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FACTORIZATION OF POSITIVE INVERTIBLE OPERATORS IN AF ALGEBRAS

HOUBEN HUANG AND TIMOTHY D. HUDSON

ABSTRACT. We examine the problem of factoring a positive invertible operator in an AF C^*-algebra as T^*T for some invertible operator T with both T and T^{-1} in a triangular AF subalgebra. A factorization theorem for a certain class of positive invertible operators in AF algebras is proven. However, we explicitly construct a positive invertible operator in the CAR algebra which cannot be factored with respect to the 2^∞ refinement algebra. Our main result generalizes this example, showing that in any AF algebra, there exist positive invertible operators which fail to factor with respect to a given triangular AF subalgebra. We also show that in the context of AF algebras, the notions of having a factorization and having a weak factorization are the same.

1. **Introduction.** Since their introduction by James Glimm over thirty years ago [7], UHF algebras have formed a very important class of C^*-algebras and the subject of extensive literature. This is due, in part, to their accessible yet highly nontrivial nature. Equally as interesting and nontrivial are the AF algebras, a generalization of UHF algebras first introduced by Ola Bratteli in the early 1970s [3]. More recently, certain triangular subalgebras of UHF algebras have become the key motivating examples in the study of limit subalgebras of C^*-algebras and von Neumann algebras; see, for example, the work of Baker [2], Hopenwasser and Power [10], Muhly, Saito, and Solel [15, 16], Muhly and Solel [14], Peters, Poon, and Wagner [18, 19], Peters and Wagner [20], Power [27, 28], and Ventura [30], to name a few; the recent monograph by Power [23] gives an excellent overview of this and other recent progress in the area. Most of this work has dealt with classifying the various limit subalgebras of a given C^*-algebra up to isometric isomorphism, although reflexivity [21], representation theory [17], and ideal structure [6, 12, 26] have also been examined.

The purpose of this note is to investigate the problem of factoring a positive invertible operator in a UHF or AF algebra as T^*T for some invertible operator T, with T and T^{-1} in a triangular limit subalgebra. We first produce a positive invertible operator in the CAR algebra which does not factor with respect to a very nice triangular limit algebra; this triangular limit algebra is even a nest subalgebra of the CAR algebra. Our construction is based on the failure of the boundedness of triangular truncation in infinite dimensions. In a more positive direction, using rather delicate estimates on the Cholesky factorizations of positive definite matrices, we are able to prove a nontrivial class of positive invertible

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operators will always factor with respect to a large class of triangular limit subalgebras. Finally, we show our construction in the CAR algebra generalizes, and thus we prove there always exist positive invertible operators which do not factor for a given triangular subalgebra.

This problem of factorization in AF algebras was motivated by the study of factorization questions among another class of triangular subalgebras of C^*-algebras, the nest algebras. The question in this setting is to determine the conditions under which a positive invertible operator factors as T^*T for an invertible element T, with both T and T^{-1} in the nest algebra of some complete nest. This has proven to be a deep problem related to the similarity theory of nests [13]. Such factorizations were first considered by Gohberg and Krein in the 1960s [8, 9], and subsequently studied by many authors, viz., Arveson [1], Davidson and Huang [5], Larson [13], Pitts [22], Power [24, 25], *etc.* (see [4] for an overview).

A C^*-algebra $\mathcal{A}$ is said to be *approximately finite* (AF) if there is an ascending sequence of finite-dimensional C^*-subalgebras of $\mathcal{A}$ whose union is dense in $\mathcal{A}$. If $\mathcal{A}$ is unital and each of these finite-dimensional subalgebras both contains the unit of $\mathcal{A}$ and is a factor, *i.e.*, is isomorphic to a full matrix algebra, then $\mathcal{A}$ is called *uniformly hyperfinite* (UHF). A norm-closed subalgebra $\mathcal{T}$ of an AF algebra $\mathcal{A}$ is called *triangular* AF if the *diagonal* $\mathcal{D} = \mathcal{T} \cap \mathcal{T}^*$ is a canonical maximal abelian self-adjoint subalgebra (masa) in $\mathcal{A}$ (see [23] for details). A *strongly maximal triangular subalgebra* $\mathcal{T}$ is a triangular subalgebra for which $\mathcal{T} + \mathcal{T}^*$ is dense in $\mathcal{A}$. Such triangular subalgebras can always be represented as direct limits of their corresponding chain of finite-dimensional subalgebras via some sequence of embeddings, and so are often referred to as triangular limit subalgebras. We will use the notation $\mathcal{M}_n$ for the $n \times n$ complex matrices, and $\mathcal{T}_n$, $\mathcal{D}_n$, and I_n for the $n \times n$ complex upper triangular, diagonal, and identity matrices, respectively.

The general setting in which we will be most interested will be the factorization of positive invertible operators of the UHF algebra of type $\{p_n\}_{n=1}^{\infty}$, where $\{p_n\}_{n=1}^{\infty}$ is strictly increasing and $p_n | p_{n+1}$ for all n. The limit subalgebras $\mathcal{T}$ we will consider are those of the form

$$\mathcal{T}_{p_1} \xrightarrow{\varphi_1} \mathcal{T}_{p_2} \xrightarrow{\varphi_2} \mathcal{T}_{p_3} \xrightarrow{\varphi_3} \cdots \longrightarrow \mathcal{T},$$

where $\varphi_n \colon \mathcal{T}_{p_n} \longrightarrow \mathcal{T}_{p_{n+1}}$ is a *regular embedding*. This basically means that the mapping can be extended to a $*$-homomorphism between the corresponding full matrix algebras and that φ_n takes matrix units in $\mathcal{T}_{p_n}$ to a sum of p_{n+1}/p_n matrix units in $\mathcal{T}_{p_{n+1}}$; a more technical definition which we will not need is given in [23]. For all $m \geq n$, the maps $\varphi_{m,n} \colon \mathcal{T}_{p_n} \longrightarrow \mathcal{T}_{p_m}$ are the compositions

$$\varphi_{m,n}(a) = \varphi_{m-1} \circ \cdots \circ \varphi_n(a),$$

for $a \in \mathcal{T}_{p_n}$.

