J. Austral. Math. Soc. (Series A) 36 (1984), 389–403

A FREE SURFACE PROBLEM ARISING IN THE DRAINAGE OF A UNIFORMLY IRRIGATED FIELD: EXISTENCE AND UNIQUENESS RESULTS

JOHN VAN DER HOEK, C. J. BARNES and J. H. KNIGHT

(Received 26 May 1982)

Communicated by R. O. Vyborny

Abstract

A field comprising uniformly porous soil overlying an impervious subsoil is drained through equally spaced tile drains placed on the boundary between the two layers of soil. When this field is subject to uniform irrigation, a free boundary forms in the porous region above the zone of saturation. We study the free boundary value problem which thus arises using the theory of variational inequalities. Existence and uniqueness results are established.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 35 R 35.

1. Introduction

In the early 1970's the Pavia school of mathematicians (see Baiocchi [1], Baiocchi *et al.* [4], Comincioli [7]) have provided a new approach to certain problems in hydrology which involve a free boundary. Their approach uses a particular class of transformations to reformulate the problems in such a way that the theory of variational inequalities can be applied to provide not only existence and uniqueness results, but numerical algorithms for the solution as well. This paper demonstrates an application of this theory to a class of drainage problems of importance to agriculture. It is shown how the original problem may be reformulated in terms of variational inequalities, with existence and uniqueness as a consequence. This work applies many of the results of Comincioli [7] and Baiocchi *et al.* [4] (particularly Chapter V), except that the more recent work of Grisvard [10] is used to stream line the argument a little. Although not directly related to our study here, variational inequalities have been employed to similar problems. We mention many of the papers of J. C. Bruck, Jr. and co-authors (for example [7], [8]) use similar methods to deal with irrigation problems. Numerical results and estimates on the free boundary have been deferred to another paper (Barnes *et al.* [5]) as are a discussion of the physics and the assumptions involved in this work. Notation used has been gathered together and explained in Section 7.

2. The physical model

The physical situation which we wish to model is that of a uniform porous soil overlying an impervious subsoil, and drained by a series of parallel, regularly spaced tile drains on the boundary between the two layers. The surface of the soil is uniformly irrigated in space and time at rate somewhat less than the saturated conductivity, so that no ponding occurs at the surface.

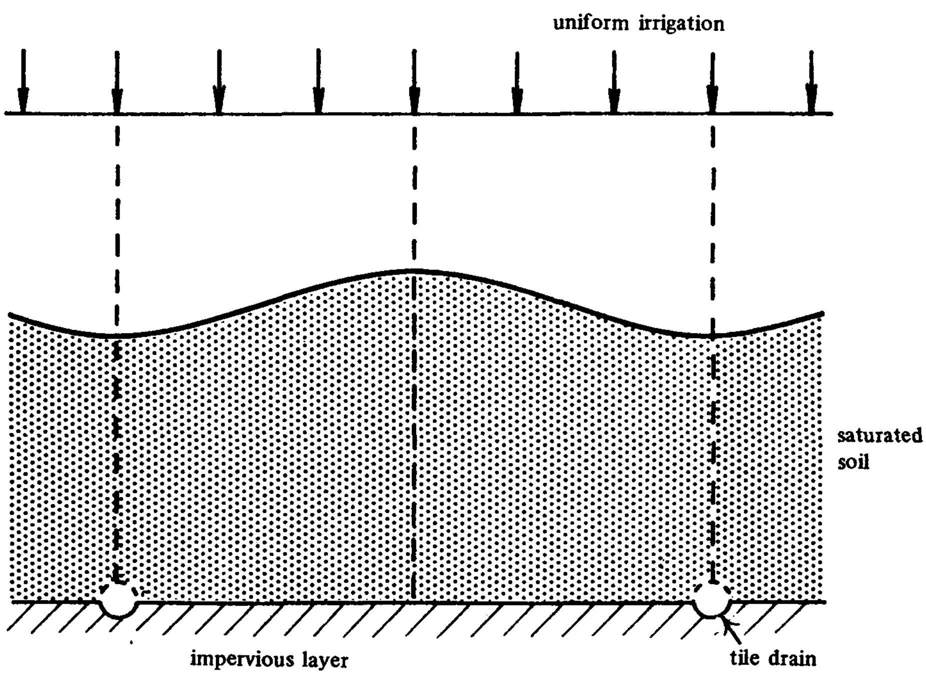

FIGURE 1

https://doi.org/10.1017/S1446788700025441 Published online by Cambridge University Press

Because of symmetry, only the unit cell involving the porous soil between a drain and the adjacent midpoint between the two drains (as shown in Figure 1) need to be considered. Since it is assumed that the soil is homogeneous and isotropic in a direction parallel to the axis of a drain, it is only necessary to consider the problem in two dimensions.

A further simplification is that the circular arc representing the boundary of the drain is replaced by a vertical slit in the side of the unit cell over the midpoint of the drain. The width h of the slit can be related to radius R of the drain by requiring that the flux of the simplified system be the same as that of the original system. The resulting net flow will not differ appreciably from the original one except in the vicinity of the drain.

In what follows we assume that the flux is sufficiently large so that the free surface does not intersect the surface of the drain. The alternative case does not differ mathematically from the simpler case of a ditch drained field, which can also be dealt with in the same manner as what follows.

