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Periods of Modular Forms and Imaginary Quadratic Base Change

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Abstract. Let f be a classical newform of weight 2 on the upper half-plane $\mathcal{H}^{(2)}$, E the corresponding strong Weil curve, K a class number one imaginary quadratic field, and F the base change of f to K. Under a mild hypothesis on the pair (f, K), we prove that the period ratio $\Omega_E/(\sqrt{|D|}\Omega_F)$ is in $\mathbb{Q}$. Here Ω_F is the unique minimal positive period of F, and Ω_E the area of $E(\mathbb{C})$. The claim is a specialization to base change forms of a conjecture proposed and numerically verified by Cremona and Whitley.

1 Introduction

Let E be an elliptic curve over an imaginary quadratic field K. For simplicity, we assume K to have class number one, and denote by D, w, and ε_K its discriminant, number of units and the associated quadratic character, respectively. By analogy with the Shimura–Taniyama conjecture over $\mathbb{Q}$, we expect the isogeny class of E to determine, in most cases, a weight 2 cusp form on $\mathrm{GL}_2(\mathbb{A}_K)$. Such a form has a unique minimal positive period Ω_F, which the Birch and Swinnerton-Dyer conjecture suggests should be related to Ω_E, the area of $E(\mathbb{C})$. Indeed, in the articles of Cremona [2] and Cremona-Whitley [4] it was conjectured that

$$\frac{1}{\sqrt{|D|}}\frac{\Omega_E}{\Omega_F} \in \mathbb{Q}. \tag{1.1}$$

In this note, we prove (1.1) in the special case when E is the base change of an elliptic curve over $\mathbb{Q}$, under a mild assumption on E and K (see Theorem 4.1 below).

In our paper [12], we proposed a conjectural p-adic construction of global points on the elliptic curve $E_{/K}$. The main ingredient in this construction is the modular symbol associated with E, obtained by dividing path integrals of the corresponding modular form F by its period Ω_F. Relating this period to Ω_E for a base change curve is the first step in relating our Stark–Heegner points to the classical Heegner points.

2 Modular Forms over Imaginary Quadratic Fields

In the relatively simple setting of an imaginary quadratic field of class number one, the adelic object conjecturally corresponding to an elliptic curve $E_{/K}$ without complex multiplication by K can be identified with a harmonic 1-form on the upper half-space $\mathcal{H}^{(3)} = \mathbb{C} \times \mathbb{R}_{>0}$. We briefly review the setup from [7].

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Gramm–Schmidt orthogonalization identifies $\mathcal{H}^{(3)}$ with the $\mathrm{PGL}_2(\mathbb{C})$-homogeneous space $\mathrm{PGL}_2(\mathbb{C})/\mathrm{PSU}_2$ via

$$(z,t) \leftrightarrow \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \bmod \mathrm{PSU}_2, \quad z \in \mathbb{C}, t \in \mathbb{R}_{>0}.$$

A basis of 1-differentials on $\mathcal{H}^{(3)}$ is given by the column vector $\beta = {}^t(-\frac{dz}{t}, \frac{dt}{t}, \frac{d\bar{z}}{t})$. For an ideal $\mathfrak{n}$ of the ring of integers $\mathcal{O}_K \subset K$, we consider the congruence subgroup

$$\Gamma_0^+(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}_K) \mid c \in \mathfrak{n} \right\}.$$

The automorphic objects with which we are concerned are defined as follows.

Definition 2.1 A plus-cusp form of weight 2 and level $\mathfrak{n}$ ("plusform" for short) is a function $F = (F_0, F_1, F_2) \colon \mathcal{H}^{(3)} \to \mathbb{C}^3$ with values in row vectors, satisfying

(i) $\Gamma_0^+(\mathfrak{n})$-invariance: The dot product $F \cdot \beta$ is a harmonic 1-form on $\mathcal{H}^{(3)}$ invariant under $\Gamma_0^+(\mathfrak{n})$;

(ii) Cuspidality: By property (i) and an explicit computation of the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathcal{H}^{(3)}$, we have $F(z,t) = F(z+w,t)$ for $w \in \mathcal{O}_K$ (see [4]). It then makes sense to require that $\int_{\mathbb{C}/\mathcal{O}_K} (\gamma^*)(F \cdot \beta) = 0$ for all $\gamma \in \mathrm{PGL}_2(\mathcal{O}_K)$, *i.e.,* the constant term in the Fourier–Bessel expansion of F at the cusp $\gamma^{-1}\infty$ (see below) is zero.

