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THREE POSITIVE PERIODIC SOLUTIONS FOR DYNAMIC EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT AND IMPULSE ON TIME SCALES*

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Abstract. In this paper, by using the Leggett–Williams fixed point theorem, the existence of three positive periodic solutions for differential equations with piecewise constant argument and impulse on time scales is investigated. Some easily verifiable sufficient criteria are established. Finally, an example is given to illustrate the results.

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1. Introduction. Impulsive differential equations, which arise in physics, population dynamics, economics, etc., are important mathematical tools for a better understanding of many real-world models, we refer the reader to [**1**–**5**] and the references therein. The study of differential equations on time scales, which has been created in order to unify the study of differential and difference equations, is an area of mathematics that has recently gained a lot of attention, moreover, many results on this issue have been well documented in the monographs [**6**–**8**]. The study of differential equations with piecewise constant arguments (EPCA) was initiated by Aftabizadeh and Wiener [**9**]. They observed that the change of sign in the argument deviation leads not only to interesting periodic properties but also to complications in the asymptotic and oscillatory behaviour of solutions. Various qualitative behaviours of solutions for EPCA have been investigated by many authors (see e.g. Refs. [**9**–**17**]).

To the best of the authors' knowledge, there have been no results about the existence of multiple solutions of impulsive differential equations with piecewise constant arguments and parameters. In this paper, by using the Leggett–Williams multiple fixed point theorem, we shall consider the following equation on time scales:

$$\begin{cases} x^{\Delta}(t) = -A(t)x^{\sigma}(t) + \lambda f\big(t, x(t), x(\beta(t))\big), \; t \in \mathbb{T}, \; t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \; t = t_k, \; k \in \mathbb{Z}, \end{cases} \tag{1}$$

where $A(t) = \operatorname{diag}[a_1(t), a_2(t), \dots, a_n(t)], f = (f_1, f_2, \dots, f_n)^T, \lambda > 0$ is a positive parameter; $\beta(t) = t_{k-1}$, if $t_{k-1} \leq t < t_k, k \in \mathbb{N}, t \in \mathbb{T}, \mathbb{T}$ is an ω-periodic time scale. For each interval I of $\mathbb{R}$, we denote $I_{\mathbb{T}} = I \cap \mathbb{T}$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and the left limits of $x(t_k)$ in the sense of time scales; in addition, if t_k is right-scattered, then $x(t_k^+) = x(t_k)$, whereas, if t_k is left-scattered, then $x(t_k^-) = x(t_k)$. There exists a positive

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integer p such that $t_{k+p} = t_k + \omega$, $I_{k+p} = I_k$, $k \in \mathbb{Z}$. Without loss of generality, we also assume that $[0, \omega]_{\mathbb{T}} \cap \{t_k, k \in \mathbb{Z}\} = \{t_1, t_2, \ldots, t_q\}$.

Throughout this paper, we assume that

(H_1) $a_i \in C(\mathbb{T}, \mathbb{R}_+)$ is ω-periodic, $i = 1, 2, \ldots, n$;

(H_2) $f \in C(\mathbb{T} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+,\ \mathbb{R}^n_+)$ is ω-periodic with respect to the first variable and $I_k \in C(\mathbb{R}^n_+,\ \mathbb{R}^n_+)$, $k \in \mathbb{Z}$.

2. Preliminaries. In this section, we shall recall some definitions, and state some preliminary results.

Definition 2.1. A function $x(t)$ is a solution of (1.1) on $\mathbb{T}$ if:

(i) $x(t)$ is continuous on $\mathbb{T}$;

(ii) the derivative $x^{\Delta}(t)$ exists at each point $t \in \mathbb{T}$, with the possible exception of the points t_{k-1}, $k \in \mathbb{Z}$, where one-sided derivatives exist;

(iii) Equation (1.1) is satisfied on each interval $[t_{k-1}, t_k)$, $k \in \mathbb{Z}$.