3. Mathematical formulation

Let D be the open rectangle

$$\{(x, y) \in \mathbf{R}^2 \colon 0 < x < a, 0 < y < b\}$$

where $2a$ is the distance between the drains, and b is the thickness of the porous layer of soils. Assume the hydraulic conductivity is unity, so that the total flux q (which we will assume throughout to satisfy $q < a$) through a unit cell has units of length. Then under certain basic assumptions it is well known that the reduced potential (or the piezometric head) and the stream function, u and v respectively, which completely determine the flow network of the system, may be defined (see Bear [6], pages 256–257). In particular, given u and v it is possible to determine the flow region Ω, and the position of the free surface, represented by the equation $y = \varphi(x)$ (see Figure 2).

In the flow region, u and v are conjugate harmonic functions, with values on the boundary depending on external conditions, so for instance, on a no flux boundary, v must be constant, while on a surface of seepage, u is equal to the gravitational potential y. Thus the potential u will have the constant value y_1 along $[AS]$ and the value y along $[SH]$. If the radius of the drains is much less than the depth of the soil (b), there is little physical difference between the case of a full drain ($S = H$) and an empty drain ($S = A$), but we shall analyze the general situation. It can be shown (see Barnes *et al.* [5]) that there exists a value $q_* < a$ depending on the parameters a, b, h, y_1, such that for $q \geqslant q_*$ the region $\overline{D}$

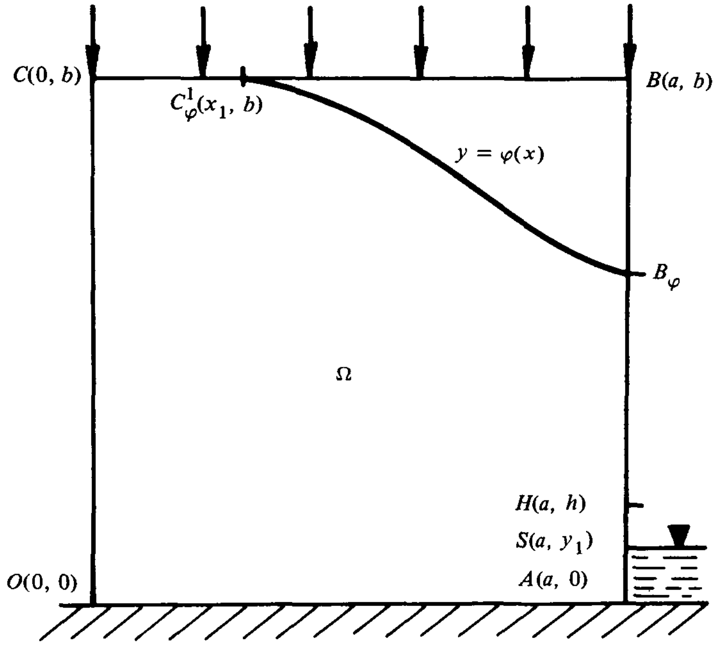

FIGURE 2a

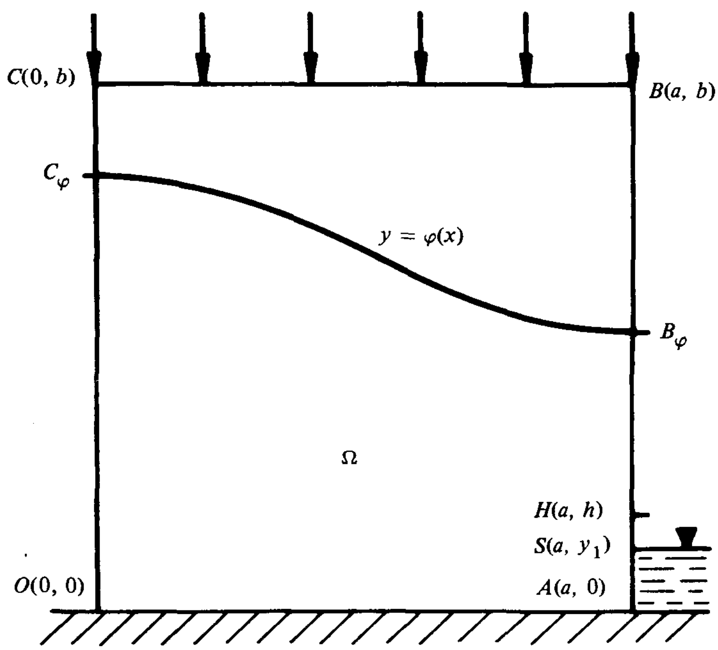

FIGURE 2b

https://doi.org/10.1017/S1446788700025441 Published online by Cambridge University Press

will be saturated and for $q < q_*$ there exists $d \in (0, a)$, which for fixed a, b, h, y_1 depends only on q so that $x_1 < d$ (see Figure 2(a)). We will henceforth assume $q < q_*$.

Formally we wish to solve

Problem A. Let a, b, h, q be positive real numbers, and let $0 \leqslant y_1 \leqslant h$. We seek $\{\varphi, \Omega, u, v\}$ such that

(3.1) φ is continuous on $[0, a]$; φ is strictly decreasing on $[x_1, a]$, $\varphi(x) = b$ for $x \in [0, x_1]$ if $x_1 > 0$ ($x_1 = 0$ in Figure 2(b));

(3.2) $\Omega = \{(x, y) \in D: 0 < x < a, 0 < y < \varphi(x)\}$;

(3.3) $u, v \in H^1(\Omega) \cap C(\bar{\Omega})$;

(3.4) $\partial u/\partial x + \partial v/\partial y = 0$ in Ω;

(3.5) $\partial u/\partial y - \partial v/\partial x = 0$ in Ω;

(3.6) $u = y$ on $\widehat{C_\varphi B_\varphi}$ or $u = y$ on $\widehat{C_\varphi^1 B_\varphi}$, $u = b$ on $[CC_\varphi^1]$;

(3.7) $u = y_1$ on $[AS]$ and $u = y$ on $[SH]$;

(3.8) $v = 0$ on $[OC_\varphi] \cup [OA]$ or $v = 0$ on $[OC] \cup [OA]$;

(3.9) $v = q$ on $[HB_\varphi]$;

(3.10) $v = qx/a$ on $\widehat{C_\varphi B_\varphi}$ or $\widehat{C_\varphi^1 B_\varphi}$.

