ON MEASURES DETERMINED BY FUNCTIONS WITH FINITE RIGHT AND LEFT LIMITS EVERYWHERE

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1. In another paper [1] measures determined from base functions, that have finite right and left limits everywhere and are of generalized bounded variation in the restricted sense, are studied and used to define non absolutely convergent integrals of Denjoy type. In this paper base functions of bounded variation and the corresponding measures are studied as a background for that paper. The results supplement parts of [2].

In both papers sequential covering classes of open intervals and functions τ on these classes defined in terms of finite right and left limits (§ 2 below) are used to associate, with each function $F \in \mathcal{J}$ (i.e., each F that has finite right and left limits everywhere), a unique Method II positive outer measure μ^* [2]. Then each function that is of bounded variation on every finite interval (BV') determines three non-decreasing functions $|F|$, F^+ and F^- corresponding to the total, positive and negative variations of F. The outer measures $\mu^*_{|F|}$, $\mu^*_{F^+}$ and $\mu^*_{F^-}$ determined by these functions coincide with the corresponding variations on open intervals and $\mu^*_{|F|} = \mu^*_{F^+} + \mu^*_{F^-}$.

It is clear from the definition that each outer measure μ^* is independent of the value of $F(x)$ at the points of discontinuity of F. On the other hand the outer measures determined by the variation functions are not independent of the values of F at the points of discontinuity unless these values remain between $F(x^-)$ and $F(x^+)$ at each x. A function F will be said to have the intermediate value property (we write F has IVP) if for each x $F(x)$ lies between $F(x^-)$ and $F(x^+)$. We show that when F has IVP μ^* coincides with $\mu^*_{|F|}$. An arbitrary F can be

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expressed in the form

(1.1) $$F = G + H ,$$

where G has IVP , $H(x^-) = H(x^+) = 0$ everywhere and μ^* (for F or G) coincides with $\mu^*_{|G|}$.

Given a signed measure ν on a σ-algebra $\mathcal{S}$, the Jordan decomposition ([5], p.11) implies the existence of positive measures ν^+ and ν^- on $\mathcal{S}$ with $\nu = \nu^+ - \nu^-$. To ν then corresponds a Hahn decomposition $X = X_o \bigcup CX_o$ with

(1.2) $$\nu(A) = \nu^+(A \cap X_o) - \nu^-(A \cap CX_o) , \quad A \in \mathcal{S} .$$

If F is BV' with finite positive and/or negative variation,

(1.3) $$\nu = \mu_{F^+} - \mu_{F^-}$$

defines a signed measure on $\mathcal{S}_F$, the $\mu^*_{|F|}$ -measurable sets. Defining $|\nu| = \nu^+ + \nu^-$, G and H as in 1.1, then

$$\mu = \mu_{|G|} = |\nu|$$

on $\mathcal{S}_F$ and the Jordan measures ν^+ and ν^- satisfy

(1.4) $$\nu^+ = \mu_{G^+} , \quad \nu^- = \mu_{G^-} .$$

If F has IVP then

$$\nu^+ = \mu_{F^+} , \quad \nu^- = \mu_{F^-}$$

and ν has a Hahn decomposition. Conversely if $F \neq G$, ν cannot have a Hahn decomposition.

2. Functions with finite right and left limits everywhere.

Let $\mathcal{J}$ denote the family of real valued functions F with $F(x^+)$ and $F(x^-)$ defined and finite for every x in $X = (-\infty, \infty)$.

THEOREM 2.1. Let $F \in \mathcal{J}$. Then F is continuous in X with the possible exception of an at most countable set of points and F is bounded on every finite interval (and on X if $F(x)$ has finite limits as $x \to \pm\infty$).

Proof. Let $S(F, x) = S(x) = \max(|F(x^+) - F(x^-)|, |F(x) - F(x^+)|, |F(x) - F(x^-)|)$. Then F is continuous at x if and only if $S(x) = 0$. Assume that there exists $d > 0$, a finite interval (a, b) and a countable set of points $\{x_i\}$ in (a, b) with $S(x_i) > d$, $i = 1, 2, \ldots$. There is then a subsequence converging to a point x', $a \leq x' \leq b$, and at x' at least one of $F(x^+)$, $F(x^-)$ fails to exist. Similarly, the assumption that $F(x)$ is unbounded leads to a sequence x_i converging to a point x with $\lim |F(x_i)| = \infty$, contradicting the existence of finite right and left limits at x.

Let $\mathcal{J}_o = \{F \in \mathcal{J} : F(x^+) = F(x^-) = 0 \text{ everywhere}\}$. Then if $F \in \mathcal{J}_o$, $F(x) = 0$ except for at most countably many points which may be dense in X. Furthermore, if x_i, $i = 1, 2, \ldots$, are the points of (a, b), $-\infty < a < b < \infty$, at which $F(x) \neq 0$, then $|F(x_i)| \to 0$ as $i \to \infty$. Note that if $F \in \mathcal{J}_o$, then F has IVP if and only if $F = 0$.

Let $F \in \mathcal{J}$ and define $H(x) = 0$ if x is a point of continuity of F or if $F(x)$ lies between $F(x^+)$ and $F(x^-)$. Elsewhere define $H(x) = F(x) - \max\{F(x^+), F(x^-)\}$ if $F(x) > F(x^+), F(x^-)$;
$= \min\{F(x^+), F(x^-)\} - F(x)$ if $F(x) < F(x^+), F(x^-)$.
Then $H \in \mathcal{J}_o$ and measures at each x the distance $F(x)$ lies above or below the interval determined by the points $F(x^+)$, $F(x^-)$. If $G = F - H$, G has IVP and completes (1.1).