We use the notation $e_{ij}^{(n)}$ for a general matrix unit in $\mathcal{T}_{p_n}$, and $e_i^{(n)}$ will denote the diagonal matrix unit $e_{ii}^{(n)}$. If A belongs to some $\mathcal{T}_{p_n}$, we shall follow the common abuse of notation by allowing A to denote both the finite-dimensional operator in $\mathcal{T}_{p_n}$ and its image in the limit algebra, $\mathcal{T}$.

Say that a positive invertible operator A in an AF algebra *factors* with respect to a limit subalgebra $\mathcal{T}$ if $A = T^*T$ for an invertible element T in $\mathcal{T}$ with T^{-1} also in $\mathcal{T}$. We will say that a positive invertible operator A has the *universal factorization property* provided A factors with respect to every strongly maximal triangular limit subalgebra. The *Cholesky factorization* of a positive definite matrix A in $\mathcal{M}_n$ is $A = T^*T$, where T is an invertible upper triangular matrix with positive diagonal elements.

We will need to make use of the representation theory of triangular AF algebras in the sequel, so we require the following definitions. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$. A *nest* $\mathcal{N}$ is a chain of closed subspaces of $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$ which is complete under the operations of intersection and closed linear span. The corresponding *nest algebra*, $\mathcal{T}(\mathcal{N})$, is the subalgebra of $\mathcal{B}(\mathcal{H})$ consisting of all operators on $\mathcal{H}$ leaving each element of $\mathcal{N}$ invariant. The *diagonal* $\mathcal{D}(\mathcal{N})$ of $\mathcal{T}(\mathcal{N})$ is $\mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$.

For convenience, we record two theorems we will need for our analysis. The first is a representation theorem for triangular AF algebras from [17]. Recall that a C^*-algebra is *primitive* if it has a faithful irreducible representation; in particular, the following theorem applies to all UHF algebras.

THEOREM 1.1 (ORR-PETERS). *Let $\mathcal{A}$ be a primitive* AF *algebra and $\mathcal{T}$ a strongly maximal triangular limit subalgebra of $\mathcal{A}$ with canonical masa $\mathcal{D}$. Then there exists a Hilbert space $\mathcal{H}$, a faithful irreducible representation π of $\mathcal{A}$, and a nest $\mathcal{N}$ in $\mathcal{H}$ such that $\overline{\pi(\mathcal{T})}^{\mathrm{wk}} = \mathcal{T}(\mathcal{N})$ and $\overline{\pi(\mathcal{D})}^{\mathrm{wk}} = \mathcal{D}(\mathcal{N})$.*

For our positive results in Section 3, we need the following estimates on the Cholesky factors of positive invertible matrices from [29].

THEOREM 1.2 (SUN). *Let A and B be two positive invertible elements of $\mathcal{M}_n$ with Cholesky factorizations T^*T and S^*S, respectively. If*

$$\|A^{-1}\|\,\|A-B\| < 1,$$

then

$$\|T-S\|_2 \leq \frac{\|A\|^{\frac{1}{2}}\|A^{-1}\|\,\|A-B\|_2}{\sqrt{2(1-\|A^{-1}\|\,\|A-B\|)}},$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.

2. **An example.** In this section we give an example of a positive invertible operator in the 2^∞ UHF algebra $\mathcal{A}$ which does not factor in the refinement limit algebra $\mathcal{T}$. The refinement embeddings $\rho_n\colon \mathcal{M}_{2^n} \longrightarrow \mathcal{M}_{2^{n+1}}$ are defined for $[a_{ij}] \in \mathcal{M}_{2^n}$ by

$$\rho_n([a_{ij}]) = [a_{ij}I_2],$$

where I_2 is the 2×2 identity matrix. Thus, the algebras $\mathcal{A}$ and $\mathcal{T}$ can be represented as the direct limits $\varinjlim(\mathcal{M}_{2^n};\rho_n)$ and $\varinjlim(\mathcal{T}_{2^n};\rho_n)$, respectively.

We need several preliminary lemmas. The first one gives the properties of a class of matrices from [4, Example 12.19]. These properties form the basis of our analysis.

LEMMA 2.1. *For any positive integer n, let T_n be the strictly upper triangular $n \times n$ Toeplitz matrix (t_{ij}), where $t_{ij} = \frac{1}{j-i}$ for $j > i$. Let $V_n = e^{iS_n}$, where S_n is defined by*

$$S_n = \frac{2T_n}{\|T_n + T_n^*\|}.$$

Then V_n is an invertible upper triangular matrix with diagonal $\Delta(V_n) = I_n$, and moreover,

$$\lim_{n\to\infty} \|V_n - I_n\| = \sin\frac{1}{2}$$

even though

$$\lim_{n\to\infty} \|V_n^* V_n - I_n\| = 0.$$

We will make use of the next two lemmas both in this section and in our more general results of Section 4.

LEMMA 2.2. *Let $\mathcal{N}$ be a nest and $\{D_n\}_{n=1}^{\infty}$ be a sequence of diagonal operators in $\mathcal{T}(\mathcal{N})$. If*

$$\mathcal{F}_n = \left\{\{0\} = N_0 < N_1 < \cdots < N_n = \mathcal{H}\right\}$$

is a finite partition of $\mathcal{N}$, then, regarding V_n as an operator in $\mathcal{T}(\mathcal{N})$,

$$\lim_{n\to\infty} \|V_n - D_n\| \geq \frac{1}{2}\sin\frac{1}{2}.$$

PROOF. Let n be any integer, and set $\Delta P_k = P(N_k) - P(N_{k-1})$ for $k = 1, \ldots, n$. If $\|I - D_n\| \geq \frac{1}{2}\sin\frac{1}{2}$, there exists at least one k, with $1 \leq k \leq n$, such that $\|\Delta P_k(I - D_n)\Delta P_k\| \geq \frac{1}{2}\sin\frac{1}{2}$. Therefore,

$$\begin{aligned}\|V_n - D_n\| &\geq \|\Delta P_k(V_n - D_n)\Delta P_k\| \\ &= \|\Delta P_k(I - D_n)\Delta P_k\| \\ &\geq \frac{1}{2}\sin\frac{1}{2}.\end{aligned}$$

