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Laplace Transform of Temperate Holomorphic Functions

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Abstract. Let $\mathbb{V}$ be n-dimensional complex vector space. The aim of this paper is to give an elementary proof of the isomorphism $(\mathcal{O}^t_{\mathbb{V}})^\wedge[n] \simeq \mathcal{O}^t_{\mathbb{V}^*}$, which quantizes the Fourier–Sato transform of the conic sheaf $\mathcal{O}^t_{\mathbb{V}}$ of temperate holomorphic functions.

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Introduction

In their paper [K-S3], M. Kashiwara and P. Schapira make the Laplace transform act on the tempered cohomology associated with conic $\mathbb{R}$-constructible sheaves, and obtain an inversion formula. More precisely, the Laplace transform on an n-dimensional complex vector space $\mathbb{V}$ (or more generally a complex vector bundle) is formally defined by the formula

$$f(z) \mapsto \widehat{f}(\zeta) = \int f(z)\exp(\langle z, \zeta\rangle)\mathrm{d}z, \tag{0.1}$$

where (z) denotes the complex coordinates on $\mathbb{V}$ and (ζ) the dual coordinates on $\mathbb{V}^*$. In order to describe this formula in the framework of algebraic $\mathcal{D}$-modules, the authors consider the projective compactification $j\colon \mathbb{V} \hookrightarrow \mathbb{P}$ and set, for F a $\mathbb{R}$-constructible, and $\mathbb{R}^+$-conic sheaf on $\mathbb{V}$,

$$\mathrm{THom}(F, \mathcal{O}_{\mathbb{V}}) = \mathrm{R}\Gamma(\mathbb{P}; T\mathcal{H}om(j_!F, \mathcal{O}_{\mathbb{P}})),$$

where $T\mathcal{H}om(\cdot, \mathcal{O})$ denotes the functor of moderate cohomology introduced in [K2]. Then, they obtain in particular the following Laplace isomorphism

$$\mathrm{THom}(F, \mathcal{O}_{\mathbb{V}}) \xleftarrow{\sim} \mathrm{THom}(F^\wedge[n], \mathcal{O}_{\mathbb{V}^*}),$$

where $F^\wedge$ denotes the Fourier–Sato transform of F.

As an application, they define the conic sheaf $\mathcal{O}^t_{\mathbb{V}}$ of tempered holomorphic functions (tempered at zero and at infinity) associated with the conic presheaf $U \mapsto \mathrm{THom}(\mathbb{C}_U, \mathcal{O}_{\mathbb{V}})$ and show it is invariant by Laplace transform. In fact, the

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Laplace kernel quantizes the Fourier-Sato transform, giving an isomorphism of conic sheaves

$$(\mathcal{O}^t_{\mathbb{V}})^{\wedge}[n] \simeq \mathcal{O}^t_{\mathbb{V}^*}. \tag{0.2}$$

Moreover, this isomorphism is linear over the Weyl algebra $D(\mathbb{V})$ (via the Fourier isomorphism $D(\mathbb{V}) \simeq D(\mathbb{V}^*)$).

Their proof is rather intricate and is based on the theory of algebraic $\mathcal{D}$-modules. The aim of this paper is to give an elementary proof of the isomorphism (0.2). Our proof is based on the observation that isomorphism (0.2) is well known when applied to convex tubes (see e.g. Faraut [F]). More precisely, let V be a n-dimensional real vector space, $\mathbb{V} = \mathbb{C} \otimes_{\mathbb{R}} V$ its complexification and λ be a subanalytic proper convex closed cone in V. One has the isomorphism of conic presheaves

$$\Gamma_\lambda(S'(V)) \xrightarrow{\sim} \mathcal{O}^{mod}(\mathrm{Int}\lambda^\circ + iV^*), \tag{0.3}$$

where $\Gamma_\lambda(S'(V))$ is the space of tempered distributions supported by λ and $\mathcal{O}^{mod}(\mathrm{Int}\lambda^\circ + iV^*)$ is the space of holomorphic functions on the dual tube $\mathrm{Int}\lambda^\circ + iV$ with growth conditions along the boundary of the tube and at infinity.

We recall the definition of the conic sheaf $\mathcal{D}b^t_V$ of [K-S3]. The sections of the Fourier–Sato transform $(\mathcal{D}b^t_V)^{\wedge}$ on a subanalytic convex open cone of V^* are the tempered distributions on V supported by the polar cone.

We define the conic sheaf $\mathcal{O}^{tt}_V$ associated with the conic presheaf $\gamma \mapsto \mathcal{O}^{mod}(\gamma + iV)$. From (0.3), we obtain an isomorphism of conic sheaves $(\mathcal{D}b^t_V)^{\wedge} \xrightarrow{\sim} \mathcal{O}^{tt}_{V^*}$. The general case may be deduced from this situation, using basic tools of sheaf and $\mathcal{D}$-module theory.

As an application, we recover a theorem of Brylinski–Malgrange–Verdier and Kashiwara–Hotta on the solutions of monodromic holonomic $\mathcal{D}$-modules: if N is a monodromic $D(\mathbb{V})$-module and $N^{\wedge}$ its Fourier transform, then there exists an isomorphism of conic sheaves

$$R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}_{\mathbb{V}})^{\wedge}[n] \simeq R\mathcal{H}om_{D(\mathbb{V}^*)}(N^{\wedge}, \mathcal{O}_{\mathbb{V}^*}).$$

As another application, we construct the complex of tempered hyperfunctions

$$\mathcal{B}^t_V = R\mathcal{H}om(\mathbb{C}_V[-n], \mathcal{O}^t_{\mathbb{V}}),$$

whose global sections are invariant by Fourier transform.

1. Notations and Review

1.1. NOTATIONS

We refer the reader to [K-S2] for a detailed exposition of sheaf theory within the framework of derived categories.

On a real analytic manifold X, we shall encounter the objects associated with X

$\mathcal{C}^\infty_X$ the sheaf of $\mathcal{C}^\infty$ functions.

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$\mathcal{D}b_X$	the sheaf of Schwartz distributions.
$\mathcal{D}_X$	the sheaf of finite order analytic differential operators.
$\mathrm{D^b}(\mathbb{C}_X)$	the bounded derived category of the category of complexes of sheaves of $\mathbb{C}$-vector spaces.
$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$	the full subcategory in $\mathrm{D^b}(\mathbb{C}_X)$ given by objects whose cohomology groups are $\mathbb{R}$-constructible sheaves.

In particular, we shall use the six Grothendieck's operations $\otimes, \mathcal{H}\mathrm{om}, f_*, f^{-1}, f_!, f^!$. We set $D'F = R\mathcal{H}om(F, \mathbb{C}_X)$.

1.2. REVIEW ON FORMAL AND MODERATE COHOMOLOGY

The functor of tempered cohomology $T\mathcal{H}om(\cdot, \mathcal{O}_X)$ has been introduced by M. Kashiwara to solve the Riemann–Hilbert problem in 1983 (see [K2]). In 1995, M. Kashiwara and P. Schapira introduced a functor dual to $T\mathcal{H}om(\cdot, \mathcal{O}_X)$ in their paper [K-S1], the functor of formal cohomology denoted by $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$. We recall some properties of these functors.

Let U be a subanalytic open subset of X and let $Z = X \setminus U$ be its complement. One sets

$$T\mathcal{H}om(\mathbb{C}_Z, \mathcal{D}b_X) = \Gamma_Z \mathcal{D}b_X, \tag{1.1}$$

$$\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X = \mathcal{I}^\infty_{X,Z}, \tag{1.2}$$

where $\Gamma_Z \mathcal{D}b_X$ denotes the sheaf of distributions with support in Z and $\mathcal{I}^\infty_{X,Z}$ denotes the subsheaf of $\mathcal{C}^\infty_X$ consisting of functions which vanish on Z up to infinite order.