Definition 2.2. [**7**] A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},\ \ \rho(t) := \sup\{s \in \mathbb{T} : s < t\} \text{ and } \mu(t) = \sigma(t) - t.$$

Definition 2.3. [**7**] For $x : \mathbb{T} \to \mathbb{R}$, then we define the delta derivative of $x(t)$, $x^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighbourhood U of t such that

$$|[x(\sigma(t)) - x(t)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

Definition 2.4. [**7**] If $X^{\Delta}(t) = x(t)$, then we define the delta integral by

$$\int_a^t x(s)\Delta s = X(t) - X(a).$$

Definition 2.5. [**18**] Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period ω if there exists a natural number n such that $\omega = np$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$ and ω is the smallest number such that $f(t + \omega) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $\omega > 0$ if ω is the smallest positive number such that $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$.

Definition 2.6. [**7**] A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$ is the graininess function. If p is regressive and right-dense continuous function, then the generalized exponential function e_p is defined by

$$e_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\},$$

for $s, t \in \mathbb{T}$, with the cylinder transformation given by

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h}, & \text{if } h > 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q).$$

Then the generalized exponential function has the following properties.

LEMMA 2.1. [7] *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, \sigma(s)) = \frac{e_p(t,s)}{1+\mu(s)p(s)}$;
(iv) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$;
(v) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(vi) $e_p(t, s)e_p(s, r) = e_p(t, r)$.

LEMMA 2.2. *The function $x(t)$ is an ω-periodic solution of (1.1) if and only if $x(t)$ is an ω-periodic solution of the following system:*

$$x(t) = \lambda \int_t^{t+\omega} G(t, s)f(s, x(s), x(\beta(s)))\Delta s + \sum_{k:t_k \in [t,t+\omega]} G(t, t_k)I_k(x(t_k)),$$

where

$$G(t, s) = \text{diag}[G_1(t, s), G_2(t, s), \dots, G_n(t, s)],$$

and

$$G_i(t, s) = \frac{e_{\ominus a_i}(t, s)}{e_{\ominus a_i}(0, \omega) - 1}, \quad s \in [t, t + \omega]_{\mathbb{T}}, \ i = 1, 2, \dots, n.$$

Proof. First, we prove the necessity. If $x(t) = (x_1(t), \dots, x_n(t))^T$ is a solution of system (1.1), then

$$x_i^{\Delta}(t) + a_i(t)x_i^{\sigma}(t) = \lambda f_i(t, x(t), x(\beta(t))), \tag{2}$$

for $t \neq t_k, \ i = 1, \dots, n$. Multiplying both sides of (2.2) by $e_{a_i}(t, 0)$, we have

$$[x_i(t)e_{a_i}(t, 0)]^{\Delta} = \lambda f_i(t, x(t), x(\beta(t)))e_{a_i}(t, 0), \tag{3}$$

for $i = 1, 2, \dots, n$. Integrating (2.3) step by step from t to $t + \omega$, we get

$$x_i(t) = \lambda \int_t^{t+\omega} \frac{e_{\ominus a_i}(t, s)}{e_{\ominus a_i}(0, \omega) - 1} f_i(s, x(s), x(\beta(s)))\Delta s + \sum_{j;t_j \in [t,t+\omega]} \frac{e_{\ominus a_i}(t, t_k)}{e_{\ominus a_i}(0, \omega) - 1} I_{ik}(x(t_k)),$$

where $i = 1, 2, \dots, n$, we find that $x(t)$ satisfies (2.1).

Second, we prove the sufficiency. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ be a solution of system (2.1). If $t \neq t_k$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n$, from (2.1), we have

$$\begin{aligned} x_i^{\Delta}(t) &= \lambda[G_i(t, t+\omega)f_i(t+\omega, x(t+\omega), x(\beta(t+\omega))) - G_i(t,t)f_i(t, x(t), x(\beta(t)))] \\ &\quad - a_i(t)x_i^{\sigma}(t) \\ &= -a_i(t)x_i^{\sigma}(t) + \lambda f_i(t, x(t), x(\beta(t))). \end{aligned}$$

If $t = t_k$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n$, then by (2.1), we get

$$\begin{aligned} x_i(t_k^+) - x_i(t_k^-) &= \sum_{j:t_j \in [t_k^+, t_k^+ + \omega)} G_i(t_k, t_j)I_{ij}(x(t_j)) - \sum_{j:t_j \in [t_k^-, t_k^- + \omega)} G_i(t_k, t_j)I_{ij}(x(t_j)) \\ &= G_i(t_k, t_k + \omega)I_{ik}(x(t_k) + \omega) - G_i(t_k, t_k)I_{ik}(x(t_k)) \\ &= I_{ik}(x(t_k)). \end{aligned}$$

So we know that, $x(t)$ is also an ω-periodic solution of (1.1). This completes the proof Lemma 2.2. □

Let $\mathbb{X}$ be a Banach space and K be a closed, nonempty subset of $\mathbb{X}$. K is a cone provided:

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and $\alpha, \beta \geq 0$;
(ii) $u, -u \in K$ imply $u = 0$.