Otherwise, if $\|I - D_n\| < \frac{1}{2}\sin\frac{1}{2}$, then

$$\|V_n - D_n\| \geq \|V_n - I\| - \|I - D_n\| > \|V_n - I\| - \frac{1}{2}\sin\frac{1}{2}.$$

Thus, by Lemma 2.1, we are done. ∎

LEMMA 2.3. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{B}(\mathcal{H})$ such that A_n converges strongly to A and A_n^{-1} converges strongly to B. If the norms of A_n and A_n^{-1} are uniformly bounded, then $B = A^{-1}$.*

PROOF. Let $\eta > 0$ be the uniform bound on the norms of A_n and A_n^{-1}. For every $x \in \mathcal{H}$,

$$\begin{aligned}\|(AB - I)x\| &= \lim_{n\to\infty} \|(A_nB - I)x\| \\ &= \lim_{n\to\infty} \|A_n(B - A_n^{-1})x\| \\ &\leq \lim_{n\to\infty} \eta\|(B - A_n^{-1})x\| = 0.\end{aligned}$$

It follows that $AB = I$. Similarly, $BA = I$, and so A and B are invertible and $B = A^{-1}$. ■

To begin our construction, first note that by Lemma 2.1, there exists an integer M such that if $n \geq M$, then $\|V_{2^n}^* V_{2^n} - I_{2^n}\| < \frac{1}{2}$. Let $X_M = V_{2^M} \oplus I_{2^M} \in \mathcal{T}_{2^{M+1}}$, and, if $n > M$, define $X_n \in \mathcal{T}_{2^{2n-M+1}}$ by

$$X_n = \rho_{2n-M,M}(V_{2^M}) \oplus \rho_{2n-M-1,M+1}(V_{2^{M+1}}) \oplus \cdots \oplus \rho_{n+1,n-1}(V_{2^{n-1}}) \oplus V_{2^n} \oplus I_{2^n}.$$

LEMMA 2.4. *Let $\{X_n\}_{n=M}^{\infty}$ be the sequence constructed above. Then*

(i) there exists a positive invertible element A of $\mathcal{A}$ such that $X_n^*X_n$ converges in norm to A, but

(ii) no subsequence of $\{X_n\}_{n=M}^{\infty}$ is Cauchy.

PROOF. Let $\varepsilon > 0$ be given. Using Lemma 2.1, choose $N \geq M$ such that if $n \geq N$, then $\|V_{2^n}^* V_{2^n} - I_{2^n}\| < \varepsilon$. Thus, for $n \geq m \geq N$, we have

$$\begin{aligned}\|X_n^*X_n - \rho_{2n-M+1,2m-M+1}(X_m^*X_m)\| &= \Big\|\sum_{i=m+1}^{n} \oplus\rho_{2n-i,i}(V_{2^i}^* V_{2^i} - I_{2^i})\Big\| \\ &= \max_{m+1\leq i\leq n} \|V_{2^i}^* V_{2^i} - I_{2^i}\| < \varepsilon.\end{aligned}$$

Hence, by completeness there is some element $A \in \mathcal{A}$ such that $X_n^*X_n$ converges to A in the norm topology. Since A is a limit of positive operators, then clearly $A \geq 0$. Observe that

$$\begin{aligned}\|X_n^*X_n - I_{2^{2n-M+1}}\| &= \Big\|\sum_{i=M}^{n} \oplus\rho_{2n-i,i}(V_{2^i}^* V_{2^i} - I_{2^i})\Big\| \\ &= \max_{M\leq i\leq n} \|V_{2^i}^* V_{2^i} - I_{2^i}\| \\ &< \frac{1}{2},\end{aligned}$$

by the choice of M. It follows that

$$\|A - I\| = \lim_{n\to\infty} \|X_n^*X_n - I_{2^{2n-M+1}}\| \leq \frac{1}{2} < 1;$$

hence, A is invertible, proving (i).

To see (ii), apply Lemma 2.1 to obtain $N_0 \geq M$ such that $\|V_k - I_k\| > \frac{1}{2}\sin\frac{1}{2}$ provided $k \geq N_0$. Then for any $n > m \geq N_0$,

$$\begin{aligned}\|X_n - \rho_{2n-M+1,2m-M+1}(X_m)\| &= \Big\|\sum_{i=m+1}^{n} \oplus\rho_{2n-i,i}(V_{2^i} - I_{2^i})\Big\| \\ &= \max_{m+1\leq i\leq n} \|V_{2^i} - I_{2^i}\| \\ &\geq \|V_{2^n} - I_{2^n}\| > \frac{1}{2}\sin\frac{1}{2} > 0,\end{aligned}$$

and so the proof is complete. ■

We next show that although X_n does not have a norm limit in $\mathcal{T}$, it does have a limit in a different topology. To do this, we make use of the usual representation of $\mathcal{A}$ onto $\mathcal{B}(\mathcal{H})$, where $\mathcal{H} = \mathcal{L}^2([0,1],\mu)$ and μ is Lebesgue measure. Let $\mathcal{N}$ denote the Volterra nest and $\mathcal{T}(\mathcal{N})$ the corresponding Volterra nest algebra. If $\pi\colon \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is the representation acting on the matrix units of $\mathcal{A}$ by

$$\big(\pi(e_{ij}^{(n)})f\big)(t) = \chi_{[\frac{i-1}{2^n},\frac{i}{2^n}]}(t)f\Big(t+\frac{j-i}{2^n}\Big)$$

for f in $\mathcal{H}$, then it is well known that π extends to a faithful irreducible representation of $\mathcal{A}$ for which $\overline{\pi(\mathcal{T})}^{\mathrm{wk}} = \mathcal{T}(\mathcal{N})$. Let P_n be the projection in $\mathcal{B}(\mathcal{H})$ given by multiplication by the characteristic function of $[1-\frac{1}{2^{n-1}}, 1-\frac{1}{2^n}]$ and Q_n the projection given by multiplication by the characteristic function of $[1-\frac{1}{2^n}, 1]$. Then P_n can be written as

(1) $$P_n = \sum_{i=1}^{2^{n-M+1}} e^{(2n-M+1)}_{2^{2n-M+1}-2^{n-M+2}+i}$$

and Q_n as

(2) $$Q_n = \sum_{i=1}^{2^{n-M+1}} e^{(2n-M+1)}_{2^{2n-M+1}-2^{n-M+1}+i}.$$

Furthermore, observe that

(3) $$X_k\big|_{\mathcal{L}^2([0,1-\frac{1}{2^n}])} = X_n\big|_{\mathcal{L}^2([0,1-\frac{1}{2^n}])}$$

for all $k \geq n \geq M$.