3.1. Lemma. *The solution u, v of Problem* A *satisfies*

(3.11) $u(x, y) > y$,

(3.12) $qx/a > v(x, y) > 0$

for $(x, y) \in \Omega$.

Proof. The proof follows by a straightforward application of the maximum principle to the harmonic functions $u - y$, v and $qx/a - v$, respectively. The arguments are like those in Kinderlehrer and Stampacchia [12], pages 229–230.

4. The Baiocchi transformation: reformulation of the problem

The difficulty with Problem A is that the flow region is *a priori* unknown, so that the boundary conditions (3.6) and (3.10) are effectively non linear. Assume that a solution $\{\varphi, \Omega, u, v\}$ of Problem A exists. Following Baiocchi [1] (see Comincioli [7]), we can define for $P \in \bar{D}$,

$$w(P) = \int_{\mathcal{C}} (\tilde{u} - y)\, dy + \left(\frac{qx}{a} - \tilde{v} \right) dx \tag{4.1}$$

https://doi.org/10.1017/S1446788700025441 Published online by Cambridge University Press

where $\mathcal{C}$ is any "smooth" path connecting P to B in $\bar{D}$, and

$$(4.2)\qquad \tilde{u}(x,y)=\begin{cases} u(x,y) & \text{for } (x,y)\in\bar{\Omega},\\ y & \text{for } (x,y)\in\bar{D}\backslash\bar{\Omega}.\end{cases}$$

$$(4.3)\qquad \tilde{v}(x,y)=\begin{cases} v(x,y) & \text{for } (x,y)\in\bar{\Omega},\\ qx/a & \text{for } (x,y)\in\bar{D}\backslash\bar{\Omega}.\end{cases}$$

Observe that w is well defined since the integral in equation (4.1) is path independent by equation (3.4). Furthermore it follows that w has the properties:

(4.4) $w\in H^2(D)\cap C^1(\bar{D})$;

(4.5) $w>0$ in Ω, $w=0$ on $\bar{D}\backslash\bar{\Omega}$;

(4.6) $\Delta w=(1-q/a)\chi_\Omega$ in D;

(4.7) $\partial w/\partial y=0$ on $[CB]$;

(4.8) $\partial w/\partial y=h-y$ on $[AH]$;

(4.9) $\partial w/\partial x=0$ on $[OC]\cup[HB]$;

(4.10) $\partial w/\partial x=-qx/a$ on $[OA]$.

It is evident from properties (4.4), (4.7)–(4.10) that the values of w are determined by the boundary conditions on $[OA]\cup[AH]$ only up to an additive constant. Let us set $\alpha=w(a,0)$. In our presentation we will write w_α to display the dependence of w on α. For each real α, let g_α be a function defined on D so that

$$(4.11)\qquad g_\alpha(x,0)=\alpha+\frac{q}{2a}(a^2-x^2)\quad\text{if } 0\leqslant x\leqslant a;$$

$$(4.12)\qquad g_\alpha(a,y)=\begin{cases}\alpha+\frac12 y^2-y_1y & \text{if } 0\leqslant y\leqslant y_1,\\ \alpha-\frac12 y_1^2 & \text{if } y_1\leqslant y\leqslant h;\end{cases}$$

$$(4.13)\qquad g_\alpha(x,b)=0\quad\text{if } d\leqslant x\leqslant a,$$

where d is the number defined in Section 3.

For example,

$$g_\alpha(x,y)=\begin{cases}\alpha+\frac{q}{2a}(a^2-x^2)+\frac12 y^2-y_1y & \text{if } 0\leqslant x\leqslant a, 0\leqslant y\leqslant 1,\\ \alpha+\frac{q}{2a}(a^2-x^2)-\frac12 y_1^2 & \text{if } 0\leqslant x\leqslant a, y_1\leqslant y\leqslant h,\\ \left\{\alpha+\frac{q}{2a}(a^2-x^2)-\frac12 y_1^2\right\}f(y) & \text{if } 0\leqslant x\leqslant a, h\leqslant y\leqslant b\end{cases}$$

where $f(y)=1-((y-h)/(b-h))^2$ if $h\leqslant y\leqslant b$, defines a function $g_\alpha\in C^1(\bar{D})$ which satisfies (4.11)–(4.13). Let $\Gamma_D=[OA]\cup[AH]\cup[EB]$ and $\Gamma_N=\partial D\backslash\Gamma_D$ where E is the point (d,b). Let

$$K_\alpha^+=\left\{v\in H^1(D): v\geqslant 0 \text{ a.e. in } D, v=g_\alpha \text{ on } \Gamma_D\right\}.$$

For $\alpha \geqslant \frac{1}{2}y_1^2$, K_α^+ is a non-empty closed convex subset of $H^1(D)$ and w_α is a solution of Problem B_α (see Baiocchi *et al.* [4], page 21).

Problem B_α. Find $w \in K_\alpha^+$ so that

$$\int_D \left\{ \nabla w . (\nabla v - \nabla w) + \left(1 - \frac{q}{a}\right)(v - w) \right\} dx\, dy \geqslant 0 \tag{4.13}$$

for all $v \in K_\alpha^+$.