One defines the sheaves $T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_X)$ and $\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$ by the short exact sequences

$$0 \to \Gamma_Z \mathcal{D}b_X \to \mathcal{D}b_X \to T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_X) \to 0, \tag{1.3}$$

$$0 \to \mathcal{I}^\infty_{X,Z} \to \mathcal{C}^\infty_X \to \mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \to 0. \tag{1.4}$$

Now one states Łojasiewicz's Theorem:

THEOREM 1.1 (Łojasiewicz [Lo]). *Let U_i $(i = 1, 2)$ be two subanalytic open subsets of X, $Z_i := X \setminus U_i$. Then the two sequences below are exact*

$$0 \to \mathcal{I}^\infty_{X,Z_1\cup Z_2} \to \mathcal{I}^\infty_{X,Z_1} \oplus \mathcal{I}^\infty_{X,Z_2} \to \mathcal{I}^\infty_{X,Z_1\cap Z_2} \to 0,$$
$$0 \to \Gamma_{Z_1\cap Z_2}\mathcal{D}b_X \to \Gamma_{Z_1}\mathcal{D}b_X \oplus \Gamma_{Z_2}\mathcal{D}b_X \to \Gamma_{Z_1\cup Z_2}\mathcal{D}b_X \to 0.$$

By this result, the functors of Whitney and Schwartz satisfy a kind of Mayer–Vietoris property, and an abstract result of [K-S1] allows one to extend these

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functors as exact functors on the category of $\mathbb{R}$-constructible sheaves:

$$\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty : \mathbb{R}\text{−cons}(X) \to \mathrm{Mod}(\mathcal{D}_X),$$
$$T\mathcal{H}om(\cdot, \mathcal{D}b_X) : (\mathbb{R}\text{−cons}(X))^{opp} \to \mathrm{Mod}(\mathcal{D}_X).$$

Moreover, given an $\mathbb{R}$-constructible sheaf F, the sheaves $F \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty$ and $T\mathcal{H}om(F, \mathcal{D}b_X)$ are soft. These functors being exact, they extend as functors on the derived categories, from $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$ to $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$.

From now on, we consider a complex analytic manifold X, of complex dimension d_X, endowed with its structural sheaf $\mathcal{O}_X$. One denotes by $\Omega_X^{(p)}$ the sheaf of holomorphic p-forms, we also write Ω_X instead of $\Omega_X^{(d_X)}$. We shall denote by $X_{\mathbb{R}}$ the real underlying analytic manifold to X and by $\overline{X}$ the complex conjugate manifold to X. By its definition, $\overline{X}$ is the topological space $X_{\mathbb{R}}$ endowed with the sheaf $\mathcal{O}_{\overline{X}}$ of anti-holomorphic functions on X. Then $X \times \overline{X}$ is a complexification of $X_{\mathbb{R}}$ by the diagonal embedding $X_{\mathbb{R}} \hookrightarrow X \times \overline{X}$. Notice that $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$ are two subrings of $\mathcal{D}_{X_{\mathbb{R}}}$ and if $P \in \mathcal{D}_X$, $Q \in \mathcal{D}_{\overline{X}}$ then $[P, Q] = 0$.

For $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$, one sets

$$F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{X_{\mathbb{R}}}), \tag{1.5}$$

$$T\mathcal{H}om(F, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, T\mathcal{H}om(F, \mathcal{D}b_{X_{\mathbb{R}}})). \tag{1.6}$$

The functors $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $T\mathcal{H}om(\cdot, \mathcal{O}_X)$ are called the functor of formal cohomology and the functor of moderate cohomology, respectively. The objects $F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $T\mathcal{H}om(F, \mathcal{O}_X)$ belong to $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$.

There are natural morphisms

$$F \otimes \mathcal{O}_X \to F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \to T\mathcal{H}om(\mathrm{D}'_X F, \mathcal{O}_X) \to R\mathcal{H}om(\mathrm{D}'_X F, \mathcal{O}_X).$$

If $\mathcal{G}$ is a locally free $\mathcal{O}_X$-module of finite rank, one sets

$$F \overset{\mathrm{w}}{\otimes} \mathcal{G} = (F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G},$$
$$T\mathcal{H}om(F, \mathcal{G}) = T\mathcal{H}om(F, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G}.$$

When X is a complexification of a real analytic manifold M, $\mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ is nothing but the sheaf $\mathcal{C}_M^\infty$ and $T\mathcal{H}om(D'_X \mathbb{C}_M, \mathcal{O}_X)$ is the sheaf $\mathcal{D}b_M$. If Y is a closed complex analytic subset of X, $\mathbb{C}_Y \overset{\mathrm{w}}{\otimes} \mathcal{O}_X = \widehat{\mathcal{O}_X|_Y}$, the formal completion of $\mathcal{O}_X$ along Y (in particular, $\mathbb{C}_Y \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ is concentrated in degree 0) and $T\mathcal{H}om(\mathbb{C}_Y, \mathcal{O}_X) = \mathrm{R}\Gamma_{[Y]}(\mathcal{O}_X)$, the algebraic cohomology of $\mathcal{O}_X$ with support in Y.

1.3. FOURIER–SATO TRANSFORM AND CONIC SHEAVES

Let $\tau : E \to M$ be a real vector bundle, of rank n, endowed with an action of $\mathbb{R}^+$ the multiplicative group of positive numbers. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_E)$ consisting of objects F such that $H^j(F)$ is locally constant

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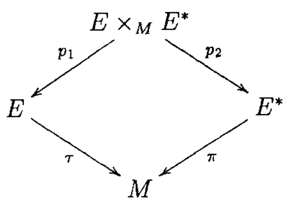
on the $\mathbb{R}^+$-orbits for all j. An object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E)$ is called a conic object. One then sets $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+,\mathbb{R}-c}(\mathbb{C}_E) = \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E) \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_E)$.

Let $\pi\colon E^* \to M$ be the dual vector bundle. Denote by p_1 and p_2 the first and second projection defined on $E \times_M E^*$, and set

$$P = \{(x, y) \in E \times_M E^*;\ \langle x, y\rangle \geqslant 0\},$$
$$P' = \{(x, y) \in E \times_M E^*;\ \langle x, y\rangle \leqslant 0\}.$$

Consider the diagram



DEFINITION 1.2. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E)$, $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{E^*})$. One sets:

$$F^{\wedge} = Rp_{2!}(p_1^{-1}F)_{P'}, \tag{1.7}$$

$$G^{\vee} = Rp_{1!}(p_2^{!}G)_{P}. \tag{1.8}$$

Let us recall the main result of this theory:

THEOREM 1.3. *The functors* $^\wedge$ *from* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E)$ *to* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{E^*})$ *and* $^\vee$ *from* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{E^*})$ *to* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E)$ *are equivalences of categories, inverse to each other. In particular, if* F_1 *and* F_2 *belong to* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E)$, *then*

$$\mathrm{RHom}(F_1, F_2) \simeq \mathrm{RHom}(F_1^{\wedge}, F_2^{\wedge}). \tag{1.9}$$

Let us summarize some properties and examples of the Fourier–Sato transformation.

