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Preregular maps between Banach lattices

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A continuous linear map from a Banach lattice E into a Banach lattice F is preregular if it is the difference of positive continuous linear maps from E into the bidual F'' of F. This paper characterizes Banach lattices B with either of the following properties:

(1) for any Banach lattice E, each map in $L(E, B)$ is preregular;

(2) for any Banach lattice F, each map in $L(B, F)$ is preregular.

It is shown that B satisfies (1) (respectively (2)) if and only if B' satisfies (2) (respectively (1)). Several order properties of a Banach lattice satisfying (2) are discussed and it is shown that if B satisfies (2) and if B is also an atomic vector lattice then B is isomorphic as a Banach lattice to $l^1(\Gamma)$ for some index set Γ.

1. Introduction

The following natural question arises in the theory of Banach lattices: Given Banach lattices E and F, is each map in the space $L(E, F)$ of continuous linear maps from E into F the difference of positive (continuous) linear maps? It is known that if F is a $C(X)$ for X an extremally disconnected, compact Hausdorff space X or if E is an AL-space and F has the monotone convergence property then the answer to

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this question is affirmative $([10]$, Chapter 4, (3.7) and (3.8)$)$. On the other hand, $L(l^2, l^2)$ contains maps that are not the difference of positive linear maps, $([10]$, Chapter 4, (3.3)$)$. Schlotterbeck [16] has shown that if F is an AM-space or if E is an AL-space then each map in $L(E, F)$ is the difference of positive linear maps into the bidual F'' of F.

The main goal of this paper is to characterize (in §3) Banach lattices B with either of the following properties.

PROPERTY I. For any Banach lattice E, each map in $L(E, B)$ is the difference of positive linear maps of E into B''.

PROPERTY II. For any Banach lattice F, each map in $L(B, F)$ is the difference of positive linear maps of B into F''.

The characterizations that we obtain indicate that such spaces are similar to AM- and AL-spaces. In particular, we show that a Banach lattice has Property I (respectively Property II) if and only if B' has Property II (respectively Property I).

In §4, we study some order properties of a Banach lattice with Property II. We also show that $L^p[0, 1]$, $1 < p < \infty$, possesses neither Property I nor Property II. Finally in §5 we show that with the additional assumption that a Banach lattice G is an atomic lattice, G is isomorphic as a Banach lattice to $l^1(\Gamma)$ for some index set Γ whenever G has Property II.

2. Preliminary material

For general terminology and notation concerning functional analysis we refer the reader to [15] while our reference for ordered locally convex spaces will be [10].

By a map between Banach spaces we will always mean a continuous linear map. A sequence in a Banach space is *summable* (respectively *absolutely summable*) if it is unconditionally convergent (respectively absolutely convergent).

A map from a Banach lattice E into a Banach space F is *order summable* if it maps positive summable sequences into absolutely summable sequences. $S_+(E, F)$ denotes the space of order summable maps from E into F. A map from a Banach space into a Banach lattice is *majorizing* if its adjoint is order summable. Majorizing maps can also be defined as maps that take null sequences into order bounded sets; (see [16], Chapter 1, and [4]).

A map from a Banach space E into a Banach space F is *absolutely summable* if it maps summable sequences into absolutely summable sequences. The space of absolutely summable maps of E into F is denoted by $S(E, F)$. A map between Banach spaces is *absolutely majorizing* (hyper-majorizing in [16]) if its adjoint is absolutely summable. The following results characterize these types of maps. For proofs see [16], (3.5), (3.6), and (3.7), or [6], (6.6), (6.7), and (6.8).

PROPOSITION 2.1. *If E and F are Banach spaces and if $T \in L(E, F)$, then the following statements are equivalent:*

(1) *T is absolutely summable;*

(2) *T' is absolutely majorizing;*

(3) *for every Banach lattice H and $S \in L(H, E)$, $T \circ S$ is order summable;*

(4) *for each $S \in L(c_0, E)$, $T \circ S$ is order summable.*

PROPOSITION 2.2. *If E and F are Banach spaces and if $T \in L(E, F)$, then the following statements are equivalent:*

(1) *T is absolutely majorizing;*

(2) *T' is absolutely summable;*

(3) *for every Banach lattice H and $S \in L(F, H)$, $S \circ T$ is majorizing;*

(4) *for every $S \in L(F, l^1)$, $S \circ T$ is majorizing.*

Absolutely summable and absolutely majorizing maps can be factored through Hilbert spaces ([16], (3.8)) and so are weakly compact.

We need the following four topologies on the tensor product $E \otimes F$ of two Banach spaces E and F.

(i) $E \tilde{\otimes}_\pi F$ is the completion of $E \otimes F$ for the norm

$$\|u\| = \inf\left\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\right\} .$$

(ii) $E \tilde{\otimes}_\varepsilon F$ is the completion of $E \otimes F$ for the norm

$$\|u\| = \sup\left\{\left|\sum_{i=1}^{n} \langle x_i, x'\rangle\langle y_i, y'\rangle\right| : x' \in E', \|x'\| \le 1, y' \in F', \|y'\| \le 1\right\} ,$$

where $u = \sum_{i=1}^{n} x_i \otimes y_i$ is any representation of $u \in E \otimes F$.