LEMMA 2.5. *The sequence $\{X_n\}_n$ converges strongly to an invertible element X in $\mathcal{T}(\mathcal{N})$ such that X^{-1} is also in $\mathcal{T}(\mathcal{N})$ and $A = X^*X$.*

PROOF. Observe that by construction, each X_n is invertible, and so by the previous lemma there is a $B > 0$ such that $\|X_n\|, \|X_n^{-1}\| \leq B$ for all $n \geq M$. Let $\varepsilon > 0$ be given, and let $f \in \mathcal{H}$. Choose $\delta > 0$ such that for any measurable subset E of $[0,1]$ with $\mu(E) < \delta$, then $\int_E |f|^2\,d\mu < (2B)^{-1}\varepsilon$. Let $N \geq M$ be such that $2^{-N} < \delta$. Then for all $k \geq n \geq N$, by (3) above,

$$\|(X_k - X_n)f\| = \|(X_k - X_n)Q_nf\| \leq 2B\int_{1-\frac{1}{2^n}}^{1} |f|^2\,d\mu < \varepsilon.$$

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Hence, $\{X_n(f)\}_{n=M}^{\infty}$ is Cauchy for all $f \in \mathcal{H}$, *i.e.*, the sequence $\{X_n\}_{n=M}^{\infty}$ is strong operator topology Cauchy; let X be the strong limit of this sequence.

An argument analogous to the one above proves that $\{X_n^{-1}\}_{n=M}^{\infty}$ converges strongly to some operator Y. By Lemma 2.3, we have

$$X^{-1} = Y \in \overline{\pi(\mathcal{T})}^{\mathrm{SOT}} \subseteq \overline{\pi(\mathcal{T})}^{\mathrm{wk}} = \mathcal{T}(\mathcal{N}).$$

It is immediate from Lemma 2.4 and the above that $A = X^*X$. ∎

The lemma which follows is the main technical tool needed to prove our construction.

LEMMA 2.6. *Let X be the operator from the previous lemma, and let U be a unitary in $\mathcal{D}(\mathcal{N})$. Then UX does not belong to the 2^∞ refinement limit algebra.*

PROOF. Suppose that UX is in the 2^∞ refinement limit algebra. Then there is a sequence $\{T_n\}_{n=1}^{\infty}$ such that T_n converges to UX in norm. For notational simplicity, we assume $T_n \in \mathcal{T}_{2^n}$. Observe that by (3), for any $n \geq M$

$$X|_{\mathcal{L}^2([0,1-\frac{1}{2^n}])} = X_n|_{\mathcal{L}^2([0,1-\frac{1}{2^n}])},$$

and so $P_nX = P_nX_n$. Furthermore, by (1) and (2), if $m \geq n+1$, then $Q_n \geq P_m$. It follows that if $\lambda_n e_{2^n}^{(n)} = e_{2^n}^{(n)} T_n$, then $Q_nT_n = \lambda_n Q_n$.

Now fix $n \geq M$. By Lemma 2.2, we can choose $N_0 \geq M$ such that if $m \geq N_0$, then $\|V_{2^m} - \lambda_n P_m U^*\| > \frac{1}{4}\sin\frac{1}{2}$. Then for any integer m such that $m \geq n+1$ and $m \geq N_0$,

$$\begin{aligned}
\|UX - T_n\| &\geq \|Q_n(UX - T_n)\| \\
&= \|UQ_nX - \lambda_n Q_n\| \\
&= \|Q_nX - \lambda_n Q_n U^*\| \\
&\geq \|P_m(Q_nX - \lambda_n Q_n U^*)\| \\
&= \|P_mX - \lambda_n P_m U^*\| \\
&= \|P_mX_m - \lambda_n P_m U^*\| \\
&= \|V_{2^m} - \lambda_n P_m U^*\| \\
&> \frac{1}{4}\sin\frac{1}{2} > 0.
\end{aligned}$$

This contradiction proves the lemma. ∎

We now give the main result of this section.

THEOREM 2.7. *The operator A constructed in Lemma 2.4 above is a positive invertible element of the 2^∞* UHF *algebra which does not factor as T^*T for any invertible element T with T and T^{-1} in the 2^∞ refinement limit algebra.*

PROOF. If $A = T^*T$ for some $T \in \mathcal{T}(\mathcal{N}) \cap \big(\mathcal{T}(\mathcal{N})\big)^{-1}$, then $X^*X = T^*T$. By Lemma 2.5, X is invertible, and so $U = TX^{-1}$ is a unitary with both U and U^* belonging to $\mathcal{T}(\mathcal{N})$; it follows that $U \in \mathcal{D}(\mathcal{N})$. Since $T = UX$, then Lemma 2.6 implies that T does not belong to the refinement limit algebra. ∎

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3. **Factorizable operators.** We saw in the last section that the Cholesky factorization is not continuous, namely, we exhibited a sequence of upper triangular matrices $\{T_n\}$ with positive diagonals such that $T_nT_n - I_n$ converges to zero while $\|T_n - D_n\| \geq \delta$ for some fixed $\delta > 0$ and any diagonal operator D_n. This failure of continuity of the Cholesky factorization is the main difficulty of factorization problems in both nest algebras and AF algebras. In this section, we overcome this obstacle for a certain class of nontrivial positive invertible operators in any UHF algebra by applying concrete estimates on the Cholesky factors to prove a factorization exists for every maximal triangular limit subalgebra. This class can be loosely described as those operators which are approximated "fast" enough by elements in the finite-dimensional factors.