Define $K_r = \{x \in K | \|x\| \leq r\}$. Let $\alpha(x)$ denote the positive continuous concave functional on K, that is, $\alpha : K \to [0, \infty)$ is continuous and satisfies

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y) \text{ for all } x, y \in K, 0 \leq \lambda \leq 1$$

and we denote the set $K(\alpha, a, b) = \{x | x \in K, a \leq \alpha(x), \|x\| \leq b\}$.

The following lemma cited from Ref. [18] is useful for the proof of our main results of this paper.

LEMMA 2.3. **[19]** *Let K be a cone of the real Banach space $\mathbb{X}$ and $A : K_c \to K_c$ be a completely continuous operator, and suppose that there exists a concave positive functional α with $\alpha(x) \leq \|x\| (x \in K)$ and numbers a, b, d with $0 < d < a < b \leq c$, satisfying the following conditions:*

(1) $\{x \in K(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ if $x \in K(\alpha, a, b)$;
(2) $\|Ax\| < d$ if $x \in K_d$;
(3) $\alpha(Ax) > a$ for all $x \in K(\alpha, a, c)$ with $\|Ax\| > b$.

Then A has at least three fixed points in K_c.

In order to apply Lemma 2.2 to system (1.1), consider the Banach space

$$\mathbb{X} = \{x | x = (x_1(t), \ldots, x_n(t))^T \in PC(\mathbb{T}, \mathbb{R}^n) : x(t+\omega) = x(t), t \in \mathbb{T}\}$$

with the norm defined by $\|x\| = \sum_{i=1}^n |x_i|_0$, where $|x_i|_0 = \sup_{t \in [0,\omega]_{\mathbb{T}}} |x_i(t)|$,

$$PC(\mathbb{T}, \mathbb{R}^n) = \{x : \mathbb{T} \to \mathbb{R}^n | x|_{(t_k, t_{k+1})_{\mathbb{T}}} \in C((t_k, t_{k+1})_{\mathbb{T}}, \mathbb{R}^n), \exists x(t_k^-) = x((t_k), k \in \mathbb{Z}\}.$$

Define the cone K in $\mathbb{X}$ by

$$K = \{x = (x_1(t), \ldots, x_n(t))^T \in \mathbb{X} : x_i(t) \geq \sigma |x_i|_0, \ t \in [0, \omega]_{\mathbb{T}}, i = 1, 2, \ldots, n\},$$

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where

$$0 < \sigma = A/B < 1$$

and

$$A = \min_{1\le i\le n}\{A_i\},\ \ B = \max_{1\le i\le n}\{B_i\},$$

$$A_i := \min\{G_i(t,s) : 0 \le t \le s \le \omega\} = \frac{1}{e_{\ominus a_i}(0,\omega) - 1} > 0,\ i = 1, 2, \dots, n,$$

$$B_i := \max\{G_i(t,s) : 0 \le t \le s \le \omega\} = \frac{e_{\ominus a_i}(0,\omega)}{e_{\ominus a_i}(0,\omega) - 1} > 0,\ i = 1, 2, \dots, n.$$

It is not difficult to verify that K is a cone in X.

Let the map Φ be defined by

$$(\Phi x)(t) = \lambda \int_t^{t+\omega} G(t,s) f(s, x(s), x(\beta(s)))\Delta s + \sum_{k:t_k\in[t,t+\omega]} G(t,t_k) I_k(x(t_k)), \tag{4}$$

for $x \in K$, $t \in \mathbb{T}$, and

$$(\Phi x) = (\Phi_1 x, \Phi_2 x, \dots, \Phi_n x)^T.$$

LEMMA 2.4. *The mapping* Φ *maps* K *into* K*,* i.e.*,* $\Phi K \subset K$.