(iii) $E \tilde{\otimes}_\sigma F$ is the completion of $E \otimes F$ for the norm

$$\|u\| = \inf\left\{\sup_{|c_i|\le 1} \left\|\sum_{i=1}^{n} x_i c_i \|y_i\|\right\| : u = \sum_{i=1}^{n} x_i \otimes y_i\right\} .$$

(iv) If E is also a Banach lattice with cone K then $E \tilde{\otimes}_{|\sigma|} F$ is the completion of $E \otimes F$ for the norm

$$\|u\| = \inf\left\{\left\|\sum_{i=1}^{n} x_i \|y_i\|\right\| : u = \sum_{i=1}^{n} x_i \otimes y_i \text{ and } x_i \in K \text{ for } i = 1, \ldots, n\right\} .$$

Jacobs [4] has shown that $\tau_\varepsilon \le \tau_\sigma \le \tau_{|\sigma|} \le \tau_\pi$, that $\left(E \tilde{\otimes}_\sigma F\right)' = S(E, F')$, and that $\left(E \tilde{\otimes}_{|\sigma|} F\right)' = S_+(E, F')$. Moreover, for maps $T \in S_+(E, F)$, the norm of T in $\left(E \tilde{\otimes}_{|\sigma|} F\right)'$ is given by

$$|\sigma|'(T) = \inf\left\{M : \sum_{i=1}^{n} \|Tx_i\| \le M \sup \sum_{i=1}^{n} |\langle x_i, x'\rangle| \text{ for all finite sets } \{x_1, \ldots, x_n\} \subset K\right\} .$$

Let E and F be Banach lattices with cones K and H respectively. In $E \otimes F$ define the *projective cone* K_p by

$K_p = \left\{\sum_{i=1}^{n} x_i \otimes y_i : x_i \in K, y_i \in H\right\}$. Then it is easy to see that for the dual system $\langle E \tilde{\otimes}_{|\sigma|} F, S_+(E, F')\rangle$, $\overline{K}_p{}'$ equals the cone of positive maps

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in $S_+(E, F')$. It follows from the definition of the dual norm $|\sigma|'$ that this latter cone is normal in $S_+(E, F')$ for the $|\sigma|'$-topology. Since this cone is also generating in $S_+(E, F')$ ([16], (2.2)), it follows from [10], Chapter 2, (1.22), that $\overline{K}_p$ is normal and generating in $E \tilde{\otimes}_{|\sigma|} F$.

If E and F are Banach lattices and if $T \in L(E, F)$, then T is *regular* (respectively *preregular*) if T is the difference of positive linear maps of E into F (respectively E into F'').

If E and F are Banach spaces and $T \in L(E, F)$ then T is *integral* if the bilinear form b_T defined on $E \times F'$ by

$$b_T(x, y') = \langle Tx, y' \rangle$$

is an element of $(E \tilde{\otimes}_\varepsilon F')'$. Integral maps are both absolutely summable and absolutely majorizing. For more information see [16], Chapter 3, or [6], Chapters 5 and 6.

A Banach lattice E is an *AM*-space if $x, y \in E$, $x, y \geq 0$, imply that $\|x \vee y\| = \|x\| \vee \|y\|$. A Banach lattice E is an *AL*-space if $x, y \in E$, $x, y > 0$ imply that $\|x+y\| = \|x\| + \|y\|$. If F is an *AL*-space then $E \tilde{\otimes}_\sigma F = E \tilde{\otimes}_\varepsilon F$ for all Banach spaces E (for a proof of this result see [6], (6.3)).

The following characterizations of *AM*- and *AL*-spaces (see [16], Chapters 1 and 4) are included so that we may compare them with our results, Theorems 3.3 and 3.7.

PROPOSITION 2.3. *The following statements about a Banach lattice* E *are equivalent:*

(1) E *is isomorphic as a Banach lattice to an AM-space;*

(2) *every null sequence in* E *is majorized;*

(3) *every order summable map from* E *into a Banach space is integral.*

PROPOSITION 2.4. *The following statements about a Banach lattice* E *are equivalent:*

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(1) *E is isomorphic as a Banach lattice to an AL-space;*

(2) *every positive summable sequence in E is absolutely summable;*

(3) *every majorizing map from a Banach space into E is integral.*

If E is a Banach lattice and $0 \leq x \in E$ we denote by E_x the linear hull of the order interval $[-x, x]$ with $[-x, x]$ as unit ball. If we order E_x by restricting the order on E then E_x is an *AM*-space.

3. Preregular maps

In this section we characterize Banach lattices with Property I or Property II and show that Property I and Property II are dual to each other.

DEFINITION. If E is a Banach space, then $l^1[E]$ will denote the space of summable sequences in E with the norm

$$\varepsilon(\{x_n\}) = \sup_{\|x'\|\leq 1} \sum_{n=1}^{\infty} |\langle x_n, x'\rangle| .$$

Pietsch [14] has shown that $l^1[E]$ is isomorphic to $l^1 \tilde{\otimes}_\varepsilon E$. If, in addition, E is a Banach lattice, then this isomorphism is an order isomorphism where we consider the cone C of positive summable sequences in $l^1[E]$ and the closure $\overline{K}_p$ of the projective cone in $l^1 \tilde{\otimes}_\varepsilon E$. In this case the norm

$$\overline{\varepsilon}(\{x_n\}) = \sup_{\substack{\|x'\|\leq 1 \\ x'\geq 0}} \sum_{n=1}^{\infty} |\langle x_n, x'\rangle|$$

is equivalent to the ε norm.

LEMMA 3.1. *If E is a Banach lattice then $(l^1[E], C)$ is a vector lattice if and only if every summable sequence in E is the difference of positive summable sequences. In this case, $(l^1[E], C)$ is a Banach lattice.*

Proof. Necessity is clear, so suppose that for every $\{x_n\} \in l^1[E]$, $\{x_n\} = \{y_n\} - \{z_n\}$ where $\{y_n\}$ and $\{z_n\}$ are positive summable sequences. Since $0 \le x_n^+ \le y_n$, it easily follows that $\{x_n^+\}$ is summable. Therefore, $\{x_n^+\} = \{x_n\}^+$ and $l^1[E]$ is a vector lattice.

When $l^1[E]$ is a vector lattice, the $\bar{\varepsilon}$ norm is clearly monotone on the cone C and so $l^1[E]$ is a Banach lattice by [8], (8.4).

EXAMPLES 3.2. $l^1 \tilde{\otimes}_\varepsilon l^1$ is not a vector lattice. For if it were, then every summable sequence in l^1 would be the difference of positive summable sequences. But then every summable sequence in l^1 would be absolutely summable (by Proposition 2.4) which would contradict the Dvoretzky-Rogers Theorem.