Let $\mathcal{C}$ and $\mathcal{C}_d$ denote the collections of all finite open intervals and all open intervals of length less than d respectively. Then $\mathcal{C}$ and $\mathcal{C}_d$ are covering classes for X in the terminology of Munroe [2]. Define $\tau(\phi) = 0$ (ϕ the empty set), $\tau(a, b) = |F(b^-) - F(a^+)|$ on $\mathcal{C}$. Then $(\mathcal{C}, \tau)$ and, for each $d > 0$, $(\mathcal{C}_d, \tau)$ determine Method I outer measures on $\mathcal{P}(X)$,

the subsets of X. We shall denote them by $\mu^*_{F,\infty}$ and $\mu^*_{F,d}$.
If $d_i \downarrow 0$ (i.e., $d_1 \geq d_2 \geq \dots$, $\lim_i d_i = 0$) then

$$\mu^*_{F,0} = \mu^*_F = \lim_{i\to\infty} \mu^*_{F,d_i}$$

determines a Method II outer measure. It is easy to verify that μ^*_F does not depend on the sequence $\{d_i\}$. When a base function F has been fixed we shall abbreviate μ^*_F to μ^*.

If $F \in \mathcal{F}_o$, μ^* vanishes identically on X. The functions F_1, F_2 determine the same outer measures if they differ by the sum of a constant and a function in $\mathcal{F}_o$.

Since for each $F \in \mathcal{F}$, $\mu^* = \mu^*_F$ is a Method II outer measure it follows from [2] that:

I. μ^* is a metric outer measure ([2], Theorem 13.3);

II. Every Borel set is Carathéodory measurable for μ^* ([2], Cor. 13.2.1);

III. If $A_n \uparrow A$ (i.e., $A_1 \subset A_2 \subset \dots$, $A = \bigcup_n A_n$) then $\mu^*(A_n) \uparrow \mu^*(A)$ ([2], Cor. 12.1.1);

If A_n, $n = 1, 2, \dots,$ are Carathéodory measurable, if $A_n \downarrow A$ and there exists n with $\mu(A_n) < \infty$, then $\mu(A_n) \downarrow \mu(A)$;

IV. Given A there exists a G_δ set $B \supset A$ with $\mu(B) = \mu^*(A)$, i.e., μ^* is a regular outer measure ([2], p.108);

V. If there exists an open set U containing A with $\mu(U) < \infty$ then, given $\varepsilon > 0$, there exists an open set $U' \supset A$ with $\mu^*(A) < \mu(U') + \varepsilon$.

When μ^* is a Method I outer measure the definition and finiteness of $\mu^*(A)$ imply the existence of U' in V from which IV follows. If $F(x) = x \sin x^{-1}$, $x \neq 0$; $F(0) = 0$, F is continuous but not of bounded variation on any open set containing the origin. Then $\mu(\{0\}) = 0$ but Theorem 4.1 implies that $\mu(U) = \infty$ for every open set containing 0. For Method II outer measures IV implies that there exists $\{U_n\}$ with each U_n open and containing A and with $U_n \downarrow B$. Then if U contains A and

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$\mu(U) < \infty$, $U \cap U_n \downarrow B'$, $B \supset B' \supset A$, $\mu(B') = \mu^*(A)$, $\mu(U \cap U_n) \downarrow \mu^*(A)$ by III, and V follows.

The outer measures μ^* and $\mu^*_{F,d}$ need not coincide. Consider $F(x) = \sin x$. Every set can be covered by intervals of length 2π (on which τ vanishes). Thus, using $\mathcal{C}$ or $\mathcal{C}_d$ with $d > 2\pi$, $\mu^*_{F,d}(X) = 0$ and $\mu^*_{F,d}$ vanishes identically. Theorem 4.1 will show that for this F, $\mu(a, b)$ coincides with the total variation of F on (a, b).

THEOREM 2.2. If F is non-decreasing, $-\infty < a < b < \infty$, then $\mu^*_{F,d}(a, b) = \tau(a, b) = F(b^-) - F(a^+)$ and μ^* coincides with $\mu^*_{F,d}$, $0 < d \leq \infty$.

Proof. If (a, b) is a covering set we obtain $\mu^*_{F,d}(a, b) \leq \tau(a, b) = F(b^-) - F(a^+)$ trivially. If $0 < d < b-a$ take a sequence of points of continuity x_i with $a < x_1 < x_2 \ldots < x_n < b$ with $x_i - x_{i-1} < d$, $x_1 - a < d$, $b - x_n < d$. There then exist points of continuity x_i' with $x_i < x_i'$, $x_1' - a$, $x_i' - x_{i-1} < d$,

$$F(x_1') - F(a^+) + \sum_{i=2}^{n} \{F(x_i') - F(x_{i-1})\} - [F(x_1) - F(a^+)$$

$$+ \sum_{i=2}^{n} \{F(x_i) - F(x_{i-1})\}] < \varepsilon .$$

Then $(a, b) \subset (a, x_1') \cup (x_n, b) \cup \{\bigcup_{i=2}^{n} (x_{i-1}, x_i')\}$ and

$$\mu^*_{F,d}(a, b) \leq F(x_1') - F(a^+) + F(b^-) - F(x_n) + \sum_{i=2}^{n} \{F(x_i') - F(x_{i-1})\}$$

$$\leq F(b^-) - F(a^+) + F(x_1) - F(x_n) + \sum_{i=2}^{n} \{F(x_i) - F(x_{i-1})\} + \varepsilon$$

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$$= F(b^-) - F(a^+) + \varepsilon .$$

Passing to the limit as $d \to 0$, $\mu(a, b) = \tau(a, b)$ as well.