Proof. For any $x \in K$, it is easy to see that $\Phi x \in PC(\mathbb{T}, \mathbb{R}^n)$. Since $x(\beta(t+\omega)) = x(\beta(t))$, in view of (2.4), for $t \in \mathbb{T}$, $i = 1, 2, \dots, n$, we obtain

$$\begin{aligned}
(\Phi_i x)(t+\omega) &= \lambda \int_{t+\omega}^{t+2\omega} G_i(t+\omega, s) f_i(s, x(s), x(\beta(s)))\Delta s + \sum_{k:t_k\in[t+\omega,t+2\omega]} G_i(t+\omega, t_k) I_{ik}(x(t_k)) \\
&= \lambda \int_t^{t+\omega} G_i(t+\omega, u+\omega) f_i(u+\omega, x(u+\omega), x(\beta(u+\omega)))\Delta u \\
&\quad + \sum_{k:t_k\in[t,t+\omega]} G_i(t+\omega, t_k+\omega) I_{ik}(x(t_k+\omega)) \\
&= \lambda \int_t^{t+\omega} G_i(t,u) f_i(u, x(u), x(\beta(u)))\Delta u + \sum_{k:t_k\in[t,t+\omega]} G_i(t,t_k) I_{ik}(x(t_k)) \\
&= (\Phi_i x)(t).
\end{aligned}$$

That is, $(\Phi x)(t+\omega) = (\Phi x)(t)$, $t \in \mathbb{T}$. So $\Phi x \in \mathbb{X}$. For any $x \in K$, $i = 1, 2, \dots, n$, we have

$$|\Phi_i x|_0 \le B_i \lambda \int_t^{t+\omega} f_i(t, x(t), x(\beta(t)))\Delta s + B_i \sum_{k:t_k\in[t,t+\omega]} I_{ik}(x(t_k))$$

and

$$(\Phi_i x)(t) \ge A_i \lambda \int_t^{t+\omega} f_i(t, x(t), x(\beta(t)))\Delta s + A_i \sum_{k:t_k\in[t,t+\omega]} I_{ik}(x(t_k)).$$

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So we get

$$(\Phi_i x)(t) \geq \frac{A}{B}|\Phi_i x|_0 = \sigma|\Phi_i x|_0.$$

Hence, $\Phi x \in K$. The proof of Lemma 2.4 is complete. □

Since the method is similar to that in the literature **[20]**, we omit the proof of the following lemma.

LEMMA 2.5. *The operator $\Phi : K \to K$ is completely continuous.*

For convenience in the following discussion, we introduce the following notations:

$$f^0 = \lim_{\|u\|\to 0} \sup_{t\in[0,\omega]} \frac{\|f(t,u(t),u(\beta(t)))\|}{\|u\|}, \quad I^0 = \limsup_{\|u\|\to 0} \sum_{k=1}^{q} \frac{\|I_k(u)\|}{\|u\|},$$

$$f^\infty = \lim_{\|u\|\to \infty} \sup_{t\in[0,\omega]} \frac{\|f(t,u(t),u(\beta(t)))\|}{\|u\|}, \quad I^\infty = \limsup_{\|u\|\to \infty} \sum_{k=1}^{q} \frac{\|I_k(u)\|}{\|u\|}$$

and for $b > 0$, we define

$$I_{(b)} = \min_{\sigma b\leq\|u\|\leq b} \sum_{k=1}^{q} \|I_k(u)\|.$$

3. Main result.

Our main result of this paper is as follows:

THEOREM 3.1. *Assume that $(H_1)-(H_2)$ hold, there exists a number $b > 0$ such that the following conditions:*

(i) $\|f(t,u,u(\beta(t)))\| + (A+1)I_{(b)} > \frac{1+A}{A}\|u\|$ *for* $\sigma b \leq \|u\| \leq b$, $t \in \mathbb{T}$;

(ii) $f^0 + I^0 < \frac{1}{B}$, $f^\infty + I^\infty < \frac{1}{B}$

hold. Then the system (1.1) has at least three positive ω-periodic solutions for

$$\frac{1}{(A+1)\omega} < \lambda < \frac{1}{\omega}.$$

Proof. By the second inequality in (ii), one can find an $\varepsilon > 0$ such that

$$\frac{\frac{1}{B} - (f^\infty + I^\infty)}{2} > \varepsilon > 0.$$

By the definitions of f^∞ and I^∞, there exists a $C_0 > b$ such that

$$\|f(s,u,u(\beta(t)))\| \leq (f^\infty + \varepsilon)\|u\|, \quad \sum_{k=1}^{q} \|I_k(u)\| \leq (I^\infty + \varepsilon)\|u\|,$$

where $\|u\| > C_0$.