First, we show there are two classes of positive invertible operators in any AF algebra which always factor.

PROPOSITION 3.1. *Let $\mathcal{A}$ be an* AF *C^*-algebra.*

(i) For any triangular limit subalgebra $\mathcal{T}$ of $\mathcal{A}$, the positive invertible operators in the diagonal $\mathcal{D} = \mathcal{T} \cap \mathcal{T}^*$ factor with respect to $\mathcal{T}$.

(ii) For any finite spectrum positive invertible operator A in $\mathcal{A}$, there is a strongly maximal triangular AF *subalgebra $\mathcal{T}$ such that A factors with respect to $\mathcal{T}$.*

PROOF. For any positive operator A in $\mathcal{D}$, since $\sqrt{A} \in \mathcal{T} \cap \mathcal{T}^*$, then (i) holds. For (ii), let A be any finite spectrum positive invertible operator in $\mathcal{A}$. By the spectral theorem, there are finite orthogonal projections $P_1, \ldots, P_n$ with $I = P_1 + \cdots + P_n$ and $P_i \in C^*(A)$, and such that $A = \sum_{i=1}^{n} \lambda_i P_i$, where the spectrum of A consists of $\lambda_1, \ldots, \lambda_n$. Under this partition of the identity, A is constant. Let $\mathcal{A}_1$ be the finite dimensional C^*-subalgebra of $\mathcal{A}$ generated by $P_1, \ldots, P_n$, so $A \in \mathcal{A}_1$. By [3, Theorem 2.2], there is an increasing sequence of finite-dimensional C^*-subalgebras $\{\mathcal{A}_n\}_{n=2}^{\infty}$ of $\mathcal{A}$ and regular embeddings ψ_n, such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A} = \varinjlim(\mathcal{A}_n; \psi_n)$. Let $\mathcal{T}$ be the triangular limit subalgebra $\mathcal{T} = \varinjlim(\mathcal{S}_n; \psi_n)$ of $\mathcal{A}$ with each $\mathcal{S}_n$ a maximal triangular subalgebra of $\mathcal{A}_n$. Clearly A factors as T^*T for some invertible T in $\mathcal{S}_1$, and so A factors as $A = T^*T$ with both T and T^{-1} in $\mathcal{T}$. ∎

We can now prove the main result of this section, a factorization theorem for a nontrivial class of operators.

THEOREM 3.2. *Let $\mathcal{A}$ be the* UHF *algebra of type $\{p_n\}_{n=1}^{\infty}$ and $\mathcal{T} = \varinjlim(\mathcal{T}_{p_n}; \varphi_n)$ a triangular limit subalgebra of $\mathcal{A}$ with respect to regular embeddings φ_n. Let A be a positive invertible operator in $\mathcal{A}$ for which there exists a sequence of positive invertible operators $\{A_n\}_{n \geq k}$, with $A_n \in \mathcal{M}_{p_n}$, such that*

$$\sqrt{p_n}\|A - A_n\| \longrightarrow 0.$$

Then A factors with respect to $\mathcal{T}$.

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PROOF. By hypothesis, A_n^{-1} converges in norm to A^{-1}, so there is a constant M such that

$$\|A_n\|^{\frac{1}{2}}\|A_n^{-1}\| \leq M$$

for all n. Furthermore, the hypotheses ensure that each A_n has a Cholesky factorization $A_n = T_n^* T_n$, with T_n invertible in $\mathcal{T}_{p_n}$. Let $\varepsilon > 0$ be given. Choose N such that if $m \geq N$, then

$$\sqrt{p_m}\|A - A_m\| < \frac{\varepsilon}{2M},$$

and if $n \geq m \geq N$, then

$$\|A_n^{-1}\|\|A_n - \varphi_{n,m}(A_m)\| < \frac{1}{2}.$$

By Theorem 1.2, if $n \geq m \geq N$ we have

$$\begin{aligned}
\|T_n - \varphi_{n,m}(T_m)\| &\leq \|T_n - \varphi_{n,m}(T_m)\|_2 \\
&\leq \frac{\|A_n\|^{\frac{1}{2}}\|A_n^{-1}\|\,\|A_n - \varphi_{m,n}(A_m)\|_2}{\sqrt{2\big(1 - \|A_n^{-1}\|\,\|A_n - \varphi_{m,n}(A_m)\|\big)}} \\
&< M\|A_n - \varphi_{n,m}(A_m)\|_2 \\
&\leq M\|A_n - A\|_2 + M\|A - A_m\|_2 \\
&\leq M\sqrt{p_n}\|A_n - A\| + M\sqrt{p_m}\|A - A_m\| \\
&< \varepsilon.
\end{aligned}$$

Hence, $\{T_n\}_{n\geq k}$ is Cauchy and so there is a T in $\mathcal{T}$ which is the norm limit of this sequence. The same argument holds for $\{T_n^{-1}\}_{n\geq k}$, whose norm limit in $\mathcal{T}$ is necessarily T^{-1}. It is elementary to check that $A = T^*T$, and so we are done. ∎

The following corollary is immediate.

COROLLARY 3.3. *Let $\mathcal{A}$ be an* AF *C^*-algebra with presentation $\varinjlim(\mathcal{A}_n; \varphi_n)$, where $\mathcal{A}_n$ is a finite-dimensional C^*-subalgebra of the form*

$$\mathcal{M}_{m_1^{(n)}} \oplus \mathcal{M}_{m_2^{(n)}} \oplus \cdots \oplus \mathcal{M}_{m_{q_n}^{(n)}},$$

with $m_1^{(n)} \leq \cdots \leq m_{q_n}^{(n)}$, and each φ_n is a regular embedding. Let A be an operator in $\mathcal{A}$ for which there is a sequence of positive invertible operators A_n, with $A_n \in \mathcal{A}_n$, such that

$$\lim_{n\to\infty} \sqrt{m_{q_n}^{(n)}}\|A_n - A\| = 0.$$

If $\mathcal{T}$ is a maximal triangular limit subalgebra of $\mathcal{A}$ of the form $\varinjlim(\mathcal{S}_n; \varphi_n)$, where $\mathcal{S}_n$ is maximal triangular in $\mathcal{A}_n$, then A factors with respect to $\mathcal{T}$.