To prove that $\geq$ holds, let a', b' be points of continuity of F, $a < a' < b' < b$ with $F(a') - F(a^+) < \varepsilon/2$ and $F(b^-) - F(b') < \varepsilon/2$. Then $\mu^*_{F,d}(a, b) \geq \mu^*_{F,d}[a', b']$ and it is sufficient to show that

$$\mu^*_{F,d}[a', b'] \geq F(b') - F(a') \geq F(b^-) - F(a^+) - \varepsilon .$$

By compactness, any covering of $[a', b']$ by open intervals in $\mathcal{C}_d$ can be replaced by a finite subcovering, and simple arithmetic shows that

$$\mu[a', b'] \geq \mu^*_{F,d}[a', b'] \geq F(b') - F(a') .$$

It follows that μ^* coincides with $\mu^*_{F,d}$, $0 < d \leq \infty$, on the Borel sets. Both μ^* and $\mu^*_{F,d}$ are regular outer measures, which implies that $\mu^* = \mu^*_{F,d}$.

We note that when F is not monotone the Borel sets, and in fact open intervals, need not be $\mu^*_{F,d}$-measurable, $0 < d < \infty$. Let $F(x) = 0$, $x \leq 0$ and $x \geq 1$; $= x$, $0 < x < 1$. Then F is BV and has IVP, but $\mu^*_{F,1/2}(0, 1) = 1$, $\mu^*_{F,1/2}[1, 2) = 1/2$, $\mu^*_{F,1/2}(0, 2) = 1 < \mu^*_{F,1/2}(0, 1) + \mu^*_{F,1/2}[1, 2)$ and $(0, 1)$ is not $\mu^*_{F,1/2}$-measurable. However, we have

THEOREM 2.3. If $F \in \mathcal{F}$ is continuous, then the Borel sets are Carathéodory measurable for each outer measure $\mu^*_{F,d}$, $d > 0$.

Proof. It is sufficient to show that for an arbitrary open interval (a, b), $\varepsilon > 0$, and any set B with $\mu^*_{F,d}(B) < \infty$,

$$\mu^*_{F,d}(B) \geq \mu^*_{F,d}(B \cap (a, b)) + \mu^*_{F,d}(B \cap C(a, b)) - \varepsilon .$$

Given B, there exist intervals (a_i, b_i) in C_d covering B with

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$$\mu^*_{F,d}(B) \geq \Sigma_1^\infty \tau(a_i, b_i) - \varepsilon/2 .$$

We shall show that the collection $\{(a_i, b_i)\}$ can be replaced by collections $\{(a_i', b_i')\}$ and $\{(a_i'', b_i'')\}$ with the first of these covering $B \cap (a, b)$, the second covering $B \cap C(a, b)$. If $(a_i, b_i) \subset (a, b)$, set $(a_i', b_i') = (a_i, b_i)$, $(a_i'', b_i'') = \phi$, the empty set. If $(a_i, b_i) \subset C(a, b)$, set $(a_i', b_i') = \phi$, $(a_i'', b_i'') = (a_i, b_i)$.

Assume that $a_i < a < b_i < b$, $F(a_i) < F(b_i) < F(a)$. The continuity of $F(x)$ implies the existence of a point a_i^*, $a_i < a_i^* < a$ with $F(a_i^*) = F(b_i)$. We replace (a_i, b_i) by (a_i', b_i') and (a_i'', b_i'') with $a_i' = a_i^*$, $b_i' = b_i'' = b_i$, $a_i'' = a_i$. Then $\tau(a_i, b_i) = \tau(a_i', b_i') + \tau(a_i'', b_i'')$.

If $F(a_i) < F(a) < F(b_i)$ we can take $a_i'' = a_i$, $b_i' = b_i$ and determine $b_i'' > a$, $a_i' < a$ with

$$\tau(a_i', b_i') + \tau(a_i'', b_i'') - \tau(a_i, b_i) < \varepsilon/2^{i+2} .$$

Other possibilities can be treated similarly. Then

$$\mu^*_{F,d}(B) \geq \Sigma_1^\infty \tau(a_i, b_i) - \varepsilon/2 \geq \Sigma\, \tau(a_i', b_i') + \Sigma\, \tau(a_i'', b_i'') - \varepsilon ,$$

$$\geq \mu^*_{F,d}(B \cap (a, b)) + \mu^*_{F,d}(B \cap C(a, b)) - \varepsilon .$$

COROLLARY. If $F \in \mathcal{F}$ is continuous, then for each $d > 0$ Properties I-V hold for $\mu^*_{F,d}$.

Since open intervals are measurable the Borel sets are measurable and ([2], Exercise (a), p.104) gives I. IV comes from ([2], Corollary 12.3.1), III from Corollary 12.1.1.