(i) Let λ be a closed proper convex cone in E with $M \subset \lambda$. We denote by $\lambda^\circ = \{y \in E^*;\ \langle x, y\rangle \geqslant 0 \text{ for all } x \in \lambda\}$ the polar cone to λ and $\mathrm{Int}\lambda^\circ$ its interior. Then one gets

$$(\mathbb{C}_\lambda)^{\wedge} \simeq \mathbb{C}_{\mathrm{Int}\lambda^\circ}. \tag{1.10}$$

(ii) Let γ be an open convex cone in E. We denote by $\gamma^a = -\gamma$ the opposite cone. We denote by $\mathrm{or}_{E^*/M}$ the relative orientation sheaf of E^* over M. Then one has

$$(\mathbb{C}_\gamma)^{\wedge} \simeq \mathbb{C}_{\gamma^{\circ a}} \otimes \mathrm{or}_{E^*/M}[-n]. \tag{1.11}$$

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(iii) Let U be a convex open subset of E^*. Then one obtain

$$\mathrm{R}\Gamma(U; F^\wedge) \simeq \mathrm{R}\Gamma_{U^\circ}(E; F). \tag{1.12}$$

One of the functorial properties of the Fourier–Sato transform is that it commutes to base change. More precisely, consider a morphism of manifolds $f\colon N \to M$ and the associated morphisms of vector bundles

$$\begin{array}{ccc} f^*E & \xrightarrow{f_\tau} & E \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ N & \xrightarrow{f} & M \end{array} \qquad \begin{array}{ccc} f^*E & \xrightarrow{f_\pi} & E^* \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ N & \xrightarrow{f} & M \end{array}$$

Then, for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E)$ and $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{f^*E})$, there are natural isomorphisms

$$(f_\tau^{-1}F)^\wedge \simeq f_\pi^{-1}(F^\wedge), \tag{1.13}$$

$$(Rf_{\tau!}G)^\wedge \simeq Rf_{\pi!}(G^\wedge). \tag{1.14}$$

1.4. THE WEYL ALGEBRA

Let $\mathbb{V}$ be a n-dimensional complex vector space.

We denote by $D(\mathbb{V})$ the Weyl algebra on $\mathbb{V}$ (i.e. the ring of differential operators with polynomial coefficients on $\mathbb{V}$). Note that $D(\mathbb{V}) = \Gamma(\mathbb{V}; \mathcal{D}_{\mathbb{V}})$, where $\mathcal{D}_{\mathbb{V}}$ is the sheaf of algebraic differential operators on the algebraic variety $\mathbb{V}$. There exists a correspondence between the category $\mathrm{Mod}_f(D(\mathbb{V}))$ of finitely generated $D(\mathbb{V})$-modules and the category $\mathrm{Mod}_{coh}(\mathcal{D}_{\mathbb{V}})$ of coherent $\mathcal{D}_{\mathbb{V}}$-modules.

We can consider the Fourier isomorphism $\wedge\colon D(\mathbb{V}) \xrightarrow{\sim} D(\mathbb{V}^*)$, and its inverse $\vee\colon D(\mathbb{V}^*) \to D(\mathbb{V})$.

Let $(z_1, \ldots, z_n)$ and $(\zeta_1, \ldots, \zeta_n)$ be dual systems of linear coordinates on $\mathbb{V}$ and $\mathbb{V}^*$. Recall that the Fourier transform is defined by

$$(z_i)^\wedge = -\frac{\partial}{\partial \zeta_i}, \qquad \left(\frac{\partial}{\partial z_i}\right)^\wedge = \zeta_i.$$

Naturally, its inverse is defined by

$$(\zeta_i)^\vee = \frac{\partial}{\partial z_i}, \qquad \left(\frac{\partial}{\partial \zeta_i}\right)^\vee = -z_i.$$

These are Fourier 'inverse' transforms.

1.5. THE CONIC SHEAF OF TEMPERED HOLOMORPHIC FUNCTIONS

Let $\mathbb{V}$ be a n-dimensional complex vector space and $\mathbb{V}_{\mathbb{R}}$ the real underlying vector space. If there is no risk of confusion, we write $\mathbb{V}$ instead of $\mathbb{V}_{\mathbb{R}}$. Let γ be a convex open cone in $\mathbb{V}$.

DEFINITION 1.4. We denote by $\mathcal{O}^{mod}(\gamma)$ the space of holomorphic functions f in the cone γ which satisfy the following property: there exist M, α, and $\beta \geqslant 0$ such that

$$|f(z)| \leqslant M(1+|z|)^{\alpha}(1+d(z, \mathbb{V}\setminus\gamma)^{-\beta}),$$

where $d(z, \mathbb{V}\setminus\gamma)$ is the distance from z to $\mathbb{V}\setminus\gamma$ (for some euclidean structure on $\mathbb{V}_{\mathbb{R}}$).

Let $j\colon \mathbb{V}\hookrightarrow\mathbb{P}$ denote the projective compactification of $\mathbb{V}$. For $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{V}})$, one sets

$$\mathrm{THom}(F, \mathcal{D}b_{\mathbb{V}}) = \mathrm{R}\Gamma(\mathbb{P}; \mathcal{THom}(j_!F, \mathcal{D}b_{\mathbb{P}})),$$
$$\mathrm{THom}(F, \mathcal{O}_{\mathbb{V}}) = \mathrm{R}\Gamma(\mathbb{P}; \mathcal{THom}(j_!F, \mathcal{O}_{\mathbb{P}})).$$

In particular, if λ is a subanalytic convex closed cone in $\mathbb{V}$, we have

$$\mathrm{THom}(\mathbb{C}_{\lambda}, \mathcal{D}b_{\mathbb{V}}) = \Gamma_{\lambda}(S'(\mathbb{V})),$$

where $\Gamma_{\lambda}(S'(\mathbb{V}))$ is the space of tempered distributions supported by λ.

Note that the functor of moderate cohomology $\mathrm{THom}(F, \mathcal{O}_{\mathbb{V}})$ belongs to $\mathrm{D}^{\mathrm{b}}(D(\mathbb{V}))$, for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{V}})$.

LEMMA 1.5. *Let γ be a subanalytic convex open cone in $\mathbb{V}$. Then*

$$\mathcal{O}^{mod}(\gamma) \simeq \mathrm{THom}(\mathbb{C}_{\gamma}, \mathcal{O}_{\mathbb{V}}).$$

Proof. The sections of $\mathcal{O}^{mod}(\gamma)$ are holomorphic functions which are tempered along the boundary of the cone γ and at infinity. Moreover, if $f \in \mathcal{O}^{mod}(\gamma)$ then all its derivations are also tempered (see [Siu]).

On the other hand, the complex $\mathcal{THom}(j_!\mathbb{C}_{\gamma}, \mathcal{O}_{\mathbb{P}})$ can also be calculated by the Dolbeault complex of $\mathcal{THom}(j_!\mathbb{C}_{\gamma}, \mathcal{C}^{\infty}_{\mathbb{P}})$ (see [K-S1]). Let us recall that a section of $\Gamma(U; \mathcal{THom}(j_!\mathbb{C}_{\gamma}, \mathcal{C}^{\infty}_{\mathbb{P}}))$, for an open subset $U \subset \mathbb{P}$, is a $\mathcal{C}^{\infty}$ function on $U \cap j(\gamma)$ which is tempered on U (all its derivations are with polynomial growth on $U \cap j(\gamma)$). □

We shall consider conic sheaves associated with conic presheaves. Let us give their constructions.

LEMMA 1.6 (see [K-S3]). *Let $\mathcal{T}$ be a family of open cones satisfying: for each $z \in \mathbb{V}$, and each open conic neighborhood γ of z, there exists $\gamma' \in \mathcal{T}$ with $z \in \gamma' \subset \gamma$.*

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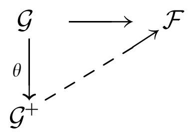
Let $\mathcal{G}$ be a presheaf of $\mathbb{C}$-vector spaces on $\mathcal{T}$. The classical construct of a sheaf associated with a presheaf gives a conic sheaf $\mathcal{G}^+$ and a morphism of presheaves $\theta: \mathcal{G} \to \mathcal{G}^+$ such that any morphism $\mathcal{G} \to \mathcal{F}$ with a conic sheaf $\mathcal{F}$ factorizes through θ:



Moreover, we have $\mathcal{G}_z \xrightarrow{\sim} \mathcal{G}_z^+$ for all $z \in \mathbb{V}$.