4.1. REMARK. For each $\alpha \geqslant \frac{1}{2}y_1^2$, Problem B_α admits a unique solution. See Kinderlehrer and Stampacchia [12] for coercivity consult Donaghue [8].

5. Uniqueness

5.1. THEOREM. *There is at most one solution to Problem* A.

PROOF. Let Q be the set of $\alpha \geqslant \frac{1}{2}y_1^2$ with the property that Problem A has a solution with the property that via the Baiocchi transform it gives rise to a solution of Problem B_α. It is sufficient in the light of Remark 4.1 to show that Q has at most one element. Assume to the contrary that $\alpha, \beta \in Q$ with $\alpha > \beta$. Let $\{\varphi_\alpha, \Omega_\alpha, u_\alpha, v_\alpha\}$ and $\{\varphi_\beta, \Omega_\beta, u_\beta, v_\beta\}$ be the corresponding solutions to Problem A and let w_α and w_β be their Baiocchi transforms. Put $w_1 = w_\alpha - w_\beta$ and

$$U = \{(x, y) \in D: w_1(x, y) < 0\}.$$

If $U \neq \varnothing$, then since $w_\alpha \geqslant 0$ on $\bar{D}$, $w_\beta > 0$ in U. Then in $U \subset \Omega_\beta$,

$$\Delta w_1 = \Delta w_\alpha - \left(1 - \frac{q}{a}\right) \leqslant 0$$

and

$$\begin{aligned} w_1 &= 0 && \text{on } \partial U \cap D; \\ w_1 &\geqslant 0 && \text{on } \partial U \cap \Gamma \text{ as } \alpha > \beta; \\ \frac{\partial w_1}{\partial n} &= 0 && \text{on } \partial U \cap (\partial D \backslash \Gamma), \end{aligned}$$

where $\Gamma = [OA] \cup [AH]$. So $w_1 \geqslant 0$ (see Kinderlehrer and Stampacchia [12], page 245), a contradiction. We thus conclude that $w_\alpha \geqslant w_\beta$ and hence that $\Omega_\beta \subset \Omega_\alpha$. Now let

$$V = \{(x, y) \in D: \tilde{u}_\alpha(x, y) < \tilde{u}_\beta(x, y)\},$$

then $V \subset \Omega_\beta$ as $\tilde{u}_\beta(x, y) = y \leqslant \tilde{u}_\alpha(x, y)$ (Lemma 3.1) for $(x, y) \in D \backslash \Omega_\beta$. Whence $u = \tilde{u}_\alpha - \tilde{u}_\beta = u_\alpha - u_\beta$ is harmonic on V and so V is empty as before.

We conclude that $\tilde{u}_\alpha \geqslant \tilde{u}_\beta$, and likewise that $\tilde{v}_\alpha \leqslant \tilde{v}_\beta$. We claim now that

$$\text{(5.1)} \qquad \tilde{v}_\alpha(a, y) = \lim_{x \to a-} (a - x)^{-1} \int_0^y (\tilde{u}_\alpha(x, t) - \tilde{u}_\alpha(a, t))\, dt$$

for $0 \leqslant y \leqslant b$. In fact

$$\int_0^y (\tilde{u}_\alpha(x, t) - \tilde{u}_\alpha(a, t))\, dt = w_\alpha(x, 0) - w_\alpha(a, 0) - \int_x^a \left(\frac{q\xi}{a} - \tilde{v}_\alpha(\xi, y) \right) d\xi$$

from which (5.1) follows using $\tilde{v}_\alpha \in C(\overline{D})$ and $w_\alpha \in C^1(\overline{D})$ and (4.10).

Likewise

$$\text{(5.2)} \qquad \tilde{u}_\alpha(x, 0) = -y_1 + \lim_{y \to 0+} \frac{1}{y} \int_x^a \tilde{v}_\alpha(\xi, y)\, d\xi$$

for $0 \leqslant x \leqslant a$ follows from

$$\int_x^a \tilde{v}_\alpha(\xi, y)\, d\xi = \int_0^y (\tilde{u}_\alpha(x, t) - t)\, dt + w_\alpha(a, 0) - w_\alpha(a, y).$$

Combining this with our previous results, we conclude that

$$\text{(5.3)} \qquad \tilde{v}_\alpha(a, y) = \tilde{v}_\beta(a, y), \qquad 0 \leqslant y \leqslant h,$$

and

$$\text{(5.4)} \qquad \tilde{u}_\alpha(x, 0) = \tilde{u}_\beta(x, 0), \qquad 0 \leqslant x \leqslant a.$$

The following is a consequence of the fact that w satisfies (4.4)–(4.10) and $v \in C^1(\overline{D})$,

$$\text{(5.5)} \qquad \int_{\partial D} (v - w) \frac{\partial w}{\partial n}\, ds = \left(1 - \frac{q}{a}\right) \int_\Omega (v - w)\, dx\, dy + \int_D \nabla w . (\nabla v - \nabla w)\, dx\, dy.$$

We apply (5.5) first with $v = w_\beta$, $w = w_\alpha$, $\Omega = \Omega_\alpha$ and then with $v = w_\alpha$, $w = w_\beta$, $\Omega = \Omega_\beta$ and add the results. Now using

$$\frac{\partial w_\alpha}{\partial n} = \frac{\partial w_\beta}{\partial n} \quad \text{on } \Gamma$$

which is a consequence of (5.3) and (5.4) we obtain

$$\text{(5.6)} \qquad \int_D |\nabla w_\beta - \nabla w_\alpha|^2\, dx\, dy = \left(1 - \frac{q}{a}\right) \int_{\Omega_\alpha \backslash \Omega_\beta} (w_\beta - w_\alpha)\, dx\, dy \leqslant 0.$$

It now follows that $w_\alpha = w_\beta$ and hence that $\alpha = \beta$, a contradiction.