3. <u>Measures determined by functions of bounded variation.</u> For $X = (-\infty, \infty)$ we define

$$\mathcal{F}_{BV} = \{F \in \mathcal{F} : F \text{ is BV on } X\} ;$$

$$\mathcal{F}_{BV'} = \{F \in \mathcal{F} : F \text{ is BV on every finite interval}\} ;$$

$$\check{\mathcal{F}} = \{F \in \mathcal{F} : F \text{ has IVP}\} .$$

For $A \subset X$ define

$$VF(A) = \sup \Sigma \, |F(b_i) - F(a_i)| ,$$

$$PVF(A) = \sup \Sigma \, \{F(b_i) - F(a_i)\} ,$$

$$NVF(A) = \sup \Sigma \, \{F(a_i) - F(b_i)\} ,$$

where the suprema are taken over all finite collections of non-overlapping intervals (a_i, b_i), $b_i > a_i$ with all a_i, b_i in A. These values will be called the total, positive and negative variations of F over A respectively.

(3.1) $VF(A) = PVF(A) + NVF(A)$ for every $A \subset X$. (Compare [4], Theorem 6.24.)

(3.2) If $a \in A$, $VF(A) = VF(A \cap (-\infty, a]) + VF(A \cap [a, \infty))$. There are analogous results for PVF and NVF.

(3.3) If $b < \infty$,
$$VF(a, b] = VF(a, b) + |F(b) - F(b^-)| ,$$
$$PVF(a, b] = PVF(a, b) + \max\{0, F(b) - F(b^-)\} ,$$
$$NVF(a, b] = NVF(a, b) + \max\{0, F(b^-) - F(b)\} .$$
Analogous results hold for $[a, b)$.

(3.4) $$VF(a, b) = \lim_{\alpha \to a^+, \beta \to b^-} VF(\alpha, \beta) = \lim_{\alpha \to a^+, \beta \to b^-} VF[\alpha, \beta] .$$

Assume $F \in \mathcal{F}$ with $F(0) = 0$. If $F(0) \neq 0$ we consider $F(x) - F(0)$. Define

$$|F|(0) = F^+(0) = F^-(0) = 0 ; \; |F|(x) = VF[0, x], \; F^+(x) = PVF[0, x],$$

$$F^-(x) = NVF[0, x], \quad x > 0 ;$$

$$|F|(x) = -VF[x, 0], \; F^+(x) = -PVF[x, 0], \; F^-(x) = -NVF[x, 0], \; x < 0 .$$

From (3.1)

$$|F|(x) = F^+(x) + F^-(x) \leq \infty ,$$

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with both sides finite for every x when $F \in \mathcal{F}_{BV'}$. Below we shall assume that $F \in \mathcal{F}_{BV'}$ unless we specify otherwise.

$$(3.5) \qquad F(x) = F^+(x) - F^-(x)$$

for every x whence

$$F^+(x) = \frac{1}{2}\{|F|(x) + F(x)\} \;,\; F^-(x) = \frac{1}{2}\{|F|(x) - F(x)\} .$$

The equalities in (3.5) are immediate if $x = 0$. Assume $x > 0$. Given $\varepsilon > 0$ there is a partition $0 = x_o < x_1 < \ldots < x_n = x$ with

$$0 \leq PVF[0,x] - \Sigma_P\{F(x_i) - F(x_{i-1})\} < \varepsilon/2 ,$$

$$0 \leq NVF[0,x] - \Sigma_N\{F(x_{i-1}) - F(x_i)\} < \varepsilon/2 .$$

Then

$$F(x) = \Sigma_1^n\{F(x_i) - F(x_{i-1})\} ; \; |\sum_1^n \{F(x_i) - F(x_{i-1})\}$$

$$- (PVF[0,x] - NVF[0,x])| = |F(x) - \{F^+(x) - F^-(x)\}| < \varepsilon .$$

A similar argument holds if $x < 0$. We next observe

$$(3.6) \qquad \begin{aligned} &VF[a,b] = |F|(b) - |F|(a) , \; PVF[a,b] = F^+(b) - F^+(a) , \\ &NVF[a,b] = F^-(b) - F^-(a) . \end{aligned}$$

If $a < 0 < b$, $VF[a,b] = VF[a,0] + VF[0,b]$ by (3.2)

$$= |F|(b) - |F|(a) .$$

If $0 < a < b$, $VF[0,b] = VF[0,a] + VF[a,b]$ leads to the result.

$$|F|(x^-) = \lim_{x' \to x^-} VF[0,x'] = VF[0,x) , \; x > 0 , \text{ using } (3.4),$$

$$|F|(x^+) = \lim_{x' \to x^+} |F|(x') = \lim_{x' \to x^+} VF[0,x'] = VF[0,x] +$$

$$(3.7) \qquad \lim_{x' \to x^+} VF[x,x'] = |F|(x) + |F(x) - F(x^+)| , \; x \geq 0 .$$

$$|F|(x^+) = - VF(x, 0] , \quad x < 0 ,$$

$$|F|(x^-) = - VF[x, 0] - |F(x) - F(x^-)| =$$
$$- |F|(x) - |F(x) - F(x^-)| , \quad x < 0 .$$