We can now introduce the conic sheaf of tempered holomorphic functions.

DEFINITION 1.7. We denote by $\mathcal{O}_{\mathbb{V}}^t$ the conic sheaf associated with the conic presheaf $U \longmapsto \mathcal{O}^{mod}(U)$, for U a subanalytic convex open cone in $\mathbb{V}$.

The conic sheaf $\mathcal{D}b_V^t$ of [K-S3].

Let V be a n-dimensional real vector space, and $j: V \hookrightarrow P$ its projective compactification. Let us recall the definition of the conic sheaf $\mathcal{D}b_V^t$ introduced in [K-S3].

DEFINITION 1.8. One denotes by $\mathcal{D}b_V^t$ the conic sheaf associated with the conic presheaf

$$U \longmapsto \mathrm{THom}(\mathbb{C}_U, \mathcal{D}b_V) = \mathrm{R}\Gamma(P; T\mathcal{H}om(j_!\mathbb{C}_U, \mathcal{D}b_P)),$$

for U a subanalytic open cone in V.

The main properties of this sheaf are the following.

PROPOSITION 1.9. (i) *The conic sheaf $\mathcal{D}b_V^t$ is conically soft (i.e. its direct image on $V/\mathbb{R}^+$ is flabby), and in particular*

$$H^j(U; \mathcal{D}b_V^t) = 0 \textit{ for all } j \neq 0 \textit{ and } U \textit{ open cone.}$$

(ii) *$\mathcal{D}b_V^t$ is a $D(\mathbb{V})$-module.*
(iii) $\mathrm{R}\Gamma(V; \mathcal{D}b_V^t) \simeq \mathrm{THom}(\mathbb{C}_V, \mathcal{D}b_V)$.
(iv) $\mathrm{R}\Gamma_{\{0\}}(V; \mathcal{D}b_V^t) \simeq \mathrm{THom}(\mathbb{C}_{\{0\}}, \mathcal{D}b_V)$.
(v) *For any cone U, one has in the category of vector spaces*

$$\Gamma(U; \mathcal{D}b_V^t) = \varprojlim_{U'} \Gamma(U'; \mathcal{D}b_V^t),$$

where the projective limit is taken over the subanalytic open cones U' such that $\overline{U}' \subset U \cup \{0\}$.

The link with the conic sheaf $\mathcal{O}^t_{\mathbb{V}}$ of [K-S3]

We have a natural description of the conic sheaf $\mathcal{O}^t_{\mathbb{V}}$ by taking the Dolbeault resolution of the conic sheaf $\mathcal{D}b^t_{\mathbb{V}_{\mathbb{R}}}$.

LEMMA 1.10. $\mathcal{O}^t_{\mathbb{V}} = R\mathcal{H}om_{D(\overline{\mathbb{V}})}(\mathcal{O}(\overline{\mathbb{V}}), \mathcal{D}b^t_{\mathbb{V}_{\mathbb{R}}})$.

Proof. By Lemma 1.5, $\mathcal{O}^t_{\mathbb{V}}$ is the conic sheaf associated with the presheaf

$$U \longmapsto \mathrm{THom}(\mathbb{C}_U, \mathcal{O}_{\mathbb{V}}),$$

for U a subanalytic convex open cone in $\mathbb{V}$. □

Thanks to the preceding lemma, we obtain the same conic sheaf of tempered holomorphic functions which has been introduced in [K-S3]. Let us remark that the sections of $\mathcal{O}^t_{\mathbb{V}}$ are tempered at zero and at infinity and

$$\mathrm{R}\Gamma(\mathbb{V}; \mathcal{O}^t_{\mathbb{V}}) \simeq \mathrm{THom}(\mathbb{C}_{\mathbb{V}}, \mathcal{O}_{\mathbb{V}}) \simeq \mathbb{C}[\mathbb{V}], \tag{1.15}$$

$$\mathrm{R}\Gamma_{\{0\}}(\mathbb{V}; \mathcal{O}^t_{\mathbb{V}}) \simeq \mathrm{THom}(\mathbb{C}_{\{0\}}, \mathcal{O}_{\mathbb{V}}) \simeq \mathbb{C}[\mathbb{V}^*][-n]. \tag{1.16}$$

2. The Laplace Transform

2.1. REAL FRAMEWORK

Let V be a n-dimensional real vector space and $\mathbb{V} = \mathbb{C} \otimes_{\mathbb{R}} V$ its complexification.

DEFINITION 2.1. We denote by $\mathcal{O}^{tt}_V$ the conic sheaf on V associated with the presheaf $\gamma \longmapsto \mathcal{O}^{mod}(\gamma + iV)$, for γ a subanalytic convex open cone in V.

Let us remark that the sections of $\mathcal{O}^{mod}(\gamma + iV)$ are tempered along the boundary of the tube $\gamma + iV \subset \mathbb{V}$ and at infinity, while sections of $\mathcal{O}^{tt}_V$ are only tempered at infinity and along iV.

We can now recall the well-known result which describes the isomorphism between the tempered distributions supported by a convex closed cone λ and tempered holomorphic functions in the dual tube.

THEOREM 2.2 (see [F]). *Let λ be a subanalytic proper convex closed cone in V. The Laplace transform induces an isomorphism between the spaces $\Gamma_\lambda(S'(V))$ and $\mathcal{O}^{mod}(\mathrm{Int}\lambda^\circ + iV^*)$.*

In order to obtain Theorem 2.5 below, we need a couple of lemma. The following lemma ensures that the Fourier transform of the sheaf $\mathcal{D}b^t_V$ is concentrated in degree 0.

LEMMA 2.3. *The complex $(\mathcal{D}b^t_V)^\wedge \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{V^*})$ is concentrated in degree 0.*

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Proof. We have to show that $(H^j(\mathcal{D}b^t_V)^\wedge)_\xi = 0$ for all $j \neq 0$ and $\xi \in V^*$.

By definition, $(H^j(\mathcal{D}b^t_V)^\wedge)_\xi = \varinjlim_U H^j(U; (\mathcal{D}b^t_V)^\wedge)$, where U ranges over the family of subanalytic proper convex conic neighborhood of ξ in V^*.

(i) If $\xi = 0$, it is a direct consequence of Proposition 1.9. (iv).
(ii) Now, we consider the case where $\xi \neq 0$.

For λ a subanalytic proper convex closed cone in V, we have

$$\begin{aligned} \mathrm{R}\Gamma_\lambda(V; \mathcal{D}b^t_V) &= \mathrm{RHom}(\mathbb{C}_\lambda, \mathcal{D}b^t_V) \\ &= \mathrm{RHom}(\mathbb{C}_\lambda{}^\wedge, (\mathcal{D}b^t_V)^\wedge) \\ &= \mathrm{RHom}(\mathbb{C}_{\mathrm{Int}\lambda^\circ}, (\mathcal{D}b^t_V)^\wedge) \\ &= \mathrm{R}\Gamma(\mathrm{Int}\lambda^\circ; (\mathcal{D}b^t_V)^\wedge). \end{aligned} \tag{2.1}$$

We shall prove that $\varinjlim_\lambda H^j_\lambda(V; \mathcal{D}b^t_V) = 0$ for all $j \neq 0$, where λ ranges over the family of subanalytic proper convex closed cones of V satisfying $\mathrm{Int}\lambda^\circ \ni \xi$.