This definition is simplified by the assumption that $h(K) = 1$, as that requires us to consider only one copy of $\mathcal{H}^{(3)}$ and makes the action of $\mathrm{PGL}_2(\mathcal{O}_K)$ on the cusps $\mathbb{P}^1(K)$ transitive. The space of all plus-cusp forms of weight 2 and level $\mathfrak{n}$ is denoted $S_2^+(\mathfrak{n})$.

As in the classical case, conditions (i) and (ii) mean that an element of $S_2^+(\mathfrak{n})$ can be identified with a harmonic differential without poles on the compact three-dimensional manifold $X_0(\mathfrak{n}) = \Gamma_0^+(\mathfrak{n}) \backslash \mathcal{H}^{(3)*}$. Here the extended upper half-space $\mathcal{H}^{(3)*} = \mathcal{H}^{(3)} \cup \mathbb{P}^1(K)$ depends on K. Note that $X_0(\mathfrak{n})$ does not have the structure of an algebraic variety (its complex dimension would be 1.5), which makes the modularity theory almost entirely conjectural.

The invariance condition (i) applied to matrices $\gamma = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), t \in \mathcal{O}_K$ and $\gamma = \left(\begin{smallmatrix} \eta & 0 \\ 0 & 1 \end{smallmatrix}\right), \eta \in \mathcal{O}_K^\times$ implies that the cusp form F has a "Fourier–Bessel" series expansion at the cusp ∞ (see [7]):

$$F(z,t) = \sum_{0 \neq (\alpha) \subset \mathcal{O}_K} c_{(\alpha)} t^2 \mathbf{K}\Big(\frac{4\pi\,|\alpha|\,t}{\sqrt{|D|}}\Big) \sum_{\eta \in \mathcal{O}_F^\times} e^{2\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{\eta\alpha z}{\sqrt{D}}\right)}. \tag{2.1}$$

The sum is over proper ideals of $\mathcal{O}_K$, and $\mathbf{K}(t) = \frac{i}{2}(-\mathbf{K}_1(t), -2i\mathbf{K}_0(t), \mathbf{K}_1(t))$. The function $\mathbf{K}_r(t)$, $r = 0$ or 1, is the ($\mathbb{R}$-valued) hyperbolic Bessel function that satisfies the differential equation

$$\frac{d^2\mathbf{K}_r}{dt^2} + \frac{1}{t}\frac{d\mathbf{K}_r}{dt} - \Big(1 + \frac{1}{t^{2r}}\Big)\mathbf{K}_r = 0$$

and decreases rapidly at infinity.

The theory of Hecke operators carries over verbatim from classical modular forms to plusforms. For a prime (π) of $\mathcal{O}_K$, the Hecke operator $T_{(\pi)}$ sends F to a form with coefficients $c'(\alpha) = N_{K/\mathbb{Q}}(\pi)c(\alpha\pi) + c(\alpha/\pi)$, the second term being understood to be 0 if $\pi \nmid \alpha$. A new plusform of level $\mathfrak{n}$ is an eigenvector for all the Hecke operators $T_{(\pi)}$ with prime index $(\pi) \nmid \mathfrak{n}$, which is not induced from a lower level.

In this setting we have the following version of the Shimura–Taniyama conjecture.