We have shown that a certain class of positive invertible operators in AF C^*-algebras always factors with respect to a nice family of triangular limit subalgebras. For specific examples of such operators, see [11].

The problem of determining which positive invertible operators factor with respect to *every* maximal triangular limit subalgebra is more difficult. The factorization results in the nest algebra setting show that the operators with the universal factorization property are precisely those of the form scalar plus Macaev ideal. Since UHF algebras contain no compact operators, intuition suggests the only operators in our setting having the universal factorization property are the constants.

4. **Nonfactorable operators in** UHF **algebras.** In Section 2 we saw that the CAR algebra contains a positive invertible operator which does not factor with respect to the refinement limit subalgebra. Our construction there appears to be very embedding-specific, *i.e.*, it appears to depend heavily on the refinement embeddings, and on the fact that we can represent the CAR algebra on $\mathcal{L}^2(0,1)$ in such a manner that the refinement limit subalgebra is weakly dense in the Volterra nest algebra. In this section, we show that the operator given in Section 2 is actually a very simple special case of a more complicated general construction.

THEOREM 4.1. *Let $\mathcal{A}$ be the* UHF *algebra of type $\{p_n\}_{n=1}^{\infty}$ and let $\mathcal{T} = \varinjlim(\mathcal{T}_{p_n};\varphi_n)$ be a triangular limit subalgebra with respect to regular embeddings φ_n. Then there exists a positive invertible operator in $\mathcal{A}$ which does not factor as T^*T for any invertible T in $\mathcal{T}$ with T^{-1} also in $\mathcal{T}$.*

PROOF. We first construct the positive invertible operator A as the limit of a sequence of finite-dimensional operators. To begin, note that by Lemma 2.1, there exists an integer M such that if $n \geq M$, then $\|V_n^*V_n - I_n\| < \frac{1}{2}$.

Define a sequence $\{k_n\}_{n=M}^{\infty}$ as follows: let k_M be the smallest integer such that $p_{k_M} > M$, and for each $n > M$, let k_n be the smallest integer larger than k_{n-1} such that

$$\frac{p_{k_{n+1}}}{p_{k_n}} > n+1; \tag{4}$$

such an integer k_n exists since $\{p_n\}_{n=1}^{\infty}$ is strictly increasing, and so k_n is defined for all n. Set

$$q_n = \frac{p_{k_n}}{p_{k_{n-1}}}$$

for all $n \geq M+1$.

Let $X_M = V_M \oplus I_{p_{k_M}-M} \in \mathcal{T}_{p_{k_M}}$, and define $P_M \in \mathcal{D}_{p_{k_M}}$ by $P_M = e_1^{(k_M)} + \cdots + e_M^{(k_M)}$.

Suppose for some positive integer n, that $X_M,\ldots,X_n$ and $P_M,\ldots,P_n$ have been defined so that for all $M \leq m \leq n$,

(i) X_m is invertible in $\mathcal{T}_{p_{k_m}}$, $e_{p_{k_m}}^{(k_m)}X_m = e_{p_{k_m}}^{(k_m)}$, and $X_m e_{p_{k_m}}^{(k_m)} = e_{p_{k_m}}^{(k_m)}$,

(ii) P_m is a projection in $\mathcal{D}_{p_{k_m}}$ satisfying $P_m e_{p_{k_m}}^{(k_m)} = 0$ and

$$\varphi_{k_m,k_{m-1}}(e_{p_{k_{m-1}}}^{(k_{m-1})})P_m = P_m,$$

(iii) $\varphi_{k_j,k_m}(P_m)X_j = \varphi_{k_j,k_m}(P_mX_m)$ for $m \leq j \leq n$,

(iv) $\|X_n - \varphi_{k_n,k_m}(X_m)\| \geq \|V_n - I_n\|$, and

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(v) there exists a unitary $U_n \in \mathcal{M}_{p_{k_n}}$ such that

$$U_n^* X_n U_n = \Big(\sum_{i=M}^{n} \oplus V_i^{(n,i)}\Big) \oplus I_{j_n},$$

where

$$V_i^{(n,i)} = \sum_{k=1}^{(n,i)} \oplus V_i,$$

(n,i) is an integer depending on i and the embedding $\varphi_{k_n,k_{n-1}}$, and $j_n = p_{k_n} - \sum_{i=M}^{n}(n,i)$.

We now proceed to construct X_{n+1} satisfying the above properties. Since the embeddings $\varphi_{r,s}$ are regular, then φ_{k_{n+1},k_n} maps the matrix unit $e_{p_{k_n}}^{(k_n)}$ to a sum of q_{n+1} matrix units in $\mathcal{T}_{p_{k_{n+1}}}$, say,

$$\varphi_{k_{n+1},k_n}(e_{p_{k_n}}^{(k_n)}) = \sum_{r=1}^{q_{n+1}} e_{i_r}^{(k_{n+1})},$$

where the positive integers i_r are listed in ascending order. Set

$$P_{n+1} = \sum_{r=1}^{n+1} e_{i_r}^{(k_{n+1})},$$

so $P_{n+1} \in \mathcal{D}_{p_{k_{n+1}}}$. Since regular embeddings preserve block structure ([12, Lemma 1.1]), then we have

$$e_{i_r}^{(k_{n+1})} \varphi_{k_{n+1},k_n}(X_{k_n}) e_{i_s}^{(k_{n+1})} = 0$$

for all $r < s$. Define $Y_{n+1} \in \mathcal{T}_{p_{k_{n+1}}}$ by $Y_{n+1} = [a_{ij}]_{1 \le i \le j \le p_{k_{n+1}}}$, where

$$a_{ij} = \begin{cases} e_r^{(n+1)} V_{n+1} e_s^{(n+1)}, & \text{if } i = i_r, j = i_s \text{ for some } 1 \le r < s \le n+1 \\ 0, & \text{otherwise} \end{cases},$$

and let $X_{n+1} = \varphi_{k_{n+1},k_n}(X_n) + Y_{n+1}$.