If X is a compact Hausdorff space, then $l^1[C(X)]$ is a vector lattice. This is contained in Remarks 3.4, but can easily be shown directly by noting that $l^1 \tilde{\otimes}_\varepsilon C(X)$ is order and topologically isomorphic to $C(X, l^1)$, the space of continuous functions from X into l^1, and that the cone of positive functions in $C(X, l^1)$ is a lattice cone.

We now characterize Banach lattices with Property I.

THEOREM 3.3. *If H is a Banach lattice with cone K then the following assertions about H are equivalent:*

(1) *every summable sequence in H is the difference of positive summable sequences;*

(2) *$\left(l^1 \tilde{\otimes}_\varepsilon H, K_p\right)$ is a vector lattice;*

(3) *H has Property I, that is, every map from a Banach lattice into H is preregular;*

(4) *each map in $L(c_0, H)$ is preregular;*

(5) *every order summable map from H into a Banach space is absolutely summable.*

Proof. That (1) is equivalent to (2) follows from Lemma 3.1 and that (3) implies (4) is clear.

(2) implies (3). Let E be a Banach lattice and $T \in L(E, H)$. Since H'' is a Banach lattice all positive linear maps from E into H'' are continuous ([*10*], Chapter 2, (2.16)), and so it suffices to show that $T^+ : E \to H''$ exists. To do this we must show that for each $x \in E$, $x \geq 0$, the set

$$B_x = \left\{ \sum_{n=1}^{k} (Tx_n)^+ : x = \sum_{n=1}^{k} x_n,\ x_n \in K \text{ for all } n \right\}$$

has a supremum in H''; (see [*10*], Chapter 2, Section 2, equation (9)).

By Lemma 3.1, $l^1 \check{\otimes}_\varepsilon H$ is a Banach lattice and hence the map $\phi : l^1 \check{\otimes}_\varepsilon H \to l^1 \check{\otimes}_\varepsilon H$ defined by $\phi(\{y_n\}) = \{y_n^+\}$ is continuous. Let $\psi : l^1 \check{\otimes}_\varepsilon H \to H$ be defined by $\psi(\{y_n\}) = \sum_{n=1}^{\infty} y_n$. Then ψ is continuous, since

$$\begin{aligned} \|\psi(\{y_n\})\| &= \sup_{\|y'\| \leq 1} \left| \sum_{n=1}^{\infty} \langle y_n, y' \rangle \right| \\ &\leq \sup_{\|y'\| \leq 1} \sum_{n=1}^{\infty} |\langle y_n, y' \rangle| \\ &= \phi(\{y_n\}) . \end{aligned}$$

For each $0 \leq x \in E$, define

$$A_x = \left\{ \{x_n\} \in l^1[E] : \text{for some } k,\ x_n = 0 \text{ for } n \geq k+1,\ x_n \geq 0 \text{ for all } n \text{ and } \sum_{n=1}^{k} x_n = x \right\} .$$

Then for $\{x_n\} \in A_x$,

$$\bar{\varepsilon}\{x_n\} = \sup_{\substack{\|x'\| \leq 1 \\ x' \geq 0}} \sum_{n=1}^{k} \langle x_n, x' \rangle = \|x\| .$$

Therefore, the set A_x is bounded in $l^1 \tilde{\otimes}_\varepsilon E$ and hence the set

$$[\psi\circ\phi\circ(1\otimes T)](Ax) = \left\{\sum_{n=1}^{k} \left(Tx_n\right)^+ : \{x_n\} \in A_x\right\} = B_x$$

is topologically bounded in H. A standard argument using the decomposition lemma shows that B_x is also directed $(\leq)$. By [10], Chapter 4, (1.8), H'' is boundedly order complete (that is, every topologically bounded, directed $(\leq)$ subset has a supremum) and so B_x has a supremum in H''.

(4) implies (5). Suppose that F is a Banach space, that $T : H \to F$ is order summable and that $S \in L(c_0, H)$. Then $S = S_1 - S_2$ where $0 \leq S_1$, $S_2 \in L(c_0, H'')$. Therefore, $T'' \circ S = T'' \circ S_1 - T'' \circ S_2$ is order summable and hence T'' is absolutely summable by Proposition 2.1. It follows from Propositions 2.1 and 2.2 that T is absolutely summable.

(5) implies (2). If F is a Banach space, then

$$\left(H \otimes_\sigma F\right)' = S(H, F) = S_+(H, F) = \left(H \otimes_{|\sigma|} F\right)' .$$

Since σ and $|\sigma|$ are both norm topologies it follows that $\sigma = |\sigma|$. Hence, in particular,

$$H \tilde{\otimes}_{|\sigma|} l^1 = H \tilde{\otimes}_\sigma l^1 = H \tilde{\otimes}_\varepsilon l^1 .$$

Therefore, by the discussion in §2, the cone $\overline{K}_p$ is generating in $l^1 \tilde{\otimes}_\varepsilon H$ and so the latter space is a vector lattice.

This completes the proof of Theorem 3.3.

REMARKS 3.4. (1) It follows from the proof of Theorem 3.3 that if H is boundedly order complete (for example, if H is a dual Banach lattice and satisfies any of the conditions of Theorem 3.3) then each map in $L(E, H)$ is regular and so $L(E, H)$ is a vector lattice for every Banach lattice E.

(2) If H is an AM-space then H has Property II. This follows from [10], Chapter 4, (3.7), and the fact that H is isomorphic as a Banach lattice to a $C(X)$ for a stonean space X.

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(3) In Theorem 3.3, (2) is equivalent to (3) in more general circumstances. In particular, if H is a Fréchet lattice such that H'_β is barrelled or such that H is boundedly order complete then this equivalence holds. If $\{H_n\}$ is a sequence of such spaces with Property I then one can show that $\prod_{n=1}^{\infty} H_n$ is also a Frechet lattice with Property I. Therefore, $\prod_{n=1}^{\infty} C(X_n)$, where each X_n is a compact, Hausdorff space, is an example of a Fréchet lattice with Property I.