Obviously, since the sheaf $\mathcal{D}b^t_V$ is conically soft, the groups $H^j_\lambda(V; \mathcal{D}b^t_V)$ vanish for $j > 1$. Let us consider the following diagram with exact rows:

$$\begin{array}{ccccccc} \Gamma(V; \mathcal{D}b^t_V) & \xrightarrow{\rho} & \Gamma(V \setminus \lambda_1; \mathcal{D}b^t_V) & \longrightarrow & H^1_{\lambda_1}(V; \mathcal{D}b^t_V) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \Gamma(V; \mathcal{D}b^t_V) & \xrightarrow{\rho} & \Gamma(V \setminus \lambda_2; \mathcal{D}b^t_V) & \longrightarrow & H^1_{\lambda_2}(V; \mathcal{D}b^t_V) & \longrightarrow & 0 \end{array}$$

where $\lambda_1 \subset \lambda_2$ are subanalytic proper convex closed cones of V, with $\lambda_1 \setminus \{0\} \subset \mathrm{Int}\lambda_2$.

The morphism $H^1_{\lambda_1}(V; \mathcal{D}b^t_V) \longrightarrow H^1_{\lambda_2}(V; \mathcal{D}b^t_V)$ is zero. Indeed, for every section $\phi \in \Gamma(V \setminus \lambda_1; \mathcal{D}b^t_V)$, there exists a section $\psi \in \Gamma(V; \mathcal{D}b^t_V)$ such that $\rho(\psi)_{|V \setminus \lambda_2} = \phi_{|V \setminus \lambda_2}$.

Therefore, we conclude that $\varinjlim_\lambda H^1_\lambda(V; \mathcal{D}b^t_V) = 0$. □

We need also to describe the sections of the sheaf $\mathcal{D}b^t_V$.

LEMMA 2.4. *Let λ be a subanalytic proper convex closed cone in V.*

$$\Gamma_\lambda(V; \mathcal{D}b^t_V) = \Gamma(\mathrm{Int}\lambda^\circ; (\mathcal{D}b^t_V)^\wedge).$$

Proof. If we take the isomorphism (2.1) in degree 0, we obtain

$$H^0\mathrm{R}\Gamma_\lambda(V; \mathcal{D}b^t_V) = H^0\mathrm{R}\Gamma(\mathrm{Int}\lambda^\circ; (\mathcal{D}b^t_V)^\wedge).$$

Moreover, the complex $(\mathcal{D}b^t_V)^\wedge$ is a sheaf by Lemma 2.3. Then, the result follows. □

Now, we can state the

THEOREM 2.5. *Let us consider the sheaf $\mathcal{O}^{tt}_{V^*}$ as a sheaf of $D(\mathbb{V})$-modules, via the Fourier isomorphism $D(\mathbb{V}) \simeq D(\mathbb{V}^*)$. The Laplace transform induces an isomorphism of sheaves of $D(\mathbb{V})$-modules*

$$(\mathcal{D}b^t_V)^\wedge \simeq \mathcal{O}^{tt}_{V^*}. \tag{2.2}$$

Proof. First, we shall define a morphism of conic sheaves $(\mathcal{D}b^t_V)^\wedge \to \mathcal{O}^{tt}_{V^*}$.

Let λ be a subanalytic proper convex closed cone in V. By Theorem 2.2, the Laplace transform induces an isomorphism

$$\Gamma_\lambda(S'(V)) \xrightarrow{\sim} \mathcal{O}^{mod}(\mathrm{Int}\lambda^\circ + iV^*). \tag{2.3}$$

By definition, $\mathcal{O}^{tt}_{V^*}$ is the conic sheaf associated with $\mathrm{Int}\lambda^\circ \mapsto \mathcal{O}^{mod}(\mathrm{Int}\lambda^\circ + iV^*)$ (indeed, for every subanalytic convex open cone γ in V^*, there exists a subanalytic convex closed cone λ in V such that $\gamma = \mathrm{Int}\lambda^\circ$).

Let us check that the conic sheaf $(\mathcal{D}b^t_V)^\wedge$ is associated with $\mathrm{Int}\lambda^\circ \mapsto \Gamma_\lambda(S'(V))$. First, we have by Lemma 2.4 $\Gamma(\mathrm{Int}\lambda^\circ; (\mathcal{D}b^t_V)^\wedge) = \Gamma_\lambda(V; \mathcal{D}b^t_V)$. Let ξ be a vector of V^*. We consider Λ a family of subanalytic proper convex closed cone λ in V satisfying $\mathrm{Int}\lambda^\circ \ni \xi$. We get

$$\varinjlim_{\lambda\in\Lambda} \Gamma_\lambda(V; \mathcal{D}b^t_V) \simeq \varinjlim_{\lambda\in\Lambda} \Gamma_\lambda(S'(V)).$$

Then the morphism (2.3) is a morphism of presheaves defined only on a basis of conic open sets of V^* (this basis is given by the family of the open convex proper subanalytic cones and the space V^*). Thanks to Lemma 1.6, the isomorphism (2.3) of presheaves defines a morphism between the associated sheaves

$$(\mathcal{D}b^t_V)^\wedge \to \mathcal{O}^{tt}_{V^*}. \tag{2.4}$$

Moreover, we have

$$\begin{array}{ccc}
\varinjlim\limits_{\lambda\in\Lambda} \Gamma_\lambda(S'(V)) & \xrightarrow{\sim} & \varinjlim\limits_{\lambda\in\Lambda} \mathcal{O}^{mod}(\mathrm{Int}\lambda^\circ + iV^*) \\
\downarrow\wr & & \downarrow\wr \\
\varinjlim\limits_{\lambda\in\Lambda} \Gamma(\mathrm{Int}\lambda^\circ; (\mathcal{D}b^t_V)^\wedge) & & \varinjlim\limits_{\lambda\in\Lambda} \Gamma(\mathrm{Int}\lambda^\circ; \mathcal{O}^{tt}_{V^*}) \\
\downarrow\wr & & \downarrow\wr \\
((\mathcal{D}b^t_V)^\wedge)_\xi & & (\mathcal{O}^{tt}_{V^*})_\xi
\end{array}$$

Then, the morphism (2.4) is an isomorphism in the stalk. We obtain an isomorphism of conic sheaves between $(\mathcal{D}b^t_V)^\wedge$ and $\mathcal{O}^{tt}_{V^*}$.

The sheaf $\mathcal{O}^{tt}_{V^*}$ (resp. $\mathcal{D}b^t_V$) is naturally endowed with a structure of $D(\mathbb{V}^*)$-module (resp. $D(\mathbb{V})$-module). Let us verify that the action of $D(\mathbb{V}^*)$ on $(\mathcal{D}b^t_V)^\wedge$ coincides with the induced action by $\mathcal{O}^{tt}_{V^*}$. Let ϕ be a section of $\mathcal{D}b^t_V$. We have

$$\zeta_i.\phi^\wedge = (\partial_{z_i}.\phi)^\wedge, \qquad \partial_{\zeta_i}.\phi^\wedge = (-z_i.\phi)^\wedge.$$

And the two actions are clearly interchanged by the Laplace transform 0.1 □

COROLLARY 2.6. *Let M be a finitely generated $D(\mathbb{V}\times\overline{\mathbb{V}})$-module. Then the Laplace transform induces an isomorphism*

$$R\mathcal{H}om_{D(\mathbb{V}\times\overline{\mathbb{V}})}(M, \mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})^\wedge \simeq R\mathcal{H}om_{D(\mathbb{V}^*\times\overline{\mathbb{V}}^*)}(M^\wedge, \mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}).$$

2.2. COMPLEX FRAMEWORK

Now we can give the main result of this paper which is already stated in [K-S3]. In their paper, M. Kashiwara and P. Schapira establish the Laplace isomorphism for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+,\mathbb{R}-c}(\mathbb{C}_\mathbb{V})$

$$\mathrm{THom}(F, \mathcal{O}_\mathbb{V}) \xleftarrow{\sim} \mathrm{THom}(F^\wedge[n], \mathcal{O}_{\mathbb{V}^*}),$$

which is linear over the Weyl algebra $D(\mathbb{V})$. Then, the authors quantize the Fourier–Sato transform of $\mathcal{O}^t_\mathbb{V}$. Our approach is different since we use the isomorphism (2.2) (in a real framework on convex tubes) to prove the