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Let $C_1 = C_0/\sigma$, if $x \in K$, $\|x\| > C_1$, then $\|x\| > C_0$ and we have

$$\begin{aligned}
\|\Phi x\| &= \sup_{t\in[0,\omega]_{\mathbb{T}}} \sum_{i=1}^{n} |(\Phi_i x)(t)| \\
&\leq \sum_{i=1}^{n} \left[\lambda B \int_t^{t+\omega} f_i(s, x(s), x(\beta(s)))\Delta s + B \sum_{k:t_k\in[t,t+\omega]} I_{ik}(x(t_k)) \right] \\
&= \lambda B \int_t^{t+\omega} \|f(s, x(s), x(\beta(s)))\| \Delta s + B \sum_{k:t_k\in[t,t+\omega]} \|I_k(x(t_k))\| \\
&\leq \lambda B \int_t^{t+\omega} (f^{\infty} + \varepsilon)\|x\| \Delta s + B(I^{\infty} + \varepsilon)\|x\| \\
&\leq \big(\lambda B(f^{\infty} + \varepsilon)\omega + B(I^{\infty} + \varepsilon)\big)\|x\| \\
&< \|x\|. \qquad (1)
\end{aligned}$$

Take $K_{C_1} = \{x| = x \in K,\ \|x\| \leq C_1\}$, then the set K_{C_1} is a bounded set. Since Φ is completely continuous, Φ maps bounded sets into bounded sets and there exists a number C_2 such that

$$\|\Phi x\| \leq C_2 \text{ for any } x \in K_{C_1}.$$

If $C_2 \leq C_1$, we deduce that $\Phi : K_{C_1} \to K_{C_1}$ is completely continuous. If $C_1 < C_2$, by (3.1), we know that for any $x \in K_{C_2}\backslash K_{C_1}$, $\|x\| > C_1$ and $\|\Phi x\| < \|x\| < C_2$ hold. Thus, $\Phi : K_{C_2} \to K_{C_2}$ is completely continuous. Now, take $c = \max\{C_1, C_2\}$, obviously $c > b$, then $\Phi : K_c \to K_c$ is completely continuous.

Take the positive continuous concave functional $\alpha(x) = \sum_{i=1}^{n} \min_{t\in[0,\omega]_{\mathbb{T}}} |x_i(t)|$. First, we let $a = \sigma b$ and take $x_i \equiv \frac{a+b}{2}$, $x \in K(\alpha, a, b)$, $\alpha(x) > a$, then the set $\{x \in K(\alpha, a, b)\} \neq \emptyset$. By (i) if $x \in K(\alpha, a, b)$, then $\alpha(x) \geq a$, and we have

$$\begin{aligned}
\alpha(\Phi x) &= \sum_{i=1}^{n} \min_{t\in[0,\omega]_{\mathbb{T}}} \left\{ \lambda \int_t^{t+\omega} G_i(t, s) f_i\big(s, x(s), x(\beta(s))\big)\Delta s + \sum_{k:t_k\in[t,t+\omega]} G_i(t, t_k) I_{ik}(x(t_k)) \right\} \\
&\geq \sum_{i=1}^{n} \min_{t\in[0,\omega]_{\mathbb{T}}} \left\{ \lambda A \int_t^{t+\omega} f_i(s, x(s), x(\beta(s)))\Delta s + A \sum_{k=1}^{q} I_{ik}(x(t_k)) \right\} \\
&\geq \lambda A \int_0^{\omega} \|f(s, x(s), x(\beta(s)))\| \Delta s + A \sum_{k=1}^{q} \|I_k(x(t_k))\| \Big\} \\
&> \lambda A\omega \left(\frac{A+1}{A}\alpha(x) - (A+1)I_{(b)} \right) + AI_{(b)} \\
&> \alpha(x) \geq a.
\end{aligned}$$

Hence condition (1) of Lemma 2.3 holds.