(4) If H is a nuclear Fréchet lattice then $l^1 \tilde{\otimes}_\varepsilon H = l^1 \tilde{\otimes}_\pi H$ which equals the space of absolutely summable sequences in H ([14]). The latter space is a vector lattice and so $l^1 \tilde{\otimes}_\varepsilon H$ is a vector lattice. Since H'_β is barrelled, it follows that H has Property I.

(5) The characterization in Proposition 2.3 of an AM-space indicates that a Banach lattice with Property I is similar to an AM-space.

We now show that Properties I and II are dual to each other.

PROPOSITION 3.5. *A Banach lattice B has Property* II *(respectively Property* I*) if and only if its dual B' has Property* I *(respectively Property* II*).*

Proof. B has Property II implies B' has Property I. Suppose that E is a Banach lattice and that $T \in L(E, B')$. Then $T' : B'' \to E'$ and the map $S = T'|_B : B \to E'$ is regular, since B has Property II. Therefore, $S' : E'' \to B'$ is regular. If $x \in E$ and $y \in B$ then

$$\langle S'x, y\rangle = \langle x, Sy\rangle = \langle x, T'y\rangle = \langle Tx, y\rangle .$$

Therefore, $S'|_E = T$ and hence T is regular.

B' has Property I implies B has Property II. Suppose that E is a Banach lattice and that $T \in L(B, E)$. Then $T' : E' \to B'$ is regular and so $T'' : B'' \to E''$ is regular. Therefore, $T = T''|_B$ is preregular.

B has Property I implies B' has Property II. We first show that every majorizing map into B' is absolutely majorizing. Let E be a

Banach space and $S \in L(E, B')$ be majorizing. Then $S' : B'' \to E'$ is order summable and so $T = S'|_B : B \to E'$ is order summable and hence absolutely summable since B has Property I. Therefore, $T' : E'' \to B'$ is absolutely majorizing and so, by Proposition 2.2, $R \circ T'$ is majorizing for all $R \in L(B', l^1)$. Hence, $(R \circ T')|_E = R \circ (T'|_E)$ is majorizing. Since this is true for all $R \in L(B', l^1)$, Proposition 2.2 implies that $T'|_E = S$ is absolutely majorizing.

We now show that B'' has Property I. This will imply that B' has Property II by an earlier part of this theorem. By Theorem 3.3 it is enough to show that $L(c_0, B'')$ is a vector lattice and to do this it suffices to show that the projective cone in $c_0 \tilde{\otimes}_\pi B'$ is normal. Since $c_0 \tilde{\otimes}_\pi B' = B' \tilde{\otimes}_\pi c_0$, this is equivalent to showing that $L(B', l^1)$ is a vector lattice. So, let $T \in L(B', l^1)$. To see that T^+ exists in $L(B', l^1)$ let $x \in B'$, $x \geq 0$, and let B'_x be the linear hull of $[-x, x]$ with unit ball $[-x, x]$. Let $I : B'_x \to B'$ be the inclusion map. Since B'_x is an AM-space and I is a positive map, I is majorizing and hence absolutely majorizing by what we have just proved about B'. Proposition 2.2 thus implies that $T \circ I$ is majorizing. It follows that $T \circ I([-x, x]) = T([-x, x])$ is bounded above in l^1 (see [16], (1.5)). Therefore, T^+ exists and is necessarily continuous. Since T was chosen arbitrarily in $L(B', l^1)$, it follows that $L(B', l^1)$ is a vector lattice.

B' has Property II implies B has Property I. Suppose that E is a Banach lattice and $T \in L(E, B)$. Then $T' : B' \to E'$ is regular and so $T'' : E'' \to B''$ is regular. Therefore, $T''|_E = T$ is preregular.

COROLLARY 3.6. *If H is a Banach lattice with Property* I *then $L(E, H'')$ is an order complete vector lattice for any Banach lattice E.*

Proof. This follows from Proposition 3.5, (3.4) of Chapter 4 in [10], and the fact that there is a positive continuous projection from H'''' into H''.

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THEOREM 3.7. *If G is a Banach lattice then the following assertions about G are equivalent:*

(1) *for any Banach lattice F, the closure of the projective cone K_p is normal in $G \tilde{\otimes}_\pi F$;*

(2) *$\overline{K}_p$ is normal in $G \tilde{\otimes}_\pi c_0$;*

(3) *G has Property* II, *that is, every map from G into a Banach lattice is preregular;*

(4) *$L(G, l^1)$ is a vector lattice;*

(5) *every majorizing map from a Banach space into G is absolutely majorizing.*

Proof. That (1) implies (2), and that (2) is equivalent to (4) is clear. If (2) holds then $L(c_0, G')$ is a vector lattice and so G' has Property I by Theorem 3.3. Therefore, G has Property II by Proposition 3.5. (3) easily implies (1) by [*10*], Chapter 2, (1.22), and so (1), (2), (3), and (4) are equivalent.

Suppose that (3) holds, that E is a Banach space, and that $T : E \to G$ is majorizing. Then $T' : E' \to G'$ is order summable and hence absolutely summable since G' has Property I. Therefore, T is absolutely majorizing and hence (5) holds. Finally, if (5) holds then one can show that $L(G, l^1)$ is a vector lattice in a manner similar to the latter part of the proof of Proposition 3.5. This completes the proof.

We remark that Proposition 2.4 and Theorem 3.7 show that a Banach lattice with Property II has a characterization similar to that of an *AL*-space.

4. Order properties of a Banach lattice with Property II

PROPOSITION 4.1. *A Banach lattice G with Property* II *has* $\sigma(G, G')$ *compact order intervals.*

Proof. Let $x \geq 0$ in G. Since G_x is an *AM*-space, the inclusion map $I : G_x \to G$ is majorizing, hence absolutely majorizing by Property II. Therefore $I([-x, x]) = [-x, x]$ is $\sigma(G, G')$ compact in G.