There are similar results for F^+ and F^-. With (3.2) they lead to

$$VF(a, b) = |F|(b^-) - |F|(a^+) \geq |F(b^-) - F(a^+)| ,$$

(3.8) $$PVF(a, b) = F^+(b^-) - F^+(a^+) ,$$

$$NVF(a, b) = F^-(b^-) - F^-(a^+) .$$

Applying Theorem 2.2 to the non-decreasing functions $|F|$, F^+ and F^- we obtain

$$\mu_{|F|}(a, b) = VF(a, b) ,$$

(3.9) $$\mu_{F^+}(a, b) = PVF(a, b) ,$$

$$\mu_{F^-}(a, b) = NVF(a, b) , \quad -\infty \leq a < b \leq \infty .$$

We then obtain easily

$$\mu_{|F|}(\{x\}) = |F(x) - F(x^+)| + |F(x) - F(x^-)|$$

(3.10) $$= |F(x^+) - F(x^-)| \quad \text{if } F \text{ has IVP at } x ;$$

(3.11) $$\mu_{|F|}[a, b] = VF[a, b] + |F(b^+) - F(b)| + |F(a) - F(a^-)| .$$

There are equalities similar to (3.10) and (3.11) for PVF and NVF. We show

(3.12) $$\mu^*_{|F|} = \mu^*_{F^+} + \mu^*_{F^-} .$$

Using the countable additivity of the measures and (3.9), (3.12) holds on open sets and therefore, using III, on bounded G_δ sets. Since the intersection of three G_δ sets is a G_δ set, IV implies that there exists a G_δ set $B \supset A$ with

$$\mu_{|F|}(B) = \mu^*_{|F|}(A)\ , \ \mu_{F^+}(B) = \mu^*_{F^+}(A)\ , \ \mu_{F^-}(B) = \mu^*_{F^-}(A)\ .$$

Thus (3.12) holds for arbitrary bounded sets and finally, using III, for all subsets of X .

We show that if μ^* , μ_1^* , μ_2^* are outer measures with $\mu^* = \mu_1^* + \mu_2^*$ and if $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2$ denote the μ^* , μ_1^* and μ_2^*-measurable sets respectively, then $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$.

That $\mathcal{S} \supset \mathcal{S}_1 \cap \mathcal{S}_2$ is trivial. Assume that $A \in \mathcal{S}$, $\mu^*(B) < \infty$. Then $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap CA)$. The assumption of additivity implies that

$$\mu_1^*(B) - \mu_1^*(B \cap A) - \mu_1^*(B \cap CA) = -\{\mu_2^*(B) - \mu_2^*(B \cap A) - \mu_2^*(B \cap CA)\}\ .$$

Assuming one side to be different from zero implies that one of the differences is greater than zero, contradicting the countable subadditivity of the corresponding outer measure. Thus $\mathcal{S} \subset \mathcal{S}_1 \cap \mathcal{S}_2$. From (3.12) we obtain

(3.13) $$\mathcal{S}_{|F|} = \mathcal{S}_{F^+} \cap \mathcal{S}_{F^-}\ .$$

If $H \in \mathcal{F}_o$, $\mathcal{S}_{|H|} = \mathcal{P}(X)$. Thus, for the decomposition (1.1),

(3.14) $$\mathcal{S}_{|F|} = \mathcal{S}_{|G|}\ .$$

4. <u>The relation between the outer measures</u> $\mu^* = \mu_F^*$ <u>and</u> $\mu^*_{|F|}$ <u>where</u> $F \in \mathcal{F}_{BV'}$. With each $F \in \mathcal{F}_{BV'}$ we have associated outer measures $\mu^* = \mu_F^*$, $\mu^*_{|F|}$, $\mu^*_{F^+}$ and $\mu^*_{F^-}$. If F does not have IVP it follows from (3.10) that $\mu^* \neq \mu^*_{|F|}$.

THEOREM 4.1. If $F \in \mathcal{F}$ and $F = G + H$ as in (1.1) then for every interval (a, b) over which F is of bounded variation

$$\mu(a, b) = VG(a, b)\ .$$

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If $F \in \mathcal{F}_{BV'}$ then for every a, b, $-\infty \leq a < b \leq \infty$,

$$\mu(a, b) = VG(a, b) ,$$

$\mu^* = \mu^*_{|G|}$ and $\mu^* = \mu^*_{|F|}$ if and only if F has IVP .

<u>Proof.</u> To prove the first part set $\bar{G}_o(x) = G(x)$, $a < x < b$; $= G(a^-)$, $x \leq a$; $= G(b^+)$, $x \geq b$; $\bar{G}(x) = \bar{G}_o(x) - \bar{G}_o(0)$. Then $\bar{G} \in \breve{\mathcal{F}} \cap \mathcal{F}_{BV}$. Let $\bar{\tau}(\alpha, \beta) = |\bar{G}|(\beta^-) - |\bar{G}|(\alpha^+)$. Then if $a \leq \alpha < \beta \leq b$,

$$\bar{\tau}(\alpha, \beta) = |\bar{G}|(\beta^-) - |\bar{G}|(\alpha^+) \geq |\bar{G}(\beta^-) - \bar{G}(\alpha^+)| = |F(\beta^-) - F(\alpha^+)| = \tau(\alpha, \beta).$$

It follows that for $a < a' < b' < b$, $A \subset (a', b')$, $d < (b-b', a'-a)$, $\mu^*_{|\bar{G}|, d}(A) \geq \mu^*_{F, d}(A)$ and thus $\mu^*_{|\bar{G}|}(A) \geq \mu^*(A)$. Using III, § 2, $\mu^*_{|\bar{G}|} \geq \mu^*$ on subsets of (a, b) and in particular

$$\mu(a, b) \leq \mu_{|\bar{G}|}(a, b) = VG(a, b) .$$

Note that if $F \in \mathcal{F}_{BV'}$, $\mu^* \leq \mu^*_{|G|}$.