Conjecture 2.2 *Each isogeny class of elliptic curves $E_{/K}$ of conductor $\mathfrak{n}$, without complex multiplication by K, determines a unique new plusform $F \in S_2^+(\mathfrak{n})$ whose Fourier–Bessel coefficient with prime index $\mathfrak{p}$ is given by*

$$c_\mathfrak{p} = N\mathfrak{p} + 1 - \#E(\mathbb{F}_\mathfrak{p}) \in \mathbb{Z}.$$

Equivalently, we have an equality of L-functions $L(E_{/K}, s) = L(F, s)$, where

$$L(F, s) := \sum_{(\alpha)\subset\mathcal{O}_K} c_{(\alpha)}(N_{K/\mathbb{Q}}\alpha)^{-s} = (2\pi)^{2s-2}\,|D|^{1-s}\,\Gamma(s)^{-2}\,\frac{16\pi^2}{w\,|D|}\int_0^\infty t^{2s-2}F_1(0,t)\frac{dt}{t}.$$

It should be noted that not all forms in $S_2^+(\mathfrak{n})$ correspond to elliptic curves over K: some are quadratic twists of lifts of forms over $\mathbb{Q}$ with real quadratic coefficients, corresponding to abelian surfaces over $\mathbb{Q}$ with quaternionic multiplication (see [3]). A curve $E_{/K}$ *with* CM by K should correspond to an Eisenstein series.

Cremona [2] produced extensive numerical evidence for Conjecture 2.2. Taylor [11] proved a weak converse to the conjecture: starting with a newform F with Fourier–Bessel expansion (2.1), he constructed a system of l-adic Galois representations of $\mathrm{Gal}(\bar{K}/K)$ whose trace of Frobenius at $\mathfrak{p}$ is equal to $c_\mathfrak{p}$ outside a set of density zero. These l-adic representations can in turn sometimes be identified as coming from an elliptic curve by checking the equality of a finite number of traces of Frobenius, according to the method of Faltings–Serre.

We will start with a weight 2 newform $f_\mathbb{Q} = \sum_{n=1}^{\infty} a_n q^n$ on the upper half-plane $\mathcal{H}^{(2)}$ of level prime to D and without complex multiplication by K. The corresponding strong Weil curve $E_{/\mathbb{Q}}$ can be viewed as a curve over K which should, under Conjecture 2.2, correspond to the base change F_K of $f_\mathbb{Q}$ to K. The existence of the base-changed modular form F_K is known independently of any Shimura–Taniyama-type conjecture, either as a consequence of the general work of Jacquet [6], or by the explicit computations of Asai [1] and Friedberg [5]. From the L-function relation satisfied by base change (see (4.1) below), one easily deduces the Fourier–Bessel coefficients of F_K: $c_\pi = a_p$ if $p = \pi\bar{\pi}$ is split, $c_p = a_p^2 - 2p$ if p is inert in K.

3 Modular Symbols

Fix a newform $F \in S_2^+(\mathfrak{n})$ with coefficients $c_{(\alpha)} \in \mathbb{Z}$. For any two cusps $a, b \in \mathbb{P}^1(K)$, we define the modular symbol

$$\{a \to b\}_K = \frac{16\pi^2}{w\,|D|}\int_a^b F\cdot\beta. \tag{3.1}$$

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This symbol is real-valued, which is readily calculated from the Fourier–Bessel series (2.1) in the special case $b = \infty$:

$$\{a \to \infty\}_K = \frac{16\pi^2}{w\,|D|} \int_0^\infty \sum_{0 \neq (\alpha) \subset \mathcal{O}_F} c_{(\alpha)} t^2 \mathbf{K}_0\Big(\frac{4\pi\,|\alpha|\,t}{\sqrt{|D|}}\Big) \sum_{\eta \in \mathcal{O}_F^\times} e^{2\pi i \operatorname{Tr}_{K/\mathbb{Q}}\left(\frac{\eta\alpha a}{\sqrt{D}}\right)} \frac{dt}{t} \in \mathbb{R}.$$

By multiplicity one (see [7]), the values of $\{a \to b\}_K$ on closed paths in $X_0(\mathfrak{n})$ form a rank one lattice in $\mathbb{R}$, whose positive generator is the period Ω_F from the Introduction.

Let $\chi\colon (\mathcal{O}_K/(\mu))^\times/\mathcal{O}_K^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character (*i.e.*, a Hecke character with trivial archimedean component) with conductor ideal $(\mu) \subseteq \mathcal{O}_K$ (here we again use $h(K) = 1$). We define the twisted L-function by $L(F, \chi, s) = \sum_{(\alpha) \subset \mathcal{O}_K} c_{(\alpha)} \chi(\alpha)(N_{K/\mathbb{Q}}\alpha)^{-s}$. Modular symbols allow us to calculate its special values.