6. Existence

The existence of a solution to Problem A will be established in two steps:

(i) there exists $\alpha \geqslant \frac{1}{2}y_1^2$ so that the solution w_α of Problem B_α belongs to $C^1(\bar{D})$;

(ii) if $w_\alpha \in C^1(\bar{D})$ is a solution to Problem B_α with $\alpha \geqslant \frac{1}{2}y_1^2$ then we can invert the Baiocchi transform to obtain a solution to Problem A.

Let

$$K_\alpha = \{v \in H^1(D): v = g_\alpha \text{ on } \Gamma_D\}$$

where Γ_D is defined in section 4; let $v^+ = \max\{0, v\}$. Then for each $\alpha \geqslant \frac{1}{2}y_1^2$ it can be shown (see Baiocchi *et al.* [4], page 54) that Problem B_α is equivalent to

Problem C_α. Find $w \in K_\alpha$ so that

$$\int_D \left\{ \nabla w . (\nabla v - \nabla w) + \left(1 - \frac{q}{a}\right)(v^+ - w^+) \right\} dx\, dy \geqslant 0 \tag{6.1}$$

for all $v \in K_\alpha$.

6.1. THEOREM. *For each real α, a solution w_α to Problem C_α satisfies*

(6.2) $\Delta w_\alpha \in L^\infty(D)$; $0 \leqslant \Delta w_\alpha \leqslant (1 - q/a)$ *a.e. in* D;

(6.3) $\partial w_\alpha / \partial n = 0$ *on* Γ_N *in the sense of* $(H_{00}^{1/2}(\Gamma_N))'$;

(6.4) $w_\alpha \in W^{1,p}(D) \cap H^{1+\sigma}(D)$, $1 \leqslant p \leqslant 4$, $\sigma < \frac{1}{2}$.

In particular, $w_\alpha \in C(\bar{D})$ *and on setting*

$$\begin{aligned} \Omega_\alpha &= \{(x, y) \in D: w_\alpha(x, y) > 0\}, \\ \Omega_\alpha^- &= \{(x, y) \in D: w_\alpha(x, y) < 0\} \end{aligned} \tag{6.5}$$

it follows that Ω_α and Ω_α^- are open and moreover

$$\begin{aligned} \Delta w_\alpha &= \left(1 - \frac{q}{a}\right) \quad \text{in } \Omega_\alpha \quad (\text{in the sense of } \mathcal{D}'(\Omega_\alpha)), \\ \Delta w_\alpha &= 0 \quad \text{in } \Omega_\alpha^- \quad (\text{in the sense of } \mathcal{D}'(\Omega_\alpha^-)). \end{aligned} \tag{6.6}$$

PROOF. This follows as in Theorem 2.2 in Baiocchi *et al.* [4], page 55 and Grisvard [10].

6.2. THEOREM. *There exists a bounded continuous function F: $R \to R$ and two real constants m, n such that a solution w_α of Problem C_α belongs to $W^{2,p}(D)$ with* $2 < p < 4$ (*and hence on $C^1(\bar{D})$*) *if and only if*

$$F(\alpha) + m\alpha + n = 0. \tag{6.7}$$

https://doi.org/10.1017/S1446788700025441 Published online by Cambridge University Press

PROOF. Consider the map T defined on $W^{2,p}(D)$, $2 < p < 4$, by

$$Tu = \left(-\Delta u,\ u|_{\Gamma_D},\ \frac{\partial u}{\partial n}\,|_{\Gamma_N}\right). \tag{6.8}$$

The range of T is a subspace of

$$X_p = L^p(D) \times W_{p,D} \times W_{p,N}$$

where

$$W_{p,D} = W^{2-1/p,p}(]EB[\ \cup\ ]OA[\ \cup\ ]AH[),$$
$$W_{p,N} = W^{1-1/p,p}(]OC[\ \cup\ ]HB[\ \cup\ ]CE[).$$

By Theorem 1, Grisvard [10], T is a Fredholm operator with index $(T) = -1$. Since dimker $T = 0$, we conclude that there exists a continuous linear functional L on X_p such that given $f \in L^p(D)$, $g \in W_{p,D}$, $h \in W_{p,N}$ and $u \in H^1(D)$ with $Tu = (f, g, h)$ then $u \in W^{2,p}(D)$ with $2 < p < 4$ if and only if

$$L(f, g, h) = 0; \tag{6.9}$$

that is, there exists $\Phi \in L^{p'}(D)$ $(1/p + 1/p' = 1)$, $\Psi_1 \in [W_{p,D}]'$, $\Psi_2 \in [W_{p,N}]'$ such that (6.9) holds if and only if

$$\langle \Phi, f \rangle + \langle \Psi_1, g \rangle + \langle \Psi_2, h \rangle = 0. \tag{6.10}$$

In fact Φ, Ψ_1 and Ψ_2 are independent of p for $2 < p < 4$. Applying (6.10) to $f = -\Delta w_\alpha$, $g = g_\alpha$ and $h = 0$, where w_α is a solution to Problem C_α we conclude that $w_\alpha \in W^{2,p}(D)$ for $2 < p < 4$ if and only if

$$\langle \Phi, -\Delta w_\alpha \rangle + \langle \Psi_1, g_\alpha \rangle = 0. \tag{6.11}$$

Setting

$$g^* = \begin{cases} 1 & \text{on } [OA] \cup [AH], \\ 0 & \text{on } [EB] \end{cases}$$

and $g = g_\alpha - \alpha g^*$ one has

$$\langle \Psi_1, g_\alpha \rangle = \langle \Psi_1, g \rangle + \alpha \langle \Psi_1, g^* \rangle.$$

Thus (6.11) is the same as (6.7) if we set

$$F(\alpha) = \langle \Phi, -\Delta w_\alpha \rangle, \qquad m = \langle \Psi_1, g^* \rangle, \qquad n = \langle \Psi_1, g \rangle$$

where m, n do not depend on α. The proof that F is bounded and continuous follows as in Baiocchi *et al.* [1], pages 57–58.