We next show that $\mu(a, b) \geq VG(a, b)$ and obviously may assume that $\mu(a, b) < \infty$. We assume initially that G is continuous. Fixing $\varepsilon > 0$, there exist non-overlapping intervals (x_i, y_i) , $i = 1, 2, \ldots, n$, with

$$(4.1) \qquad 0 \leq VG(a, b) - \sum_{i=1}^{n-1} |G(y_i) - G(x_i)| < \varepsilon/4 .$$

Using (3.8), this implies

$$(4.2) \qquad 0 \leq V(a, b) - \sum_{i=1}^{n-1} VG(x_i, y_i) < \varepsilon/4$$

and

$$(4.3) \qquad \Sigma' NVG(x_i, y_i) + \Sigma'' PVG(x_i, y_i) < \varepsilon/4 ,$$

where Σ' denotes summation over the intervals with $G(y_i) > G(x_i)$, Σ'' over those with $G(y_i) < G(x_i)$.

By continuity there exist points x_i', y_i', $x_i < x_i' < y_i' < y_i$, $i = 1, 2, \ldots, n$, with

$$(4.4) \qquad 0 \le \left| \Sigma_i |G(y_i) - G(x_i)| - \Sigma_i |G(y_i') - G(x_i')| \right| < \varepsilon/4 .$$

Let $E = \cup_i [x_i', y_i']$. Then for $d < \frac{1}{2} \min\{y_i - y_i', x_i' - x_i, i = 1, 2, \ldots, n\}$, every covering of E by subsets of $\mathcal{C}_d$ can be replaced by a finite subcovering where each covering interval intersects one and only one of the intervals $[x_i', y_i']$ and where no point of E is in more than two of the covering intervals, no point x_i', y_i' in more than one ([3], Lemma 2, p. 57). Denote the intervals covering $[x_i', y_i']$ by $(\alpha_{ij}, \beta_{ij})$, $j = 1, 2, \ldots, n(i)$. Then

$$\Sigma_j \{G(\beta_{ij}) - G(\alpha_{ij})\} = G(y_i') - G(x_i') + G(x_i') - G(\alpha_{ij}) + G(\beta_{in(i)}) - G(y_i') + \Sigma_j \{G(\beta_{ij}) - G(\alpha_{ij})\}$$

and, if $G(y_i') > G(x_i')$,

$$(4.5) \qquad \Sigma_j |G(\beta_{ij}) - G(\alpha_{ij})| \ge G(y_i') - G(x_i') - NVG(x_i, y_i)$$

with a similar relation using PVG if $G(y_i') < G(x_i')$.

There exists such a covering with

$$\begin{aligned} \mu^*_{G,d}(E) &> \sum_{i,j} |G(\beta_{ij}) - G(\alpha_{ij})| - \varepsilon/4 \\ &> \sum_{i=1}^{n} |G(y_i') - G(x_i')| - \varepsilon/4 - \Sigma' NVG(x_i, y_i) - \Sigma'' PVG(x_i, y_i) \\ &> \sum_{1}^{n} |G(y_i) - G(x_i)| - \frac{3}{4}\varepsilon \\ &> VG(a, b) - \varepsilon ; \end{aligned}$$

$$\mu(a, b) \ge \mu(E) \ge \mu^*_{G,d}(E) > VG(a, b) - \varepsilon .$$

When G is not continuous (3.8) asserts that

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$VG(\alpha, \beta) \geq |G(\beta^-) - G(\alpha^+)|$ but may be strictly less than $|G(\beta) - G(\alpha)|$ so that (4.1) does not imply (4.2). Assuming G not continuous, let $\{x_i\}$ denote the points of discontinuity in (a, b). Then, if $S'(x_i) = |G(x_i^+) - G(x_i^-)|$,

$$\sum_i S'(x_i) \leq VG(a, b) < \infty .$$

Let $\{(a_i, b_i), \ i = 1, 2, \ldots, k+1\}$ denote the intervals in (a, b) complementary to $\{x_i, \ i = 1, 2, \ldots, k\}$. For each k there exist non-overlapping intervals (x_{ij}, y_{ij}), $j = 1, 2, \ldots, n(i)$ in (a_i, b_i), $i = 1, 2, \ldots, k+1$, with

$$(4.1') \qquad 0 \leq \sum_{i=1}^{k+1} \{VG(a_i, b_i) - \Sigma_j |G(y_{ij}) - G(x_{ij})|\} < \varepsilon/8 .$$

Now

$$\Sigma_j \big| |G(y_{ij}) - G(x_{ij})| - |G(y_{ij}^-) - G(x_{ij}^+)| \big| \leq \sum_{k+1}^{\infty} S'(x_i) ,$$

so that for k sufficiently large we have

$$(4.1'') \qquad 0 \leq \sum_1^{k+1} \{VG(a_i, b_i) - \Sigma_j |G(y_{ij}^-) - G(x_{ij}^-)|\} < \varepsilon/4 .$$

With (4.1'') replacing (4.1) the preceding argument with minor modifications gives

$$\mu\{\textstyle\bigcup_1^{k+1} (a_i, b_i)\} > \Sigma_1^k VG(a_i, b_i) - \varepsilon ,$$

$$\mu(a, b) = \sum_1^{k+1} \mu(a_i, b_i) + \sum_1^k \mu(\{x_i\}) > \sum_1^{k+1} VG(a_i, b_i) + \sum_1^k S'(x_i) - \varepsilon$$

$$= VG(a, b) - \varepsilon .$$

Since ε is arbitrary we have proved that $\mu(a, b) = VG(a, b)$.