Proposition 3.1 *There exists a $t_K(\chi) \in \mathbb{Q}(\chi)$ such that*

$$L(F, \chi, 1) = \tau_K(\bar{\chi})^{-1} t_K(\chi) \Omega_F,$$

where $\tau_K(\bar{\chi}) = \sum_{\alpha \in \mathcal{O}_K/(\mu)} \bar{\chi}(\alpha) e^{2\pi i \operatorname{Tr}_{K/\mathbb{Q}} \frac{\alpha}{\mu\sqrt{D}}}$ is the Gauss sum.

Proof For any $a, b \in \mathbb{P}^1(K)$, there exists an $r \in \mathbb{Q}$ such that $\{a \to b\}_K = r\Omega_F$. This is the Manin–Drinfeld lemma for forms over K, proved as over $\mathbb{Q}$ by using a suitable Hecke operator to "close the path". The normalization constant in (3.1) was chosen so that

$$L(F, \chi, 1) = \tau_K(\bar{\chi})^{-1} \sum_{\kappa \in \mathcal{O}_K/(\mu)} \bar{\chi}(\kappa) \left\{\frac{\kappa}{\mu} \to \infty\right\}_K,$$

a version of Birch's lemma proved analogously to the classical case. Combining these two facts gives the proposition. For details, see [7, Lemma 6]. ∎

To fix notation, we recall the analogous proposition over $\mathbb{Q}$. Let $f_{\mathbb{Q}} \in S_2(N)$ be a classical newform on $\mathcal{H}^{(2)}$, and let Ω_+, Ω_- denote the smallest positive real and imaginary parts of its periods.

Proposition 3.2 *Let $\chi\colon (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character. Set $\Omega = \Omega_+$ if χ is even, and $\Omega = i\Omega_-$ if χ is odd. There is a number $t_{\mathbb{Q}}(\chi) \in \mathbb{Q}(\chi)$ such that $L(f_{\mathbb{Q}}, \chi, 1) = \tau_{\mathbb{Q}}(\bar{\chi})^{-1} t_{\mathbb{Q}}(\chi)\Omega$, where $\tau_{\mathbb{Q}}(\bar{\chi}) = \sum_{k=0}^{m-1} \bar{\chi}(k) e^{\frac{2\pi i k}{m}}$ is the Gauss sum.*

4 Comparison of Periods

Our main result is the following.

Theorem 4.1 *Keeping the notations from the introduction, let $f_{\mathbb{Q}} \in S_2(N)$ be a newform on $\mathcal{H}^{(2)}$ with $(N, D) = 1$, and F_K on $\mathcal{H}^{(3)}$ its base change to K. Assume that the strong Weil curve E corresponding to $f_{\mathbb{Q}}$ does not have complex multiplication by K. Then*

$$\frac{1}{\sqrt{|D|}} \frac{\Omega_E}{\Omega_{F_K}} \in \mathbb{Q}.$$

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Proof Let $\chi\colon (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character with $(m, ND) = 1$, and let $\chi \circ N_{K/\mathbb{Q}}$ be its base change to K. By the coprimality assumptions, we can factor the twisted special L-value of F_K as follows:

$$L(F_K, \chi \circ N_{K/\mathbb{Q}}, 1) = L(f_{\mathbb{Q}}, \chi, 1)L(f_{\mathbb{Q}}, \chi\varepsilon_K, 1). \tag{4.1}$$

Expressing the left-hand (resp. right-hand) side in terms of Proposition 3.1 (resp. 3.2), we get two expressions for $L(F_K, \chi \circ N_{K/\mathbb{Q}}, 1)$ in terms of modular symbols:

$$\tau_K(\bar{\chi} \circ N_{K/\mathbb{Q}})^{-1} r_K(\chi)\Omega_{F_K} = \tau_{\mathbb{Q}}(\bar{\chi})^{-1}\tau_{\mathbb{Q}}(\bar{\chi}\varepsilon_K)^{-1} r_{\mathbb{Q}}(\chi) i\Omega_+\Omega_-, \tag{4.2}$$

