $|A'_\varepsilon|/\lambda_\varepsilon \to 0$, as $\varepsilon \to 0$. Then we have

$$C\varepsilon \geqslant |R_\varepsilon - R'_\varepsilon| = |\mathrm{e}^{\lambda_\varepsilon A_0} - \mathrm{e}^{A'_\varepsilon}| \geqslant |\mathrm{e}^{\lambda_\varepsilon A_0} - I| - |\mathrm{e}^{A'_\varepsilon} - I| \geqslant |\mathrm{e}^{\lambda_\varepsilon A_0} - I| - c|A'_\varepsilon|.$$

Dividing by λ_ε and sending $\varepsilon \to 0$, we obtain a contradiction, since the left-hand side vanishes by (5.9) and the right-hand side converges to $|A_0| = 1$.

Using again that $|R_\varepsilon - R'_\varepsilon| \leqslant C\varepsilon$, it is easy to see that in fact the intrinsic distance of R'_ε from $\mathcal{R}$ is of order λ_ε.

If, instead, the sequence (λ_ε) satisfies (5.9) for some $k \geqslant 4$, the previous arguments can be adapted with small changes as follows. We assume, in addition, that π is of class C^k and that for every $R_0 \in \mathcal{R}$ there exists $A_0 \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ such that

$$|A_0| = 1 \quad \text{and} \quad \mathcal{V}_j(R_0, A_0) = 0 \quad \text{for every } j = 2, \ldots, k-1, \tag{5.13}$$

where $\mathcal{V}_j(R_0, A_0)$ is the j-th variation of the functional in (2.7) at R_0 computed in the direction A_0. This is fulfilled by the pressure load in example 5.1, if φ and ψ have enough regularity and satisfy suitable boundary conditions. By expanding up to order k in (5.11), condition (5.13) guarantees that the sequence (y_ε), constructed as above, is still a sequence of almost minimizers. The bounds on the intrinsic distance from $\mathcal{R}$ can be proved as before.

REMARK 5.4. If condition (5.10) is not satisfied, that is, for every $R_0 \in \mathcal{R}$ one has

$$\mathcal{F}(R_0, A_0) = 0 \quad \Longleftrightarrow \quad A_0 = 0,$$

the phenomenon described in the previous example cannot arise. More precisely, one can show that the intrinsic distance of the approximating rotations from $\mathcal{R}$ is

at most of order $\sqrt{\varepsilon}$, as in (5.4). The argument is the same as in [**22**, theorem 5.1], combined with (5.7).

Acknowledgments

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