If $F \in \mathcal{F}_{BV'}$, $G \in \mathcal{F} \cap \mathcal{F}_{BV'}$, $\mu(a, b) = \mu_{|G|}(a, b) = VG(a, b)$ for every open interval, finite or infinite. Thus $\mu^* = \mu^*_{|G|}$ on the open sets. From V they coincide on all bounded sets and finally, from III, $\mu^* = \mu^*_{|G|}$.

If $F \in \breve{\mathcal{F}} \cap \mathcal{F}_{BV'}$, $F = G$ and $\mu^* = \mu^*_{|F|}$. If $F \neq G$ and $H(x) \neq 0$, $\mu_{|F|}(\{x\}) = \mu_{|G|}(\{x\}) + 2|H(x)| > \mu(\{x\})$.

We observe that if $A \subset X$ contains an interval (a, b) on which G is not BV , that is a union of intervals on each of which G is BV then, using III, $\mu^*(A) \geq \mu(a, b) = \infty$. In particular if $F(x) = x \sin x^{-1}$, $x \neq 0$; $F(0) = 0$, $\mu(U) = \infty$ if U is an open set containing 0 . Our methods do not prove that $VG(a, b) = \infty$ always implies that $\mu(a, b) = \infty$. There exist continuous functions (e.g. the Weierstrasse continuous non-differentiable function) for which G is not BV on any interval.

5. <u>Signed measures and Jordan and Hahn decompositions.</u> Assume that $F \in \mathcal{F}_{BV'}$ and that at least one of $PVF(x)$, $NVF(x)$ is finite. On $\mathcal{S}_F = \mathcal{S}_{|F|} = \mathcal{S}_{F^+} \cap \mathcal{S}_{F^-}$ define the set function

$$\nu = \nu(F) = \mu_{F^+} - \mu_{F^-} .$$

Then ν is a signed measure on $\mathcal{S}_F$. The Jordan decomposition ([5], p.11) implies the existence of positive measures $(\mu_F)^+$ and $(\mu_F)^-$ on $\mathcal{S}_F$ with

$$(\mu_F)^+(A) = \sup_{\substack{e \in \mathcal{S}_F \\ e \subset A}} \nu(e), \quad (\mu_F)^-(A) = \sup_{\substack{e \in \mathcal{S}_F \\ e \subset A}} [-\nu_F(e)] .$$

Set

$$|\nu| = |\nu(F)| = (\mu_F)^+ + (\mu_F)^- .$$

Then $|\nu|$ is a positive measure on $\mathcal{S}_F$ and

$$(\mu_F)^+(A) = \sup_{e \subset A} \nu(e) = \sup_{e \subset A} [\mu_{F^+}(e) - \mu_{F^-}(e)]$$

$$\leq \sup_{e \subset A} \mu_{F^+}(e) \leq \mu_{F^+}(A) ;$$

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$$(\mu_F)^-(A) \le \mu_{F^-}(A)\ ; \quad |\nu|(A) \le \mu_{|F|}(A)\ .$$

We observe that strict inequality may occur. For example let $F(0) = 1$, $F(x) = 0$, $x \neq 0$. Then $F \in \mathfrak{J}_0 \cap \mathfrak{J}_{BV}$, $\mu_{F^+}(\{0\})$ $= \mu_{F^-}(\{0\}) = 1$, $\nu(\{0\}) = 0$, $\mu_{|F|}(\{0\}) = 2$.

In general write $F = G + H$ as in (1.1). Then if $A \in \mathcal{S}_F$

$$\mu_{F^+}(A) = \mu_{G^+}(A) + \mu_{H^+}(A)\ ,$$

$$\mu_{F^-}(A) = \mu_{G^-}(A) + \mu_{H^-}(A)\ .$$

Now $\mu_{H^+}(A) = \mu_{H^-}(A) = \sum_{(i: x_i \in A)} |H(x_i)|$, where $H(x) = 0$, $x \neq x_i$, $i = 1, 2, \ldots$. It follows that

$$\nu(F)(A) = \nu(G)(A)\ ,\ (\mu_F)^+(A) = (\mu_G)^+(A),\ (\mu_F)^-(A) = (\mu_G)^-(A)\ ,$$

$$|\nu|(F)(A) = |\nu|(G)(A)\ .$$

THEOREM 5.1. If $F \in \breve{\mathfrak{J}}_{BV}$, $A \in \mathcal{S}_F$, then $|\nu|(A) = \mu_{|F|}(A)$.

Proof. We have seen that $\le$ holds. We first show that $\ge$ holds for every open interval. Let (a, b) be a finite open interval, $\varepsilon > 0$ arbitrary.