for some $r_{\mathbb{Q}}(\chi), r_K(\chi) \in \mathbb{Q}(\chi)$. Both Ω_+ and Ω_- appear since, as K is imaginary, the characters $\bar{\chi}$ and $\bar{\chi}\varepsilon_K$ have opposite parity. Since $(m, D) = 1$, the Gauss sums are related by the identity

$$\tau_K(\bar{\chi} \circ N_{K/\mathbb{Q}}) = -i\frac{\tau_{\mathbb{Q}}(\bar{\chi})\tau_{\mathbb{Q}}(\bar{\chi}\varepsilon_K)}{\sqrt{|D|}} \tag{4.3}$$

(see [9, p. 183]. We have that $\Omega_E = \delta\Omega_+\Omega_-$, where $\delta = 2$ if $E(\mathbb{R})$ is connected, and 1 otherwise. Substituting this and (4.3) into (4.2), we get

$$\delta\sqrt{|D|}\, r_K(\chi)\Omega_{F_K} = -r_{\mathbb{Q}}(\chi)\Omega_E. \tag{4.4}$$

We now need a theorem of Rohrlich [10].

Theorem 4.2 *Let g be a newform of level N on $\mathcal{H}^{(2)}$. Let S be a finite set of primes not dividing N. For all but finitely many primitive Dirichlet characters χ whose conductors are divisible only by primes in S, we have $L(g, \chi, 1) \neq 0$.*

This allows us to find a χ such that $L(f_{\mathbb{Q}}, \chi, 1) \neq 0 \neq L(f_{\mathbb{Q}}, \chi\varepsilon_K, 1)$, and hence $r_K(\chi) \neq 0 \neq r_{\mathbb{Q}}(\chi)$. We then divide by $r_{\mathbb{Q}}(\chi)$ in (4.4) to conclude that

$$\frac{1}{\sqrt{|D|}}\frac{\Omega_E}{\Omega_{F_K}} = -\frac{\delta r_K(\chi)}{r_{\mathbb{Q}}(\chi)} \in \mathbb{Q}(\chi). \tag{4.5}$$

Finally, we need to show that the ratio (4.5) in fact lies in $\mathbb{Q}$. This is strongly suggested by the fact that it is independent of χ. Indeed, choose two characters χ_1, χ_2 with non-zero special values and relatively prime conductors, so that $\mathbb{Q}(\chi_1) \cap \mathbb{Q}(\chi_2) = \mathbb{Q}$. ■

Naturally, we would like to understand the period ratio (4.5). Incidental to the computations in [12], we calculated it for pairs (E, K), where K is euclidean and $E_{/\mathbb{Q}}$ is a strong Weil curve of prime conductor ≤ 53 which remains inert in K. In all cases, we found that $\Omega_E = (w\sqrt{|D|}/2)\Omega_{F_K}$. This means that each of those strong Weil curves over $\mathbb{Q}$ remains a strong Weil curve over K in the sense of [4]. For level 11, the final remark of [4] observes that this is the case precisely for the K where 11 is inert. It would be interesting to explore whether this holds for a general curve of prime conductor.

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The numbers $r_{\mathbb{Q}}(\chi), r_K(\chi)$ are computed in terms of modular symbols. Our proof of Theorem 4.1 uses only their rationality properties, treating their actual values as a black box. In practice, one encodes a modular form in $S_2^+(\mathfrak{n})$ as a finite sequence of integers by evaluating $\{a \to b\}_K / \Omega_{F_K}$ on a basis of $H_1(X_0(\mathfrak{n}), \mathbb{Z})$. One gets a similar sequence of integers for a classical modular form on $\mathcal{H}^{(2)}$ by dividing the modular symbol $\{a \to b\}^+_{\mathbb{Q}} = \mathrm{Re}(-2\pi i \int_a^b f(z)dz)$ by Ω^+ and evaluating on a homology basis of the classical modular curve. The following natural question seems of considerable intrinsic interest.

Question Is it possible to give a recipe for computing the sequence of integers associated with the base-changed form F_K on $\mathcal{H}^{(3)}$ directly from the one associated with the original form $f_{\mathbb{Q}}$ on $\mathcal{H}^{(2)}$?

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