6.3. LEMMA. *The coefficient m in* (6.7) *is non-zero.*

PROOF. The proof is a modification of Lemma 2.1 in Baiocchi *et al.* [1], page 59. In fact $m = 0$ if and only if the solution $Z \in H^1(D)$ of

$$\Delta Z = 0 \quad \text{in} \quad D, \qquad Z|_{\Gamma_D} = g^*, \qquad \frac{\partial Z}{\partial n}|_{\Gamma_N} = 0$$

belongs to $W^{2,p}(D)$ for any $2 < p < 4$ and hence to $C^1(\bar{D})$. We show that $Z \notin C^1(\bar{D})$. Since Z is not constant, $0 \leqslant g^* \leqslant 1$ on Γ_D we have $Z < 1$ on Γ_N by the Hopf maximum principle (Hopf [11] or Gilbarg and Trudinger [9], Lemma 3.4). Now in the rectangle $G = \{(x, y) \in \mathbf{R}^2\colon 0 < x < 2a,\ h < y < b\}$ the function $\tilde{Z}$ defined by $\tilde{Z}(x, y) = Z(a - |x - a|, y)$ for $(x, y) \in \bar{G}$, satisfies $\Delta\tilde{Z} = 0$ in G; $\tilde{Z}$ has a maximum value at the point (a, h) where $\tilde{Z} = 1$. By the Hopf maximum principle the outward normal derivative to G at (a, h) must be positive, that is

$$\lim_{\varepsilon \to 0+} \frac{\partial Z}{\partial y}(a, h + \varepsilon) < 0$$

while obviously

$$\lim_{\varepsilon \to 0+} \frac{\partial Z}{\partial y}(a, h - \varepsilon) = 0$$

as $Z = 1$ on $\{a\} \times [0, h]$. Thus $Z \notin C^1(\bar{D})$ and the result follows.

6.4. THEOREM. *There exists a real number $\gamma \geqslant \frac{1}{2}y_1^2$ such that Problem C_γ has a solution w_γ which belongs to $W^{2,p}(D)$ for any $2 < p < 4$, and hence to $C^1(\bar{D})$.*

PROOF. That Problem C_γ has a unique solution for each real γ is well known (see Kinderlehrer and Stampacchia [12]) as K_γ is a non-empty, closed convex subset of $H^1(D)$. By Theorem 6.2 and Lemma 6.3 there exists a root γ of Equation (6.7); in other words the solution w_γ to Problem C_γ belongs to $W^{2,p}(D)$ for $2 < p < 4$ and as a consequence to $C^1(\bar{D})$. We now claim that $\gamma \geqslant \frac{1}{2}y_1^2$. Suppose to the contrary that $\gamma < \frac{1}{2}y_1^2$, then

$$w_\gamma(H) = \min\{w_\gamma(P)\colon P \in \Gamma_D\} \tag{6.12}$$

and

$$\frac{\partial w_\gamma}{\partial x}(H) = 0 \tag{6.13}$$

by continuity. Let Ω_H be the component of Ω_γ^- which contains in its boundary a neighbourhood of H in $]AB[$. Then $\Delta w_\gamma = 0$ in Ω_H (equation (6.6)). The negative minimum of w_γ on Ω_H is attained at the point H for $w_\gamma = 0$ on $\partial\Omega_H \cap D$ and $\partial w\gamma/\partial n = 0$ on $\partial\Omega_H \cap \Gamma_N$. But then (6.13) contradicts the conclusion of the Hopf maximum principle which would require that $\partial w_\gamma/\partial x(H) < 0$.

6.5. Definition. In the remainder of this section we shall put $w = w_\gamma$ where w_γ is defined in Theorem 6.4. Note that by virtue of the equivalence of Problems B_γ and C_γ we have $w \geqslant 0$ in $\bar{D}$. We will also set $\Omega = \Omega_\gamma$.

6.6. Lemma. *One has*

$$\frac{\partial w}{\partial x} \leqslant 0 \quad \textit{and} \quad \frac{\partial w}{\partial y} \leqslant 0 \quad \textit{in } \bar{D}.$$

Proof. Since $w \in C^1(\bar{D})$,

$$\Omega_y = \left\{ P \in D\colon \frac{\partial w}{\partial y}(P) > 0 \right\}$$

is an open subset of Ω; for if $P \in \Omega_y \backslash \Omega$ then $w(P) = 0$ and w is increasing with respect to y in a neighbourhood of P, but then w would take on negative values. If $\Omega y \neq \varnothing$, let

$$\mu = \max\left\{ \frac{\partial w}{\partial y}(P)\colon P \in \bar{D} \right\} > 0$$

then $\mu = \partial w/\partial y(P_0)$ for some $P_0 \in \bar{\Omega}_y$ and by the maximum principle $P_0 \in \partial\Omega_y$.