THEOREM 2.7. *Let N be a finitely generated $D(\mathbb{V})$-module. The Fourier–Sato transform $(\mathcal{O}^t_\mathbb{V})^\wedge[n]$ is concentrated in degree 0, where n is the dimension of $\mathbb{V}$. Moreover, we have the $D(\mathbb{V})$-linear isomorphism induced by the Laplace transform*

$$R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}^t_\mathbb{V})^\wedge \simeq R\mathcal{H}om_{D(\mathbb{V}^*)}(N^\wedge, \mathcal{O}^t_{\mathbb{V}^*})[-n]. \tag{2.5}$$

Proof. Applying Corollary 2.6 for the $D(\mathbb{V}\times\overline{\mathbb{V}})$-module $M = N \boxtimes \mathcal{O}(\overline{\mathbb{V}})$, we obtain

$$R\mathcal{H}om_{D(\mathbb{V}\times\overline{\mathbb{V}})}(N\boxtimes\mathcal{O}(\overline{\mathbb{V}}), \mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})^\wedge \simeq R\mathcal{H}om_{D(\mathbb{V}^*\times\overline{\mathbb{V}}^*)}((N\boxtimes\mathcal{O}(\overline{\mathbb{V}}))^\wedge, \mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}).$$

Concerning the left-hand side, we have

$$\begin{aligned} &R\mathcal{H}om_{D(\mathbb{V}\times\overline{\mathbb{V}})}(N\boxtimes\mathcal{O}(\overline{\mathbb{V}}), \mathcal{D}b^t_{\mathbb{V}_\mathbb{R}}) \\ &\quad\simeq R\mathcal{H}om_{D(\mathbb{V})}(N, R\mathcal{H}om_{D(\overline{\mathbb{V}})}(\mathcal{O}(\overline{\mathbb{V}}), \mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})) \\ &\quad= R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}^t_\mathbb{V}). \end{aligned}$$

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As for the right-hand side,

$$
\begin{aligned}
&R\mathcal{H}om_{D(\mathbb{V}^*\times\overline{\mathbb{V}}^*)}((N\boxtimes\mathcal{O}(\overline{\mathbb{V}}))^\wedge,\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}})\\
&\quad\simeq R\mathcal{H}om_{D(\mathbb{V}^*\times\overline{\mathbb{V}}^*)}(N^\wedge\boxtimes\mathcal{O}(\overline{\mathbb{V}})^\wedge,\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}})\\
&\quad\simeq R\mathcal{H}om_{D(\mathbb{V}^*)}(N^\wedge,R\mathcal{H}om_{D(\overline{\mathbb{V}}^*)}(\mathcal{O}(\overline{\mathbb{V}})^\wedge,\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}))\\
&\quad\simeq R\mathcal{H}om_{D(\mathbb{V}^*)}(N^\wedge,R\mathcal{H}om_{D(\overline{\mathbb{V}}^*)}(\mathcal{B}_{\{0\}|\overline{\mathbb{V}}^*},\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}))
\end{aligned}
$$

In the last isomorphism, we used the identification

$$
\mathcal{O}(\overline{\mathbb{V}})^\wedge=\left(\frac{\mathcal{D}(\overline{\mathbb{V}})}{(\partial_{\overline{z_1}},\ldots,\partial_{\overline{z_n}})}\right)^\wedge=\frac{\mathcal{D}(\overline{\mathbb{V}}^*)}{(\overline{\zeta_1},\ldots,\overline{\zeta_n})}=\mathcal{B}_{\{0\}|\overline{\mathbb{V}}^*}.
$$

$\mathcal{B}_{\{0\}|\overline{\mathbb{V}}^*}$ denotes the regular holonomic $\mathcal{D}(\overline{\mathbb{V}}^*)$-module of holomorphic hyperfunctions on $\{0\}$.

We are reduced to prove the isomorphism

$$
R\mathcal{H}om_{D(\overline{\mathbb{V}}^*)}(\mathcal{B}_{\{0\}|\overline{\mathbb{V}}^*},\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}))\simeq\mathcal{O}^t_{\mathbb{V}^*}[-n].
$$

We first consider the particular case $n=1$. The complex $R\mathcal{H}om_{D(\overline{\mathbb{V}}^*)}(\mathcal{B}_{\{0\}|\overline{\mathbb{V}}^*},\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}})$ is represented by the complex

$$
0\to\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}\underset{(\overline{\zeta})}{\longrightarrow}\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}\to 0.
$$

Since $(\overline{\zeta})$ is clearly injective, we are reduced to prove that Coker $(\overline{\zeta})=\mathcal{O}^t_{\mathbb{V}^*}$. Recall that (ζ) is the complex coordinate on $\mathbb{V}^*$ and denote by (η) the complex conjugate coordinate on $\overline{\mathbb{V}}^*$. Then $\mathbb{V}^*_\mathbb{R}$ is the diagonal of $\mathbb{V}^*\times\overline{\mathbb{V}}^*$ defined by $\zeta=\overline{\eta}$.

Let $g(\zeta,\eta)$ be a section of $\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}$. Its restriction $g(\zeta,\eta)_{|(\eta=0)}$ is tempered at zero and at infinity, and hence we get a map $\alpha\colon\mathcal{O}^{tt}_{\mathbb{V}^*_\mathbb{R}}\to\mathcal{O}^t_{\mathbb{V}^*}$. This map is surjective, a right inverse being given by $f(\zeta)\mapsto g(\zeta,\eta)=f(\zeta+\eta)$.

For general n, $\mathcal{B}_{\{0\}|\overline{\mathbb{V}}^*}$ is isomorphic to the negative Koszul complex associated to the sequence $(\cdot\overline{\zeta_1},\ldots,\cdot\overline{\zeta_n})$ of $D(\overline{\mathbb{V}}^*)$-linear endomorphisms of $D(\overline{\mathbb{V}}^*)$. Since ζ_1 is injective and ζ_p induces an injective endomorphism of the quotient $\mathcal{D}(\overline{\mathbb{V}}^*)/(\overline{\zeta_1},\ldots,\overline{\zeta_{p-1}})$, this complex is exact except in degree zero.

More precisely, let us denote by $e_1,\ldots,e_p$ the canonical basis of $\mathbb{Z}^p$ and we set $D(\overline{\mathbb{V}}^*)^{(k)}=D(\overline{\mathbb{V}}^*)\otimes\Lambda^k\mathbb{Z}^p$. Then, one has

$$
K_\bullet(D(\overline{\mathbb{V}}^*);\overline{\zeta_1},\ldots,\overline{\zeta_n})=\left(0\to D(\overline{\mathbb{V}}^*)^{(n)}\underset{\delta}{\to}\cdots\underset{\delta}{\to}D(\overline{\mathbb{V}}^*)\to 0\right),
$$

where

$$
\begin{aligned}
&\delta_k\colon D(\overline{\mathbb{V}}^*)^{(k)}\to D(\overline{\mathbb{V}}^*)^{(k-1)}\\
&m\otimes e_{i_1\ldots i_k}\mapsto\sum_{j=1}^k(-1)^{j-1}m\cdot\zeta_{i_j}\otimes e_{i_1\ldots\hat{i_j}\ldots i_k}.
\end{aligned}
$$

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And we find

$$R\mathcal{H}om_{D(\overline{\mathbb{V}}^*)}(\mathcal{B}_{\{0\}|\overline{\mathbb{V}}^*}, \mathcal{O}^{tt}_{\mathbb{V}^*_{\mathbb{R}}}) = \mathrm{Hom}_{D(\overline{\mathbb{V}}^*)}(K_\bullet(D(\overline{\mathbb{V}}^*); \overline{\zeta_1}, \ldots, \overline{\zeta_n}), \mathcal{O}^{tt}_{\mathbb{V}^*_{\mathbb{R}}})$$
$$\simeq \mathcal{O}^t_{\mathbb{V}^*}[-n]. \qquad \square$$

In the particular case where $N = D(\mathbb{V})$, we recover the result of [K-S3].