By construction, it is easy to see that X_{n+1} is invertible, and by (4), we have both $X_{n+1} e_{p_{k_{n+1}}} = e_{p_{k_{n+1}}}$ and $e_{p_{k_{n+1}}} X_{n+1} = e_{p_{k_{n+1}}}$. Furthermore, also by construction we see that $e_{p_{k_n}}^{(k_n)} P_{n+1} = P_{n+1}$, and again by (4), $P_{n+1} e_{p_{k_{n+1}}}^{(k_{n+1})} = 0$. Note that by (ii), for $M \le m \le n$, $\varphi_{k_n,k_m}(P_m) e_{p_{k_n}}^{(k_n)} = 0$. Then since $Y_{n+1} = P_{n+1} Y_{n+1}$, by induction hypotheses (ii) and (iii) we have

$$\begin{aligned} \varphi_{k_{n+1},k_m}(P_m) X_{n+1} &= \varphi_{k_n,k_m}(P_m) X_n + \varphi_{k_{n+1},k_n}(P_m) Y_{n+1} \\ &= \varphi_{k_n,k_m}(P_m X_m) + \varphi_{k_{n+1},k_m}(P_m) P_{n+1} Y_{n+1} \\ &= \varphi_{k_n,k_m}(P_m X_m) + \varphi_{k_{n+1},k_m}(P_m) e_{p_{k_n}}^{(k_n)} P_{n+1} Y_{n+1} \\ &= \varphi_{k_n,k_m}(P_m X_m) + \varphi_{k_{n+1},k_n}\big(\varphi_{k_n,k_m}(P_m) e_{p_{k_n}}^{(k_n)}\big) P_{n+1} Y_{n+1} \\ &= \varphi_{k_n,k_m}(P_m X_m) \end{aligned}$$

for all $M \leq m < n+1$. This proves that X_{n+1} and P_{n+1} satisfy conditions (i), (ii), and (iii). To see (iv), first note by the previous paragraph, for any $M \leq m \leq n+1$, $P_{n+1} = P_{n+1}\varphi_{k_{n+1},k_m}(e^{(k_m)}_{p_{k_m}})$. Thus,

$$\begin{aligned}\|X_{n+1} - \varphi_{k_{n+1},k_m}(X_m)\| &\geq \left\|P_{n+1}\left(X_{n+1} - \varphi_{k_{n+1},k_m}(X_m)\right)\right\| \\ &= \|P_{n+1}X_{n+1} - P_{n+1}\varphi_{k_{n+1},k_m}(e^{(k_m)}_{p_{k_m}}X_m)\| \\ &= \|P_{n+1}(X_{n+1} - I_{n+1})\| \\ &= \|V_{n+1} - I_{n+1}\|.\end{aligned}$$

Finally, to see (v), observe that by construction, interchanging the appropriate rows and columns of X_{n+1} yields an operator of the form

$$V_M \oplus \cdots \oplus V_M \oplus V_{M+1} \oplus \cdots \oplus V_{M+1} \oplus \cdots \oplus V_n \oplus \cdots \oplus V_n \oplus V_{n+1} \oplus \cdots \oplus V_{n+1} \oplus I_j,$$

where the number of copies of each V_i, $i = M, \ldots n+1$ depends on the embedding φ_{k_{n+1},k_n} and i, and j is a positive integer. Hence (v) follows. Thus, by induction X_k satisfying conditions (i) through (v) above is defined for all positive integers $k \geq M$.

It is immediate from Lemma 2.1 and condition (iv) above that the sequence $\{X_n\}_{n=M}^{\infty}$ is not norm Cauchy in $\mathcal{T}$. From (v), we see that for $m, n \geq M$,

$$\begin{aligned}\|X_n^*X_n - \varphi_{k_n,k_m}(X_m^*X_m)\| &= \left\|\sum_{k=m+1}^{n} \oplus\left(\sum_{j=1}^{(n,k)} \oplus(V_k^*V_k - I_k)\right)\right\| \\ &= \max_{m+1\leq i\leq n} \|V_i^*V_i - I_i\|.\end{aligned}$$

But again by Lemma 2.1, the last term above can be made arbitrarily small for sufficiently large n and m. Hence, $\{X_n^*X_n\}_{n=1}^{\infty}$ is a norm Cauchy sequence in $\mathcal{A}$. Let $A \in \mathcal{A}$ be the norm limit of the operators $X_n^*X_n$, so clearly, $A \geq 0$. Next, note that

$$\begin{aligned}\|X_n^*X_n - I_{p_{k_n}}\| &= \left\|\sum_{i=M}^{n} \oplus\left(\sum_{j=1}^{(n,i)} \oplus(V_i^*V_i - I_i)\right)\right\| \\ &= \max_{M\leq i\leq n} \|V_i^*V_i - I_i\| < \frac{1}{2},\end{aligned}$$

by the choice of M. It follows that

$$\|A - I\| = \lim_{n\to\infty} \|X_n^*X_n - I_{p_{k_n}}\| \leq \frac{1}{2} < 1;$$

hence, A is invertible.