$P_0 \notin [AH] \cup [CB]$ as $\partial w/\partial y \leqslant 0$ there; if $P_0 \in]OC[$ then $\partial\Omega_y \cap]OC[$ contains a neighbourhood of P_0 in $]OC[$ and hence using results like Lemma 6.18 of Gilbarg and Trudinger [9] $\partial(\partial w/\partial y)/\partial n(P_0) = \partial(\partial w/\partial y)/\partial x(P_0) = 0$ contradicting the Hopf maximum principle which would have required $\partial(\partial w/\partial y)/\partial n(P_0) > 0$; likewise $P_0 \notin]HB[$ and $P_0 \notin]OA[$ for then $\partial(\partial w/\partial y)/\partial n(P_0) = -\partial^2 w/\partial y^2(P_0) = -1 < 0$; $P_0 \neq 0$ for there w attains its maximum and so $\partial w(O)/\partial y \leqslant 0$.

As $P_0 \notin \partial\Omega_y \cap D$ we must conclude that $\Omega_y = \varnothing$. Also

$$\Omega_x = \left\{ P \in D\colon \frac{\partial w}{\partial x}(P) > 0 \right\}$$

is an open subset of Ω. If $\Omega_x \neq \varnothing$ let

$$\nu = \max\left\{ \frac{\partial w}{\partial x}(P)\colon P \in \bar{D} \right\} > 0;$$

then $\nu = \partial w/\partial x(P_1)$ for some $P_1 \in \partial\Omega_x$. As above $P_1 \notin [OC] \cup [HB] \cup [OA] \cup [CB] \cup]AS[$. Suppose that $P_1 \in]SH[$. Then there exists a point $Q \in]P_1H[$ such that $[P_1Q] \subset \partial\Omega_x$. As $\partial w/\partial y = 0$ on $[SH]$ and $\partial w/\partial y \leqslant 0$ in $\bar{D}$ we conclude that $\partial(\partial w/\partial x)/\partial y = \partial(\partial w/\partial y)/\partial x \geqslant 0$ on $]P_1Q[$, hence $\partial w/\partial x = \nu$ on $[P_1Q]$ by the definition of ν. Whence

$$w(x, y) = \gamma + \nu(x - a) + \frac{1}{2}\left(1 - \frac{q}{a}\right)(x - a)^2$$

for (x, y) in a connected component of Ω which contains $[OA]$ in its boundary. This follows from the uniqueness of the Cauchy problem

$$\Delta w = (1 - q/a),$$
$$w = \gamma \quad \text{on } [P_1 Q],$$
$$\frac{\partial w}{\partial x} = \nu \quad \text{on } [P_1 Q]$$

in that component of Ω. Since this cannot be true and as $P_1 \notin \partial\Omega_x \cap D$ we again conclude that $\Omega_x = \varnothing$ and the lemma is proved.

6.7. LEMMA. *$\partial w/\partial y = 0$ on $[EB]$.*

PROOF. As in Lemma 2.5 of Baiocchi *et al.* [4], page 28, the function

$$x \to \frac{\partial w}{\partial y}(x, b)$$

is non decreasing on $[d, a]$. This is a corollary to Lemma 6.6. As $\partial w/\partial y(d, b) = 0$ and $\partial w/\partial y \leqslant 0$ in $\overline{D}$ by Lemma 6.6, the conclusion follows.

As in Baiocchi *et al.* [4], page 28, we set for $P_0 = (x_0, y_0) \in D$,

$$(6.14) \qquad \begin{cases} Q^+_{P_0} = \{(x, y) \in D \colon x > x_0, y > y_0\}, \\ Q^-_{P_0} = \{(x, y) \in D \colon x < x_0, y < y_0\}. \end{cases}$$

6.8. LEMMA. *For every $P \in \overline{D}\backslash\overline{\Omega}$ one has $\overline{Q^+_P} \subset \overline{D}\backslash\overline{\Omega}$; for every $P \in \overline{D} \cap \partial\Omega$ one has $\overline{Q^-_P} \subset \overline{\Omega}$. Furthermore $\partial\Omega \cap D$ contains no vertical or horizontal line segments and $]EB]$ does not intersect $\overline{\Omega}$.*

PROOF. See Lemma 2 of Comincioli [7], page 234 which is based on Baiocchi *et al.* [4], pages 28–29 and Baiocchi [2], pages 118–119. The proofs make use of Lemmas 6.6 and 6.7 above. If $]EB] \cap \overline{\Omega} \neq \varnothing$ then there exists $Q \in\ ]EB]$ so that $[EQ] \subset \partial\Omega$. However on $[EQ]$ we have $w = \partial w/\partial y = 0$ and hence $w = 0$ in Ω which is false.

6.9. LEMMA. *Ω is a set of the form* (3.2) *with φ satisfying* (3.1).

PROOF. This follows from the above results with the same arguments as in Baiocchi *et al.* [4].

6.10. THEOREM. *Let $\{\varphi, \Omega, u, v\}$ be defined by the following*:
(6.15) $\Omega = \{(x, y) \in D \colon w(x, y) > 0\}$;

(6.16) $\varphi(x) = \sup\{y \in (0, b)\colon\ (x, y) \in \Omega\}$; $\varphi(0) = \lim_{x\to 0+} \varphi(x)$, $\varphi(a) = \lim_{x\to a+} \varphi(x)$;
(6.17) $u = U|_{\bar\Omega}$, $U(x, y) = y - \partial w/\partial y(x, y)$ *if* $(x, y) \in \bar{D}$;
(6.18) $v = V|_{\bar\Omega}$, $V(x, y) = qx/a + \partial w/\partial x(x, y)$ *if* $(x, y) \in \bar{D}$;
then $\{\varphi, \Omega, u, v\}$ *is a solution to Problem* A.