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DEFINITIONS. A sequence $\{x_n\}$ in a Banach space E is called *weakly summable* if $\sum_{n=1}^{\infty} |\langle x_n, x'\rangle| < \infty$ for all $x' \in E'$.

It follows from the fact that weakly bounded sets in E are norm bounded that if $\{x_n\}$ is a weakly summable sequence then

$$\varepsilon(\{x_n\}) = \sup_{\|x'\|\le 1} \sum_{n=1}^{\infty} |\langle x_n, x'\rangle|$$

is finite. Note that $\{x_n\}$ is summable in E if it is weakly summable and the net $\{\{x_n\}_{n\in\sigma} : \sigma \text{ is a finite subset of } N\}$, where N is the set of natural numbers, converges to $\{x_n\}_{n=1}^{\infty}$ for the ε-topology on $l^1[E]$. For more details see [12], Chapter 1.

A Banach lattice has the *monotone convergence property* if the filter of sections of every directed $(\le)$ topologically bounded subset converges to its supremum.

PROPOSITION 4.2. *For a Banach lattice E the following statements are equivalent:*

(1) *E has the monotone convergence property;*

(2) *every positive weakly summable sequence is summable;*

(3) *every increasing, topologically bounded sequence converges to its supremum.*

Proof. (1) implies (2). Let $\{x_n\}$ be a positive weakly summable sequence in E. For each finite subset σ of N,

$$\Big\|\sum_{n\in\sigma} x_n\Big\| = \sup_{\substack{\|x'\|\le 1\\ x'\ge 0}} \Big\langle \sum_{n\in\sigma} x_n, x'\Big\rangle$$

$$\le \sup_{\substack{\|x'\|\le 1\\ x'>0}} \sum_{n=1}^{\infty} \langle x_n, x'\rangle \le \varepsilon(\{x_n\}) < \infty .$$

Therefore $\Big\{\sum_{n\in\sigma} x_n : \sigma \text{ is a finite subset of } N\Big\}$ is a topologically bounded, directed $(\le)$ subset of E and so converges by the monotone

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convergence property. Hence $\{x_n\}$ is summable.

(2) implies (3). Let $\{x_n\}$ be an increasing, topologically bounded sequence in E. Without loss of generality we can assume that $x_n \geq 0$ for all n. Define $y_n = x_n - x_{n-1}$ (where $x_0 = 0$). Then $\sum_{n=1}^{k} y_n = \sum_{n=1}^{k} x_n - x_{n-1} = x_k$ and so for $0 \leq x' \in E'$ we have that

$$\sum_{n=1}^{k} \langle y_n, x' \rangle = \langle x_k, x' \rangle \leq \|x_k\| \|x'\|$$
$$\leq \left(\sup_n \|x_n\|\right) \|x'\| < \infty .$$

This is true for all k so that the sequence $\{y_n\}$ is weakly summable and hence summable by (2). Since E is complete, $\sum_{n=1}^{\infty} y_n$ exists, that is, the sequence $\{x_n\}$ converges to some $x \in E$, and $x = \sup\{x_n\}$ since the cone in E is closed.

(3) implies (1). Assume that E does not have the monotone convergence property. Then there exists a topologically bounded, directed $(\leq)$ net $\{x_\alpha : \alpha \in A\}$ such that $\alpha \geq \beta$ if and only if $x_\alpha \geq x_\beta$ and such that $\{x_\alpha\}$ does not converge. Hence there is a $\delta > 0$ such that there is no $\alpha_0 \in A$ with the property that $\|x_\alpha - x_\beta\| < \delta$ for all $\alpha, \beta \geq \alpha_0$. Let $\alpha_1 \in A$ and choose $\alpha_2 > \alpha_1$ such that $\|x_{\alpha_2} - x_{\alpha_1}\| \geq \frac{\delta}{2}$. Now choose $\alpha_3 > \alpha_2$ such that $\|x_{\alpha_3} - x_{\alpha_2}\| \geq \frac{\delta}{2}$. Continuing in this way we get a monotone increasing, topologically bounded sequence $\left\{x_{\alpha_n}\right\}$ that does not converge.

PROPOSITION 4.3. *A Banach lattice G with Property* II *has the monotone convergence property.*

Proof. Let $\{x_n\}$ be a weakly summable positive sequence in G. By (1.3.5) in [12], $(\lambda_n x_n)$ is summable for each $\{\lambda_n\} \in c_0$ and so we can

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define a map $T : c_0 \to G$ by $T(\{\lambda_n\}) = \sum_{n=1}^{\infty} \lambda_n x_n$. T is continuous since

$$\sup_{\|\{\lambda_n\}\|\le 1} \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| = \sup_{\|\{\lambda_n\}\|\le 1} \sup_{\|x'\|\le 1} \left| \left\langle \sum_{n=1}^{\infty} \lambda_n x_n , x' \right\rangle \right|$$

$$\le \sup_{\|x'\|\le 1} \sum_{n=1}^{\infty} |\langle x_n , x' \rangle|$$

$$= \varepsilon(\{x_n\}) < \infty .$$

T is positive since $\{x_n\}$ is positive, and hence T is majorizing since c_0 is an AM-space. Therefore, by Property II, T is absolutely majorizing and hence weakly compact.

Let U be the unit ball in c_0 . Then $T(U)$ is weakly relatively compact and contains the net $A = \left\{ \sum_{n\in\sigma} x_n : \sigma \text{ is a finite subset of } N \right\}$.

Hence every subnet of A contains a convergent subnet which necessarily must converge to $\sup A$ ([*10*], Chapter 2, (3.1)). By a standard property of net convergence this implies that A converges to $\sup A$ for $\sigma(G, G')$ and hence for the norm topology of G by [*10*], Chapter 2, (3.4). It follows that $\{x_n\}$ is summable and so G has the monotone convergence property by Proposition 4.2.

In an AL-space every summable sequence is absolutely summable. A Banach lattice with Property II has a weaker property. In order to describe this property we need the following definitions.