There exist points x_i, $i = 1, 2, \ldots, n$, with $\nu_i < \nu_{i+1}$,

$$(5.1) \qquad 0 \le VF(a, b) - \sum_{1}^{n-1} |F(x_i) - F(x_{i-1})| < \varepsilon\ ,$$

and the inequalities remain valid if additional points are added to the sequence. We show that the IVP($F \in \breve{\mathfrak{J}}_{BV}$) implies that we can assume the points x_i to be points of continuity. If x_i is a point of discontinuity we can assume (adding points if necessary) that x_{i-1}, x_{i+1} are points of continuity with

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$$|F(x_{i+1}) - F(x_i^+)| + |F(x_{i-1}) - F(x_i^-)| < \varepsilon/2n .$$

Then, since $F \in \check{\mathfrak{F}}$,

$$||F(x_{i+1}) - F(x_{i-1})| - S'(x_i)| < \varepsilon/2n ,$$

$$S'(x_i) - \varepsilon/2n \le |F(x_{i+1}) - F(x_i)| + |F(x_i) - F(x_{i-1})|$$

$$< S'(x_i) + \varepsilon/2n ,$$

$$||F(x_{i+1}) - F(x_{i-1})| - |F(x_{i+1}) - F(x_i)| - |F(x_i) - F(x_{i-1})||$$

$$< \varepsilon/n .$$

It follows that we can drop x_i from the sequence without changing the sum by more than ε/n and thus can remove all points of discontinuity of $\{x_i\}$, $i = 1, 2, \ldots, n$, without changing the sums by more than ε.

Where Σ^+ and Σ^- denote summation over the terms with positive and negative increments respectively, (5.1) implies that

$$|\Sigma^+ |F(x_i) - F(x_{i-1})| - PVF(a, b)| < \varepsilon ,$$

$$|\Sigma^- |F(x_i) - F(x_{i-1})| - NVF(a, b)| < \varepsilon .$$

We can assume that the intervals in Σ^+ are disjoint and let U denote their union. Then

$$\mu_{F^+}(U) = \Sigma^+ \mu_{F^+}(x_{i-1}, x_i) = \Sigma_1^{+n} [F^+(x_i) - F^+(x_{i-1})]$$

$$\ge \Sigma^+ [F(x_i) - F(x_{i-1})]$$

$$\ge \Sigma^+ PVF(x_{i-1}, x_i) - 2\varepsilon = \mu_{F^+}(U) - 2\varepsilon$$

$$\ge \mu_{F^+}(a, b) - 3\varepsilon ;$$

$$\mu_{F^-}(U) < \varepsilon ,$$

$$(\mu_F)^+(a, b) \ge \mu_{F^+}(U) - \mu_{F^-}(U) \ge \mu_{F^+}(a, b) - 4\varepsilon .$$

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Since ε is arbitrary $(\mu_F)^+(a,b) \geq \mu_{F^+}(a,b)$. By a similar argument we show that $(\mu_F)^-(a,b) \geq \mu_{F^-}(a,b)$ and conclude that $|\nu|(a,b) = \mu_{|F|}(a,b)$.

The additivity of the measures $(\mu_F)^+$ and μ_{F^+} implies that they coincide on all open sets. Similarly $(\mu_F)^- = \mu_{F^-}$ on open sets and thus $|\nu|(U) = \mu_{|F|}(U)$ for every open set. If $U_n \downarrow A$, with each set U_n open, $(\mu_F)^+(U_n) \downarrow (\mu_F)^+(A)$ ([2], Corollary 10.3.1), $\mu_{F^+}(U_n) \downarrow \mu_{F^+}(A)$ by II and $(\mu_F)^+(A) = \mu_{F^+}(A)$. Finally, if B is an arbitrary measurable set there exists a G_δ set A that is a measurable cover for B, $A = B \cup (A-B)$, $\mu_{F^+}(A) = \mu_{F^+}(B)$, $\mu_{F^+}(A-B) = 0$. Then

$(\mu_F)^+(A) = \mu_{F^+}(A)$, $(\mu_F)^+(A-B) \leq \mu_{F^+}(A-B) = 0$ and $(\mu_F)^+(B) = \mu_{F^+}(B)$. Similarly $(\mu_F)^-(B) = \mu_{F^-}(B)$ whence $|\nu|(B) = \mu_{|F|}(B)$ for every $B \in \mathcal{S}_F$.

COROLLARY. If $F \in \mathcal{F}_{BV'}$ and one of $PVF(x)$, $NVF(x)$ is finite then always $\mu = |\nu| = \mu_{|G|}$ and they coincide with $\mu_{|F|}$ if and only if $F \in \breve{\mathcal{F}}_{BV}$.

If $F \in \mathcal{F}_{BV'} - \mathcal{F}_{BV}$ and $PVF(X) = NVF(X) = \infty$, $\mu_{F^+} - \mu_{F^-}$ need not be defined on unbounded sets. Writing ν and $|\nu|$ as before ν need not be a signed measure on $\mathcal{S}_F$. However the above discussion and equalities are valid where ν is defined.

Again let $F \in \mathcal{F}_{BV'}$ and assume one of $PVF(X)$, $NVF(X)$ to be finite. Then the Hahn-Lebesgue decomposition ([5], p.32) gives the existence of a measurable set X' with

$$\nu(A) = (\mu_F)^+(A), \quad (\mu_F)^-(A) = 0, \quad A \subset X';$$

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$\nu(A) = -(\mu_F)^-(A)$, $(\mu_F)^+(A) = 0$, $A \subset CX'$. Thus, if F has IVP , there exists a measurable set X' such that for every $A \in \mathcal{S}_F$,

$$\mu(A) = |\nu|(A) = \mu_{|F|}(A) = \mu_{F^+}(A \cap X') + \mu_{F^-}(A \cap CX')$$

$$\nu(A) = |\nu|(A \cap X') - |\nu|(A \cap CX') .$$

For $0 \neq F \in \mathcal{F}_o$ such a decomposition is not possible for $\mu_{|F|}$.

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