PROOF. This theorem summarizes the results of this section.

7. Notation

For any subset G of $\mathbf{R}^2$, $\bar{G}$ will denote its closure, ∂G its boundary. For two sets G_1 and G_2, $G_1 \backslash G_2 = \{g \in G_1\colon g \notin G_2\}$. For $P, Q \in \mathbf{R}^2$, $[PQ]$ ($]PQ[$) will denote the closed (open) line segment joining P to Q. In equation (3.6) $\widehat{B_\varphi C_\varphi}$ denotes the set of points $\{(x, \varphi(x))\colon 0 < x < a\}$. χ_G will denote the characteristic function of G, that is, $\chi_G(P) = 1$ if $P \in G$, $\chi_G(P) = 0$ if $P \notin G$.

For an open subset G of $\mathbf{R}^2$; $C(G)$ ($C(\bar{G})$), $C^1(G)$ ($C^1(\bar{G})$) will denote the space of functions continuous on G ($\bar{G}$) and functions continuously differentiable on G ($\bar{G}$) respectively. We denote by $L^p(\Omega)$ ($1 \leqslant p \leqslant \infty$) the usual space of the (classes of) real valued functions, defined a.e. (almost everywhere) on Ω, measurable and p-summable on Ω (or a.e. bounded on Ω if $p = \infty$); $W^{k,p}(\Omega)$ ($k = 1, 2, \ldots$; $1 \leqslant p \leqslant \infty$) denotes the Banach space (Sobolev space) of all elements of $L^p(\Omega)$ whose derivatives (in the sense of distributions) up to order k are again elements of $L^p(\Omega)$ and equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left\{ \sum_{\substack{l\geqslant 0,\, h\geqslant 0\\ h+l\leqslant k}} \int_\Omega \left| \frac{\partial^{l+h} u}{\partial x^l \partial y^h}(x, y) \right|^p dx\, dy \right\}^{1/p}.$$

We denote by $H^\alpha(\Omega)$ (α real $\geqslant 0$) the space of restrictions to Ω of functions of $H^\alpha(\mathbf{R}^2)$, with the quotient norm. These spaces and the space $H^{1/2}_{00}(\Gamma_N)$ which appears in equation (6.3) are discussed in Lions and Magenes [13], Chapter 1. $\mathcal{D}'(\Omega)$ denotes the space of (Schwartz) distributions on Ω. Given a Banach space X, X' will denote its continuous dual and $\langle\ ,\ \rangle$ will denote the dual pairing between X and X'. For a function u defined on a subset of $\mathbf{R}^2$, $u\,|\,\Omega$ will denote its restriction to the subset Ω. For $u \in H^1(\Omega)$, Ω an open subset of $\mathbf{R}^2$, $\partial u/\partial n$ will denote the outward normal derivative (understood if necessary in a distributional sense—see Lions and Magenes [13], Chapter 1). For $u, v \in H^1(\Omega)$,

$$\int_\Omega \nabla u . \nabla v\, dx\, dy \equiv \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx\, dy.$$

For $u \in \mathcal{D}'(\Omega)$, Δu denotes the distribution

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

References

[1] C. Baiocchi and E. Magenes, 'On free-boundary problems associated with the flow of a liquid through porous materials,' *Uspehi Mat. Nauk* **29**, 2(1974), 50–69.

[2] C. Baiocchi, 'Su un problema di frontiera libera connesso a questioni di idraulica,' *Ann. Mat. Pura Appl.* (IV) **92** (1972), 107–127.

[3] C. Baiocchi, 'Problèmes à frontière libre en hydraulique,' *C. R. Acad. Sci. Paris* **278** (1974), 1201–1204.

[4] C. Baiocchi, V. Comincioli, E. Magenes and G. A. Pozzi, 'Free problems in the theory of fluid flow through porous media: existence and uniqueness theorems,' *Ann. Mat. Pura. Appl.* (IV) **99** (1973), 1–82.

[5] C. Barnes, J. Knight and John van der Hoek, (1982).

[6] J. Bear, *Dynamics of fluids in porous media* (American Elsevier, to appear).

[7] V. Comincioli, 'A theoretical and numerical approach to some free boundary problems,' *Ann. Mat. Pura Appl.* (IV) **100** (1974), 211–238.

[8] W. F. Donaghue, 'A coerciveness inequality,' *Ann. Mat. Pura Appl.* (III) **20** (1966), 589–593.

[9] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order* (Springer Verlag, 1977).

[10] P. Grisvard, 'Behaviour of the solutions of an elliptic boundary value problem in a polygonal domain,' *Numerical solution of partial differential equations* (III) (Synspade 1975) B. Hubbard, editor, 207–274 (Academic Press, 1975).

[11] E. Hopf, 'A remark on linear elliptic differential equations of second order,' *Proc. Amer. Math. Soc.* **3** (1952), 791–793.

[12] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and applications* (Academic Press, 1980).

[13] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Vol. I (Springer Verlag, 1972).

[14] J. C. Bruch, Jr. and J. M. Sloss, 'A variational inequality method applied to free surface seepage from a triangular ditch,' *Water Resources Research* **14** (1978), 119–124.

[15] J. C. Bruch, Jr., 'A numerical solution of an irrigation flowfield,' *Internat. J. Numer. Anal. Methods Geomech.* **3** (1979), 23–36.

Department of Pure Mathematics
University of Adelaide
Adelaide, S. A. 5000
Australia

CSIRO Division of Soils
Waite Road
Urbrae, S. A. 5064
Australia

CSIRO Division of Mathematics and Statistics
PO Box 218
Linfield, N.S.W. 2070
Australia