DEFINITIONS (see [*13*]). A sequence $\{x_n\}$ in a normed space E is called (*weakly*) *p-summable* $(p \ge 1)$ if $\sum_{n=1}^{\infty} |\langle x_n , x' \rangle|^p < \infty$ for all $x' \in E'$. $\{x_n\}$ is called *absolutely p-summable* if $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. A continuous linear map T from a normed space E into a normed space F is called *absolutely p-summable* if T maps p-summable sequences into absolutely p-summable sequences.

PROPOSITION 4.4. *Every positive summable sequence in a Banach*

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lattice G *with Property* II *is absolutely* 2-*summable.*

Proof. Let $\{x_n\}$ be a positive summable sequence in G, let $x = \sum_{n=1}^{\infty} x_n$, and consider the *AM*-space G_x. The sequence $\{x_n\}$ is weakly summable in G_x. For, let $a \in G_x'$, $a \geq 0$. Then

$$\begin{aligned} \sum_{n=1}^{k} |\langle x_n, a\rangle| &= \sum_{n=1}^{k} \langle x_n, a\rangle \\ &= \left\langle \sum_{n=1}^{k} x_n, a \right\rangle \leq \langle x, a\rangle . \end{aligned}$$

Since k is arbitrary it follows that $\sum_{n=1}^{\infty} |\langle x_n, a\rangle| < \infty$ and so $\{x_n\}$ is weakly summable in G_x. Therefore $\{x_n\}$ is 2-summable in G_x.

Consider the inclusion map $I : G_x \to G$. As we have seen before in Proposition 4.2, I is absolutely majorizing and hence I can be factored through a Hilbert space H, that is, there exist continuous linear maps $I_1 : G_x \to H$ and $I_2 : H \to G$ such that the following diagram commutes;

$$\begin{tikzcd} G_x \arrow[rr, "I"] \arrow[dr, "I_1"'] & & G \\ & H \arrow[ur, "I_2"'] & \end{tikzcd}$$

By [5], (4.3), I_1 is absolutely 2-summable. Therefore I is absolutely 2-summable and so $\{I(x_n)\} = \{x_n\}$ is absolutely 2-summable in G.

EXAMPLE 4.5. $L^p[0, 1]$, $1 < p \leq \infty$, does not have Property II. For $p = \infty$ this follows immediately since $L^\infty[0, 1]$ does not have the monotone convergence property. For $p < \infty$ we consider two cases. First assume that $p > 2$. Let $\{E_n\}$ be a sequence of disjoint sets of positive measure. Let $\phi_n(x)$ be that positive multiple of the characteristic function of E_n such that $\int_0^1 |\phi_n(x)|^p = 1$. Choose a sequence $\{a_n\}$

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such that $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n^2 = \infty$, and $\sum_{n=1}^{\infty} a_n^p < \infty$. Then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ converges unconditionally in $L^p[0, 1]$ but

$$\sum_{n=1}^{\infty} \|a_n \phi_n(x)\|^2 = \sum_{n=1}^{\infty} \left(\int_0^1 |a_n \phi_n(x)|^p \right)^{2/p}$$
$$= \sum_{n=1}^{\infty} a_n^2 = \infty .$$

Hence $L^p[0, 1]$ cannot have Property II by Proposition 4.4.

Now assume that $1 < p \leq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose on the contrary that $L^p[0, 1]$ does have Property II. Let $T : l^2 \to l^2$ be a nonregular map (see [*10*], pp. 171-172). We note that for $r > 1$, l^2 is isomorphic to the complemented subspace of $L^r[0, 1]$ generated by the Rademacher functions and, moreover, this isomorphism sends the nth unit vector in l^2 into the nth Rademacher function in $L^r[0, 1]$ (see [*9*]). So, let $S : l^2 \to L^q[0, 1]$ be this isomorphism and let $R : L^p[0, 1] \to l^2$ be a continuous projection. Then the map T can be factored as

$$\begin{tikzcd}
 & l^2 \arrow[r, "T"] \arrow[dl, "S"'] & l^2 & \\
L^q[0, 1] \arrow[r, "I"] & L^p[0, 1] \arrow[r, "R"] & l^2 \arrow[ul, "T"'] &
\end{tikzcd}$$

where I is the continuous, positive injection of $L^q[0, 1]$ into $L^p[0, 1]$ which exists since $q \geq p$. Since we are assuming that $L^p[0, 1]$ has Property II, $T \circ R$ is a regular map and S is a regular map by Proposition 3.5. Hence $T = T \circ R \circ I \circ S$ is regular, a contradiction.

5. The atomic case

DEFINITIONS. A positive element x in a vector lattice is called an atom if $0 \leq y \leq x$ implies that $y = \alpha x$ for some $\alpha \in [0, 1]$. An order

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complete vector lattice E is called *atomic* if the band generated by the atoms is equal to E .

The l^p spaces, $1 \le p < \infty$, are examples of atomic lattices.

In this section we prove that if G is a Banach lattice with Property II, and, in addition, G is atomic, then G is isomorphic as a Banach lattice to $l^1(\Gamma)$ for some index set Γ .

First, suppose that G is any Banach lattice with Property II and that $\{x_n\}$ is a positive summable sequence in G that is disjoint; that is, $x_n \wedge x_m = 0$ for $n \neq m$. Define the functional x_m'' on $\text{L.H.}\{x_n\}$ (where L.H. denotes linear hull) by

$$\left\langle x_m', \sum \alpha_n x_n \right\rangle = \alpha_m \|x_m\| ,$$

where, of course, $\alpha_n = 0$ for all but a finite number of n . Then x_m' is continuous on $\text{L.H.}\{x_n\}$ since

$$\begin{aligned}\left|\left\langle x_m, \sum \alpha_n x_n \right\rangle\right| &= |\alpha_m| \|x_m\| \\ &= \| |\alpha_m| x_m \| \le \left\| \sum |\alpha_m| x_n \right\| \\ &= \left\| \left| \sum \alpha_n x_n \right| \right\| = \left\| \sum \alpha_n x_n \right\| ,\end{aligned}$$

the next to last equality resulting from the fact that the x_n's are disjoint. It follows that each x_m' can be extended to a continuous linear functional x_m' of norm 1 on $X = \overline{\text{L.H.}\{x_n\}}$.