By Theorem 1.1, we can represent $\mathcal{A}$ as the bounded linear operators on some Hilbert space $\mathcal{H}$ in such a manner that $\mathcal{T}$ is weakly dense in a nest algebra $\mathcal{T}(\mathcal{N})$ in $\mathcal{B}(\mathcal{H})$. To show that A does not factor in $\mathcal{T}$, we first show A has a strong factorization, *i.e.*, $A = X^*X$

for some operator X such that X and X^{-1} belong to $\mathcal{T}(\mathcal{N})$ and X_n converges to X in the strong operator topology. By condition (i), the invertibility of A, and the fact that A is the norm limit of the $X_n^*X_n$, there is a constant $B > 0$ such that $\|X_n\|, \|X_n^{-1}\| \leq B$ for all n. For each $n \geq M$, define $Q_n = e_{p_n}^{(p_n)}$. Then by the construction of X_n, for each $k \geq n \geq M$,

$$(5) \qquad X_n(I - Q_n) = X_k(I - Q_n) \quad \text{and} \quad X_n^{-1}(I - Q_n) = X_k^{-1}(I - Q_n).$$

Let τ represent the normalized trace functional on $\mathcal{A}$. Clearly $\tau(Q_n)$ converges to zero as n goes to infinity. However, it is a known fact that $\tau(Q_n) \longrightarrow 0$ as $n \longrightarrow \infty$ if and only if $Q_n \longrightarrow 0$ in the strong operator topology (see, for instance, [11, Proposition 1.7.2]. Thus, if $f \in \mathcal{H}$ and $n \geq m \geq M$, then by (5)

$$\|(X_n - X_m)f\| = \|(X_n - X_m)Q_mf\| \leq 2B\|Q_mf\|,$$

which becomes arbitrarily small as m becomes large. Hence, $\{X_n(f)\}_{n=M}^{\infty}$ is Cauchy in the strong operator topology, and so has a strong limit X. A similar argument proves that $\{X_n^{-1}\}_{n=M}^{\infty}$ has a strong operator limit, Y. By Lemma 2.3, $X^{-1} = Y$, and so both X and X^{-1} are in $\mathcal{T}(\mathcal{N})$. Using the facts that $X_n^*X_n \longrightarrow A$ in norm and $X_n \longrightarrow X$ in the strong operator topology, it is an elementary calculation to verify that $A = X^*X$.

Now suppose that A factors as T^*T for some invertible T in $\mathcal{T}$. Then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ of finite-dimensional operators converging in norm to T. For notational simplicity, we assume that $T_n \in \mathcal{T}_{p_{k_n}}$. Since $A = T^*T = X^*X$, then $U = TX^{-1}$ is a unitary with both U and U^* belonging to $\mathcal{T}(\mathcal{N})$. Hence, $U \in \mathcal{D}(\mathcal{N})$. Let $e_{p_{k_n}}^{(k_n)}T_n = \lambda_n e_{p_{k_n}}^{(k_n)}$ for all $n \geq M$.

Note that by (iii) and the strong convergence of X_n to X, $P_kX = P_kX_k$ for all $k \geq M$. Fix $n \geq M$. By Lemma 2.2, we can choose $N_0 \geq M$ such that if $m \geq N_0$, then $\|V_m - \lambda_nP_mU^*\| > \frac{1}{4}\sin\frac{1}{2}$. Then for any such m, by (ii) and (iii), we have

$$\begin{aligned}
\|UX - T_n\| &\geq \|e_{p_{k_n}}^{(k_n)}(UX - T_n)\| \\
&= \|Ue_{p_{k_n}}^{(k_n)}X - \lambda_n e_{p_{k_n}}^{(k_n)}\| \\
&= \|e_{p_{k_n}}^{(k_n)}X - \lambda_n e_{p_{k_n}}^{(k_n)}U^*\| \\
&\geq \|P_m(e_{p_{k_n}}^{(k_n)}X) - \lambda_nP_me_{p_{k_n}}^{(k_n)}U^*\| \\
&= \|P_mX - \lambda_nP_mU^*\| \\
&= \|P_mX_m - \lambda_nP_mU^*\| \\
&= \|V_m - \lambda_nP_mU^*\| \\
&> \frac{1}{4}\sin\frac{1}{2},
\end{aligned}$$

a contradiction. Hence, A does not factor with respect to $\mathcal{T}$, as desired. ∎

One of Larson's main results in [13] (Theorem 4.7) was that in every complete uncountable nest, there exist positive invertible operators which do not factor with respect to the nest algebra. However, he showed that every positive invertible operator admits a *weak factorization*, *i.e.*, factors as T^*T for T in the nest algebra, without requiring T^{-1} to belong to the nest algebra.

Surprisingly, in AF C^*-algebras the two notions of having a factorization and having a weak factorization with respect to a strongly maximal triangular AF algebra are equivalent. The key observation is that strongly maximal triangular AF subalgebras are inverse closed, *i.e.*, if an operator belongs to such a subalgebra and is invertible, then its inverse is also in the subalgebra.

THEOREM 4.2. *Let $\mathcal{T}$ be a strongly maximal triangular* AF *subalgebra of an* AF C^*-*algebra $\mathcal{A}$. Then a positive invertible operator A in $\mathcal{A}$ factors as T^*T with both T and T^{-1} in $\mathcal{T}$ if and only if A factors as T^*T with only T in $\mathcal{T}$.*

PROOF. Let $\mathcal{T} = \varinjlim(\mathcal{S}_n; \varphi_n)$, where each $\mathcal{S}_n$ is a finite-dimensional upper triangular matrix algebra. Suppose A has a weak factorization as T^*T for some T in $\mathcal{T}$. Then we can choose a sequence $\{T_n\}_n$, with T_n in $\mathcal{S}_n$, so that T_n converges in norm to T. Moreover, since the set of invertibles in $\mathcal{B}(\mathcal{H})$ is open, we can take each T_n to be invertible.

Since the mapping $A \longmapsto A^{-1}$ is norm-continuous, then T_n^{-1} converges in norm to T^{-1}. Furthermore, since the inverse of an upper triangular matrix is upper triangular, then T_n^{-1} belongs to $\mathcal{S}_n$ for each n. It follows that T^{-1} is in $\mathcal{T}$, and so A factors as T^*T with T and T^{-1} in $\mathcal{T}$. ∎

COROLLARY 4.3. *The non-factorizable positive invertible operators constructed in Theorems 2.7 and 4.1 do not have weak factorizations as T^*T for any T in $\mathcal{T}$.*

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Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
N2L 3G1

Current address of first author:
Department of Mathematics
University of Saskatchewan
Saskatoon, Saskatchewan
e-mail: huang@snoopy.usask.ca

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
N2L 3G1

Current address of second author:
Department of Mathematics
East Carolina University
Greenville, North Carolina 27858-4353
U.S.A.
e-mail: mahudson@ecuvax.cis.ecu.edu