THEOREM 2.8. $(\mathcal{O}^t_{\mathbb{V}})^\wedge[n] \simeq \mathcal{O}^t_{\mathbb{V}^*}$.

These last two theorems allow us to recover a result of Brylinski–Malgrange–Verdier [B-M-V] and Hotta–Kashiwara [H-K]. This gives the transformation between the sheaf of solutions of N and that of $N^\wedge$, in the case where N is a monodromic $D(\mathbb{V})$-module. Recall that a finitely generated $D(\mathbb{V})$-module N is monodromic if $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty$ for any $u \in N$ and θ the Euler vector field on $\mathbb{V}$. Roughly speaking, a sheaf on $\mathbb{V}$ is monodromic if it is locally constant on the $\mathbb{C}^*$-orbits.

COROLLARY 2.9 ([B-M-V], [H-K], [Ma1]). *Let N be a monodromic $D(\mathbb{V})$-module. Then*

(i) *$N^\wedge$ is monodromic.*
(ii) *$R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}_{\mathbb{V}})$ is monodromic.*
(iii) *$R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}^t_{\mathbb{V}}) \xrightarrow{\sim} R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}_{\mathbb{V}})$.*
(iv) *$R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}_{\mathbb{V}})$ is a conic sheaf (i.e. belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{\mathbb{V}})$).*
(v) *The Laplace morphism induces an isomorphism*

$$R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}_{\mathbb{V}})^\wedge[n] \simeq R\mathcal{H}om_{D(\mathbb{V}^*)}(N^\wedge, \mathcal{O}_{\mathbb{V}^*}).$$

Proof. (i) If $u \in N$ and $b \in \mathbb{C}[s]$ such that $b(\theta).u = 0$ then $b(\theta)^\wedge = b(-n-\theta')$ where θ' is the Euler vector field on $\mathbb{V}^*$, and $b(-n-\theta').u^\wedge = 0$.

(ii) is obvious.

(iii) Resolving N by modules like $D(\mathbb{V})/b(\theta)$, one can reduce to the case where N is like this. By induction, one can suppose that $b(\theta) = \theta - \lambda$, with $\lambda \in \mathbb{C}$. Then the result follows.

(iv) Since $R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}^t_{\mathbb{V}})$ is a conic sheaf.

(v) follows from (iii) and the isomorphism (2.5). $\square$

Remark 2.10 ([B-M-V]). The Riemann–Hilbert correspondence permits to state: if $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{V})$ such that $Rj_*F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{P})$ (where $j\colon \mathbb{V} \hookrightarrow \mathbb{P}$), then there exists a complex N of $D(\mathbb{V})$-modules with regular holonomic cohomology and $R\mathcal{H}om_{D(\mathbb{V})}(N, \mathcal{O}_{\mathbb{V}}) = F$. If moreover F is monodromic, then N is monodromic.

But $\wedge$ exchanges monodromic regular holonomic $D(\mathbb{V})$-modules with monodromic regular holonomic $D(\mathbb{V}^*)$-modules (this statement becomes false if we do not require the hypothesis of monodromy). Finally, the Fourier–Sato transform

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interchanges monodromic perverse sheaves on $\mathbb{V}$ with monodromic perverse sheaves on $\mathbb{V}^*$.

As another consequence of Theorem 2.8, we can give a vanishing theorem for the sections of $\mathcal{O}^t_{\mathbb{V}}$. First, we have the obvious lemma.

LEMMA 2.11. *Let γ be a subanalytic open cone in $\mathbb{V}$. Then the groups $H^j(\gamma; \mathcal{O}^t_{\mathbb{V}})$ vanish for $j \geqslant n+1$.*

Proof. Naturally, the complex $\mathrm{R}\Gamma(\gamma; \mathcal{O}^t_{\mathbb{V}})$ is concentrated in degree $\geqslant 0$. Moreover, the sheaf $\mathcal{O}^t_{\mathbb{V}}$ admits the following conically soft resolution

$$0 \to \mathcal{D}b^t_{\mathbb{V}_\mathbb{R}} \underset{\bar\partial}{\to} (\mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})^{(0,1)} \underset{\bar\partial}{\to} \cdots \underset{\bar\partial}{\to} (\mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})^{(0,n)} \to 0.$$

By the $\Gamma(\gamma; \cdot)$-acyclicity of $\mathcal{D}b^t_{\mathbb{V}_\mathbb{R}}$, we obtain $H^j(\gamma; \mathcal{O}^t_{\mathbb{V}}) = 0 \ \ \forall j \geqslant n+1$. □

If, moreover, the cone γ is convex, one can use the isomorphism $(\mathcal{O}^t_{\mathbb{V}})^\wedge[n] \simeq \mathcal{O}^t_{\mathbb{V}^*}$ to prove the

PROPOSITION 2.12. *Let γ be a subanalytic convex open cone in $\mathbb{V}$. Then the complex $\mathrm{R}\Gamma(\gamma; \mathcal{O}^t_{\mathbb{V}})$ is concentrated in degree 0 and 1.*

Proof. First, the Laplace transform induces the isomorphism

$$\mathrm{R}\Gamma(\gamma; \mathcal{O}^t_{\mathbb{V}}) \simeq \mathrm{R}\Gamma_{\gamma^\circ}(\mathbb{V}^*; \mathcal{O}^t_{\mathbb{V}^*})[n].$$

On one hand, the complex $\mathrm{R}\Gamma(\gamma; \mathcal{O}^t_{\mathbb{V}})$ is naturally concentrated in degree $\geqslant 0$. On the other hand, the complex $\mathrm{R}\Gamma_{\gamma^\circ}(\mathbb{V}^*; \mathcal{O}^t_{\mathbb{V}^*})[n]$ is concentrated in degree $\leqslant 1$. Indeed, we have the exact long sequence

$$\cdots \to H^n(\mathbb{V}^*; \mathcal{O}^t_{\mathbb{V}^*}) \to H^n(\mathbb{V}^* \setminus \gamma^\circ; \mathcal{O}^t_{\mathbb{V}^*}) \to H^{n+1}_{\gamma^\circ}(\mathbb{V}^*; \mathcal{O}^t_{\mathbb{V}^*}) \to \cdots$$

Let us recall that $\mathrm{R}\Gamma(\mathbb{V}; \mathcal{O}^t_{\mathbb{V}}) \simeq \mathbb{C}[\mathbb{V}]$. Then, we obtain the isomorphism

$$H^j(\mathbb{V}^* \setminus \gamma^\circ; \mathcal{O}^t_{\mathbb{V}^*}) \simeq H^{j+1}_{\gamma^\circ}(\mathbb{V}^*; \mathcal{O}^t_{\mathbb{V}^*}) \text{ for } j > 0.$$

By Lemma 2.11, we know that for any U subanalytic open cone in $\mathbb{V}^*$ the groups $H^j(U; \mathcal{O}^t_{\mathbb{V}^*})$ vanish for $j \geqslant n+1$. Then the groups $H^j_{\gamma^\circ}(\mathbb{V}^*; \mathcal{O}^t_{\mathbb{V}^*})$ vanish for $j \geqslant n+2$, and the complex $\mathrm{R}\Gamma_{\gamma^\circ}(\mathbb{V}^*; \mathcal{O}^t_{\mathbb{V}^*})$ is concentrated in degree $\leqslant n+1$. □