If we define $z_n = x_n/\|x_n\|$, then $\{z_n, x_n'\}$ is a biorthogonal system (that is, $\langle z_n, x_m' \rangle = \delta_{nm}$) in X . Moreover, $\{z_n\}$ is an unconditional basis for X such that if $\sum_{n=1}^{\infty} \alpha_n z_n$ is convergent in X then $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. To see this define U_m on X by

$$U_m(x) = \sum_{n=1}^{m} \langle x_n', x\rangle z_n , \quad x \in X .$$

U_m is continuous since x_n' is continuous for each n and if $\sum \alpha_n z_n \in \text{L.H.}\{z_n\} = \text{L.H.}\{x_n\}$ then

$$\begin{aligned} \left\|U_m\left(\sum \alpha_n z_n\right)\right\| &= \left\|\sum_{n=1}^{m} \alpha_n z_n\right\| \\ &= \left\|\sum_{n=1}^{m} |\alpha_n| z_n\right\| \le \left\|\sum |\alpha_n| z_n\right\| \\ &= \left\|\sum \alpha_n z_n\right\| . \end{aligned}$$

Since $U_m(z_m) = z_m$ it follows that $\|U_m\| = 1$. It now follows from [7], Corollary 3, p. 31, that $\{z_n\}$ is a basis for X.

If $x \in X$ and $x = \sum_{n=1}^{\infty} \alpha_n z_n$ then $\alpha_n = \langle x, x_n'\rangle$ for all n and so $x = \sum_{n=1}^{\infty} \langle x, x_n'\rangle z_n$. Then the disjointness of the z_n's and continuity of the lattice operations imply that $|x| = \sum_{n=1}^{\infty} |\langle x, x_n'\rangle| z_n$. Hence, given $\delta > 0$ there exists an n_0 such that

$$\left\|\sum_{n=n_0}^{\infty} |\langle x, x_n'\rangle| z_n\right\| < \delta .$$

If σ is any finite subset of N such that σ is disjoint from $\{1, 2, \ldots, n_0\}$ then

$$\begin{aligned} \left\|\sum_{n\in\sigma} \langle x, x_n'\rangle z_n\right\| &= \left\|\sum_{n\in\sigma} |\langle x, x_n'\rangle| z_n\right\| \\ &\le \left\|\sum_{n=n_0}^{\infty} |\langle x, x_n'\rangle| z_n\right\| \\ &< \delta . \end{aligned}$$

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Hence, $\sum_{n=1}^{\infty} \langle x, x_n' \rangle z_n$ converges unconditionally to x, $\sum_{n=1}^{\infty} |\langle x, x_n' \rangle| z_n$ converges unconditionally to $|x|$, and it follows from Proposition 4.4 that

$$\sum_{n=1}^{\infty} \|\langle x, x_n' \rangle z_n\|^2 = \sum_{n=1}^{\infty} \|\alpha_n z_n\|^2 \le \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty .$$

We make use of these results in the following proposition.

PROPOSITION 5.1. *Suppose that G is a Banach lattice with Property* II *and that $\{x_n\}$ is a positive, disjoint, summable sequence in G. If $X = \overline{\mathrm{L.H.}\{x_n\}}$ is complemented in G then $\{x_n\}$ is absolutely summable.*

Proof. Let P be a continuous projection of G onto X. By the considerations preceding the statement of this proposition, we can define a map $T : X \to l^2$ by

$$T\left(\sum_{n=1}^{\infty} \langle x, x_n' \rangle z_n\right) = \{\langle x, x_n' \rangle\} .$$

A simple application of the Closed Graph Theorem shows that T is continuous (since convergence implies coordinate convergence in X and in l^2). Let $R : l^2 \to L^1[0, 1]$ be the isomorphism that sends e_n into the nth Rademacher function ϕ_n. If $S = R \circ T \circ P$, then $S : G \to L^1[0, 1]$ and $S(x_n) = \|x_n\|\phi_n$. Moreover, since G has Property II, S is a regular map and hence S is order summable. Therefore, the positive sequence $\{x_n\}$ gets mapped into the absolutely summable sequence $\{\|x_n\|\phi_n\}$. Since $\|\phi_n\|_{L^1} = 1$, it follows that

$$\sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \big\| \|x_n\|\phi_n \big\| < \infty$$

and so $\{x_n\}$ is absolutely summable.

COROLLARY 5.2. *Suppose that G is a Banach lattice with Property* II

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and that $\{x_n\}$ *is a positive, disjoint, summable sequence of atoms in* G *. Then* $\{x_n\}$ *is absolutely summable.*

Proof. First note that in Banach lattices E with the monotone convergence property the closure of a lattice ideal is a band. For, let L be a lattice ideal in E and let $x \geq 0$ be an element of the band generated by L. Then $x = \sup\{y \in L : 0 \leq y \leq x\}$. The set $\{y \in L : 0 \leq y \leq x\}$ is directed $(\leq)$, and so by the monotone convergence property its filter of sections converges to x. Therefore x is in the closure of L.

If we now consider our original sequence $\{x_n\}$ then $\text{L.H.}\{x_n\}$ is a lattice ideal in G. For if $0 \leq y \leq \sum_{n=1}^{k} \alpha_n x_n$, then $\alpha_n \geq 0$ and

$$
\begin{aligned}
y &= y \wedge \sum_{n=1}^{k} \alpha_n x_n \\
&= y \wedge \left(\alpha_1 x_1 \vee \alpha_2 x_2 \vee \ldots \vee \alpha_k x_k\right) \\
&= \left(y \wedge \alpha_1 x_1\right) \vee \left(y \wedge \alpha_2 x_2\right) \vee \ldots \vee \left(y \wedge \alpha_k x_k\right) \\
&= \sum_{n=1}^{k} \beta_n x_n \in \text{L.H.}\{x_n\} ,
\end{aligned}
$$

since each x_n is an atom. Therefore $X = \overline{\text{L.H.}\{x_n\}}$ is a band in G and so is complemented in G by [10], Chapter 2, (4.9). The result now follows from Proposition 5.1.