Secondly, by the first inequality of condition of (ii), one can find $\varepsilon > 0$ such that

$$\frac{\frac{1}{B} - (f^0 + I^0)}{2} > \varepsilon > 0$$

and there exists a $0 < d < a$ such that

$$\|f(s,u,u(\beta(t)))\| \leq (f^0+\varepsilon)\|u\|, \quad \sum_{k=1}^{q}\|I_k(u)\| \leq (I^0+\varepsilon)\|u\|,$$

where $0 \leq \|u\| \leq d$. If $x \in K_d = \{x \mid \|x\| \leq d\}$, we have

$$\begin{aligned}
\|\Phi x\| &= \sup_{t\in[0,\omega]_{\mathbb{T}}} \sum_{i=1}^{n} |(\Phi_i x)(t)| \\
&\leq \sum_{i=1}^{n}\left[\lambda B\int_t^{t+\omega} f_i(s,x(s),x(\beta(s)))\Delta s + B\sum_{k:t_k\in[t,t+\omega]} I_{ik}(x(t_k))\right] \\
&= \lambda B\int_t^{t+\omega} \|f(s,x(s),x(\beta(s)))\|\Delta s + B\sum_{k:t_k\in[t,t+\omega]} \|I_k(x(t_k))\| \\
&\leq \lambda B\int_t^{t+\omega} (f^0+\varepsilon)\|x\|\Delta s + B(I^0+\varepsilon)\|x\| \\
&\leq (\lambda B(f^0+\varepsilon)\omega + B(I^0+\varepsilon))\|x\| \\
&< \|x\| \leq d.
\end{aligned}$$

That is, condition (2) of Lemma 2.3 holds.

Finally, if $x \in K(\alpha, a, c)$ with $\|\Phi x\| \geq b$, by the definition of the cone K, we get

$$\begin{aligned}
b < \|\Phi x\| &\leq \sum_{i=1}^{n}\left[\lambda B\int_t^{t+\omega} f_i(s,x(s),x(\beta(s)))\Delta s + B\sum_{k:t_k\in[t,t+\omega]} I_{ik}(x(t_k))\right] \\
&= \lambda B\int_0^{\omega} \|f(s,x(s),x(\beta(s)))\|\Delta s + B\sum_{k:t_k\in[t,t+\omega]} \|I_k(x(t_k))\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\alpha(\Phi x) &= \sum_{i=1}^{n}\min_{t\in[0,\omega]_{\mathbb{T}}}\left\{\lambda\int_t^{t+\omega} G_i(t,s)f_i(s,x(s),x(\beta(s)))\Delta s + \sum_{k:t_k\in[t,t+\omega]} G_i(t,t_k)I_{ik}(x(t_k))\right\} \\
&\geq \sum_{i=1}^{n}\min_{t\in[0,\omega]_{\mathbb{T}}}\left\{\lambda A\int_0^{\omega} f_i(s,x(s),x(\beta(s)))\Delta s + A\sum_{k=1}^{q} I_{ik}(x(t_k))\right\} \\
&\geq \lambda A\int_0^{\omega} \|f(s,x(s),x(\beta(s)))\|\Delta s + A\sum_{k=1}^{q}\|I_k(x(t_k))\| \\
&\geq \sigma\|\Phi x\| > \sigma b = a.
\end{aligned}$$

So the condition (3) of Lemma 2.3 holds. Therefore, by Lemma 2.3, we obtain that the operator Φ has at least three fixed points in K_c. The proof of Theorem 3.1 is complete. □

COROLLARY 3.1. *Using the following*

$$(ii^*) \quad f^0 = 0, \ I^0 = 0, \ f^\infty = 0, \ I^\infty = 0$$

instead of (ii) in Theorem 3.1, the conclusion of Theorem 3.1 remains true.

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4. An example.
Consider the following equation:

$$\begin{cases} x'(t) = -\frac{1}{2}x(t) + \lambda x^2(t)e^{-x(\beta(t))}, & t \neq t_k, \\ \Delta x(t_k) = 0.1x^2(t_k)e^{-2x(t_k)}, & t = t_k = 2k, \ k \in \mathbb{Z}, \end{cases} \tag{1}$$

where λ is a non-negative parameter. Clearly, $A > 0$, $B > 0$. According to Corollary 3.1, (4.1) has at least three positive periodic solutions.

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