PROBLEM 2.13. We do not know whether $H^1(\gamma; \mathcal{O}^t_{\mathbb{V}}) = 0$ for a subanalytic convex open cone γ. More generally, it would be interesting to show that $H^j(\gamma; \mathcal{O}^t_{\mathbb{V}}) = 0$ for $j \geqslant n$ for a subanalytic (not necessarily convex) open cone γ. The vanishing of the group $H^n(\gamma; \mathcal{O}^t_{\mathbb{V}})$ is equivalent to the surjectivity of

$$\Gamma(\gamma; (\mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})^{(0,n-1)}) \underset{\bar\partial}{\to} \Gamma(\gamma; (\mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})^{(0,n)}). \tag{2.6}$$

Following a technique of B. Malgrange [Ma2], it is enough to show the surjectivity of the Laplace operator Δ

$$\Gamma(\gamma; \mathcal{D}b^t_{\mathbb{V}_\mathbb{R}}) \underset{\Delta}{\to} \Gamma(\gamma; \mathcal{D}b^t_{\mathbb{V}_\mathbb{R}}).$$

Indeed, let us give $\omega = \alpha d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n \in \Gamma(\gamma; (\mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})^{(0,n)})$. If there exists $\beta \in \Gamma(\gamma; \mathcal{D}b^t_{\mathbb{V}_\mathbb{R}})$ such that

$$4\Delta\beta = \sum_{i=1}^{n} \frac{\partial^2 \beta}{\partial z_i \partial \bar{z}_i} = \alpha,$$

then the form

$$\varpi = \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial \beta}{\partial z_i} d\bar{z}_1 \wedge \ldots \widehat{d\bar{z}_i} \ldots \wedge d\bar{z}_n$$

does satisfy $\bar{\partial}\varpi = \omega$. But the surjectivity of the morphism (2.6) is an open problem.

3. Fourier Hyperfunctions

We keep the same notations as in the last section.

Recall that Sato's hyperfunctions on V are defined by

$$\mathcal{B}_V = R\mathcal{H}om(\mathbb{C}_V[-n], \mathcal{O}_\mathbb{V}).$$

Within our framework, the isomorphism $(\mathcal{O}^t_\mathbb{V})^\wedge[n] \simeq \mathcal{O}^t_{\mathbb{V}^*}$ allows us to construct a complex $\mathcal{B}^t_V$ whose global sections are invariant by Fourier transform.

DEFINITION 3.1. We define the complex of tempered hyperfunctions by

$$\mathcal{B}^t_V = R\mathcal{H}om(\mathbb{C}_V[-n], \mathcal{O}^t_\mathbb{V}).$$

PROPOSITION 3.2. *The global sections of the complex $\mathcal{B}^t_V$ is stable under Fourier transform. More precisely,*

$$\mathrm{RHom}(\mathbb{C}_V[-n], \mathcal{O}^t_\mathbb{V}) \xrightarrow{\sim} \mathrm{RHom}(\mathbb{C}_{iV^*}[-n], \mathcal{O}^t_{\mathbb{V}^*}).$$

Proof. It is a direct consequence of the Theorem 2.8.
Indeed, we have the following isomorphisms

$$\begin{aligned}\mathrm{RHom}(\mathbb{C}_V[-n], \mathcal{O}^t_\mathbb{V}) &\simeq \mathrm{RHom}(\mathbb{C}^\wedge_V[-n], \mathcal{O}^{t\,\wedge}_\mathbb{V}) \\ &\simeq \mathrm{RHom}(\mathbb{C}_{iV^*}[-n], \mathcal{O}^t_{\mathbb{V}^*}).\end{aligned} \qquad \square$$

Remark 3.3. We do not know whether the complex $\mathcal{B}^t_V$ is concentrated in degree zero.

We have only the vanishing of the groups $H^j\mathrm{RHom}(\mathbb{C}_V[-n], \mathcal{O}^t_\mathbb{V})$ for $j > 1$. Indeed, in the proof of Proposition 2.12, we get the following isomorphism for

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any λ subanalytic closed cone in $\mathbb{V}$

$$H^j(\mathbb{V} \setminus \lambda; \mathcal{O}^t_{\mathbb{V}}) \simeq H^{j+1}_{\lambda}(\mathbb{V}; \mathcal{O}^t_{\mathbb{V}}) \quad \text{for } j > 0.$$

Then, we find the vanishing of the groups $H^j_{\lambda}(\mathbb{V}; \mathcal{O}^t_{\mathbb{V}})$ for $j > n+1$.

In particular, for $\lambda = V$, we obtain that the complex $R\mathcal{H}om(\mathbb{C}_V[-n], \mathcal{O}^t_{\mathbb{V}}) = \mathrm{R}\Gamma_V(\mathbb{V}; \mathcal{O}^t_{\mathbb{V}})[n]$ is concentrated in degree $\leqslant 1$.

In the article [K], T. Kawai defines the sheaf $\mathcal{O}_{\#}$ (resp. $\mathcal{O}^{\#}$) of the germ of slowly increasing holomorphic functions on $\mathbb{D}^n \times i\mathbb{R}^n$ (resp. rapidly decreasing holomorphic functions) where $\mathbb{D}^n$ is the compactification $\mathbb{R}^n \cup S^{n-1}_{\infty}$. Let us remark that in his framework the growth condition is exponential and not polynomial. Then the author can define the sheaf $\mathcal{R}$ of hyperfunctions associated with $\mathcal{O}_{\#}$ by

$$\mathcal{R}(\Omega) = H^n_{\Omega}(\mathbb{D}^n \times i\mathbb{R}^n; \mathcal{O}_{\#}),$$

where Ω is open in $\mathbb{D}^n$.

Naturally, the sheaf $\mathcal{R}$ coincides with $\mathcal{B}$ (the sheaf of hyperfunctions of M. Sato) on $\mathbb{R}^n$. The duality between $\mathcal{R}(\mathbb{D}^n)$ and $\mathcal{O}^{\#}(\mathbb{D}^n)$ and the stability of $\mathcal{O}^{\#}(\mathbb{D}^n)$ under the Fourier transform permit to define the Fourier transform of a section μ of $\mathcal{R}(\mathbb{D}^n)$ by

$$\langle \mathcal{F}\mu, \phi \rangle = \langle \mu, \mathcal{F}\phi \rangle.$$

3.1. FOURIER MICROFUNCTIONS

Similarly, we can define a biconic sheaf on $V \times V^* \simeq T_V\mathbb{V}$ by

$$\mathcal{C}^t_V = \mu hom(\mathbb{C}_V[-n], \psi^{-1}\mathcal{O}^t_{\mathbb{V}}),$$

where $\psi\colon \mathbb{V} \to \mathbb{V}$ is the identity map from $\mathbb{V}$, endowed with the usual topology, to $\mathbb{V}$ endowed with the conic topology induced by the projection $\mathbb{V} \to S^{2n-1} \cup \{0\}$.

Using the isomorphism [K-S2, Ex. VII.2] and identifying

$$\begin{aligned} V \times V^* &\xrightarrow{\sim} V^* \times V \\ (x, \xi) &\longmapsto (-\xi, x) \end{aligned}$$

we get the isomorphism of complexes of sheaves on $V \times V^*$

$$(\mathcal{C}^t_V)^\wedge \simeq \mathcal{C}^t_V, \tag{3.1}$$

of which the result of Proposition 3.2 is a particular case.

3.2. FURTHER DEVELOPMENT

In [K-S3], the authors remark that one could construct the biconic sheaf of rings $\mathcal{E}^t_{\mathbb{V}}$ of tempered microdifferential operators on $\mathbb{V} \times \mathbb{V}^*$. This sheaf is invariant by

Fourier transform. It would be a natural task to develop systematically such a theory and, in particular, to define the action of $\mathcal{E}'_{\mathbb{V}}$ on $\mathcal{C}'_V$.

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