THEOREM 5.3. *If* G *is an atomic Banach lattice with Property* II *then* G *is order and topologically isomorphic to the Banach lattice* $l^1(\Gamma)$ *for some index set* Γ *.*

Proof. By Zorn's Lemma there exists a maximal, disjoint collection $\{z_\alpha : \alpha \in A\}$ of atoms of norm one. G is equal to the band generated by the $\{z_\alpha\}$ and, since $\text{L.H.}\{z_\alpha\}$ is a lattice ideal, if $x \geq 0$ in G then

$$x = \sup\{y \in \text{L.H.}\{z_\alpha\} : 0 \leq y \leq x\} .$$

By the monotone convergence property the filter of sections of $\{y \in \text{L.H.}\{z_\alpha\} : 0 \leq y \leq x\}$ converges to x and so, since G is a Banach space, there exists a sequence $\{y_n\} \in \text{L.H.}\{z_\alpha\}$ such that $y_n \to x$. It follows for each $x \geq 0$ in G, and hence for each x in G, that x is in the closure of the linear hull of a countable number of the $\{z_\alpha\}$. Then, by the methods similar to those used in the remarks preceding Proposition 5.1, it is easy to see that $\{z_\alpha : \alpha \in A\}$ is an unconditional basis for G and for each $x \in G$, $x = \sum_{\alpha \in A} a_\alpha z_\alpha$ where all but a countable number of the a_α's are zero. Moreover, it follows from Corollary 5.2 that $\sum_{\alpha \in A} a_\alpha z_\alpha$ is absolutely summable and so $\sum_{\alpha \in A} |a_\alpha| < \infty$. Hence we may define a map $T : G \to l^1(A)$ by $T(z_\alpha) = e_\alpha$, where e_α is the αth unit vector in $l^1(T)$. T is clearly a positive, one-to-one, onto map. The continuity of T follows from the Closed Graph Theorem and the fact that in G and in $l^1(A)$ convergence implies coordinate convergence. Hence T is an isomorphism by the Open Mapping Theorem.

COROLLARY 5.4. *Suppose that G is a Banach lattice with Property II. If G has atoms then G has a complemented subspace that is order isomorphic to an $l^1(\Gamma)$ for some index set Γ.*

Proof. Let G_1 be the band in G generated by the atoms and let $P : G \to G_1$ be the canonical, positive, continuous band projection. Let F be a Banach lattice and $T \in L(G_1, F)$. Then $T \circ P$ is preregular and so $T = T \circ P \circ I$ is preregular where $I : G_1 \to G$ is the inclusion map. Therefore, G_1 has Property II. The result now follows from Theorem 5.3.

COROLLARY 5.5. *l^p, $1 < p < \infty$ has neither Property* I *nor Property* II.

We conjecture that any Banach lattice with Property II is isomorphic as a Banach lattice to an *AL*-space.

References

[1] T. Andô, "On fundamental properties of a Banach space with a cone", *Pacific J. Math.* 12 (1962), 1163-1169.

[2] J.A. Crenshaw, "Extreme positive linear operators", *Math. Scand.* 25 (1969), 195-217.

[3] Alexandre Grothendieck, *Produits tensoriels topologiques et espaces nucléaires* (Mem. Amer. Math. Soc. 16. Amer. Math. Soc., Providence, Rhode Island, 1955).

[4] Harold Jacobs, "Ordered topological tensor products", Dissertation, University of Illinois, Urbana, Illinois, 1969.

[5] J. Lindenstrauss and A. Pełczyński, "Absolutely summing operators in L_p spaces and their applications", *Studia Math.* 29 (1968), 275-326.

[6] H.P. Lotz, *Lectures on topological tensor products, linear mappings and nuclear spaces* (University of Illinois, Urbana, Illinois, 1971).

[7] Jürg T. Marti, *Introduction to the theory of bases* (Springer Tracts in Natural Philosophy, 18. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[8] Isaac Namioka, *Partially ordered linear topological vector spaces* (Mem. Amer. Math. Soc. 24. Amer. Math. Soc., Providence, Rhode Island, 1957).

[9] A. Pełczyński, "Projections in certain Banach spaces", *Studia Math.* 19 (1960), 209-228.

[10] Anthony L. Peressini, *Ordered topological vector spaces* (Harper and Row, New York, Evanston, and London, 1967).

[11] A.L. Peressini and D.R. Sherbert, "Ordered topological tensor products", *Proc. London Math. Soc.* (3) 19 (1969), 177-190.

[12] Albrecht Pietsch, *Nukleare lokalkonvexe Räume* (Akademie-Verlag, Berlin, 1965; 2nd ed., Akademie-Verlag, Berlin, 1969). English translation of 2nd ed.: *Nuclear locally convex spaces* (translated by William H. Ruckle. Ergebnisse der Mathematik und ihrer Grenzgebiete, 66. Springer-Verlag, Berlin, Heidelberg, New York, 1972).

[13] A. Pietsch, "Absolut p-summierende Abbildungen in normierten Räumen", *Studia Math.* 28 (1967), 333-353.

[14] Albrecht Pietsch, "Zur Theorie der topologischen Tensorprodukte", *Math. Nachr.* 25 (1963), 19-30.

[15] Helmut H. Schaefer, *Topological vector spaces* (The Macmillan Company, New York; Collier-Macmillan, London, 1966).

[16] Ulf Schlotterbeck, "Über Klassen majorisierbarer Operatoren auf Banachverbänden", Dissertation, Eberhard-Karls-Universität zu Tübingen, Tübingen, 1969.

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