CONVERGENCE OF SOLUTIONS OF THIRD ORDER DIFFERENTIAL EQUATIONS*

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(received April 11, 1969)

1. Consider a system of differential equations $\frac{dx}{dt} = F(t, x)$. Solutions of this system are said to be convergent if, given any pair of solutions $x(t)$, $y(t)$, $x(t) - y(t) \to 0$ as $t \to \infty$. In this case the system is also said to be extremely stable.

In [6] a technique was developed which yielded the convergence of solutions of the forced Lienard equation. Here a similar technique is applied to forced third order equations. A critical step in [6] was to show that a certain matrix was negative definite. This could be done directly in [6] since the matrix was only 2×2. With third and higher order equations, direct use of necessary and sufficient conditions is not feasible since the computations become unwieldy.

A theorem on Gersgorin circles is used to bound the eigenvalues of the matrix. A theorem of Fan [2] is also used for the same purpose. Since these conditions are only sufficient, it can not be expected that they will reduce to the Routh-Hurwitz conditions.

The technique is theoretically applicable to higher dimensional equations, but for more than 3×3 matrices, even the computations with these techniques become unmanageable. It is to be emphasized that in the following theorems one set of inequalities for a technique is prescribed. Clearly there are other possibilities, and in any particular case one might want to investigate these other possibilities.

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* Much of this paper is a part of the author's Ph.D. dissertation at the University of Iowa. The author wishes to thank Professor P.E. Waltman for his advice and encouragement.

This work was supported in part by National Science Foundation COSIP (GY 4754).

It is to be noted that the theorems yield convergence for any forcing function $e(t)$ for which solutions exist. This is an extremely strong kind of stability and is particularly important when $e(t)$ represents an unwanted, or unplanned, disturbance.

The following equations will be studied

$$(1.1) \qquad \dddot{x} + A\ddot{x} + g(x)\dot{x} + h(x) = e(t)$$

$$(1.2) \qquad \dddot{x} + f(\dot{x})\ddot{x} + g(x)\dot{x} + h(x) = e(t).$$

2. THEOREM 1. _Assume that_ A _is a positive constant,_ $g \in C^0(R^1)$, $h \in C^1(R^1)$ _and_ $e(t)$ _is such as to guarantee the existence of solutions of_ (1.1); _then all solutions of_ (1.1) _converge provided there exist positive constants_ a_1, a_2, a_3, b_1, b_2 _and_ ε _such that_

(i) $a_1a_2a_3 > a_1b_2^2 + a_3b_1^2 + 2a_2b_1b_2 + a_2^3$

(ii) $a_1a_2 > b_1 \geqq \frac{1}{2} + \varepsilon$

(iii) $b_2 \geqq \frac{1}{2} + \varepsilon$

(iv) $[2b_1g(x) + 2a_2h'(x) - 2a_1A + (a_1 + b_1A - a_2g(x) + b_2h'(x))^2$
$+ (a_2A + b_2g(x) - a_3h'(x) - b_1)^2] \leqq - \varepsilon$ for all x.

Remark 1. If $g(x) \equiv b$ and $h(x) \equiv cx$ where b and c are positive constants, then constants a_1, a_2, a_3, b_1 and b_2 can always be found such that inequality (iv) is satisfied. Thus, for the constant coefficient differential equation $\dddot{x} + A\ddot{x} + b\dot{x} + cx = e(t)$, conditions (i) - (iv) can always be reduced to the inequalities (i) - (iii). In particular, if $A \geqq 1$ and $b \geqq 1$ then the conditions are all satisfied if the Routh-Hurwitz conditions $Ab > c$ hold, although it is not necessary to have $a \geqq 1$ or $b \geqq 1$ in order that conditions (i) - (iv) hold.

Remark 2. By taking $a_1 = 10$, $a_2 = 2$, $a_3 = 4$ and $b_1 = b_2 = 1$, it is easy to see that all of the conditions of Theorem 1 are satisfied by the equation $\dddot{x} + 6\ddot{x} + 11\dot{x} + 6x = t$. Solutions of this equation are of the form $x(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t} + \frac{t}{6} - \frac{11}{36}$. Thus, it is clear that the conditions of Theorem 1 do not imply that the solutions of (1.1) are bounded.

Proof of Theorem 1. Equation (1.1) is equivalent to the system

$$(2.1)\qquad \begin{aligned} \dot{x}_1 &= x_2 - Ax_1 \\ \dot{x}_2 &= x_3 - G(x_1) \\ \dot{x}_3 &= e(t) - h(x_1) \end{aligned}$$

where $G(x) = \int_0^x g(s)ds$.

If $\theta(t)$ is an arbitrary but fixed solution of (2.1), then any other solution of (2.1) can be written as $x(t) = \theta(t) + \eta(t)$; and for each fixed $x(t)$, $\eta(t)$ is a solution of the system

$$(2.2)\qquad \dot{\eta} = \begin{pmatrix} -A & 1 & 0 \\ -R(t) & 0 & 1 \\ -S(t) & 0 & 0 \end{pmatrix} \eta$$

where

$$R(t) = \begin{cases} \dfrac{G(x_1(t)) - G(\theta_1(t))}{x_1(t) - \theta_1(t)} & \text{if } x_1(t) \neq \theta_1(t) \\ g(\theta_1(t)) & \text{if } x_1(t) = \theta_1(t) \end{cases}$$

$$S(t) = \begin{cases} \dfrac{h(x_1(t)) - h(\theta_1(t))}{x(t) - \theta(t)} & \text{if } x_1(t) \neq \theta_1(t) \\ h'(\theta_1(t)) & \text{if } x_1(t) = \theta_1(t) \end{cases}.$$

Let $V(\eta) = \eta \cdot Q\eta$ be a scalar valued function where $\eta = (\eta_1, \eta_2, \eta_3)$ and

$$Q \equiv \begin{pmatrix} a_1 & -b_1 & -a_2 \\ -b_1 & a_2 & -b_2 \\ -a_2 & -b_2 & a_3 \end{pmatrix} .$$

The matrix, and hence the function $V(\eta)$, is positive definite if and only if

$$\text{(2.3)} \qquad \begin{aligned} &\text{(i)} && a_1 > 0 \\ &\text{(ii)} && a_1 a_2 > b_1^2 \\ &\text{(iii)} && a_1 a_2 a_3 > a_1 b_2^2 + a_3 b_1^2 + 2a_2 b_1 b_2 + a_2^3 . \end{aligned}$$

As an aid in evaluating $\dot{V}(\eta, t)$ we note that, since $g(x)$ and $h'(x)$ are continuous, $R(t)$ and $S(t)$ can be written as

$$\text{(2.4)} \qquad \begin{aligned} R(t) &= \int_0^1 g(s\eta_1(t) + \theta_1(t))ds \\ S(t) &= \int_0^1 h'(s\eta_1(t) + \theta_1(t))ds. \end{aligned}$$

The method used here in evaluating $\dot{V}(\eta, t)$ is the same as the method used by Waltman and Bridgland in [6] where a second order equation was studied.

If we set

$$B = \begin{pmatrix} -A & 1 & 0 \\ -g(s\eta_1(t) + \theta_1(t)) & 0 & 1 \\ -h'(s\eta_1(t) + \theta_1(t)) & 0 & 0 \end{pmatrix}$$

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and $C = B^TQ + QB$, then C is the matrix

$$\begin{pmatrix} 2b_1g + 2a_2h' - 2a_1A & a_1 + b_1A - a_2g + b_2h' & a_2A + b_2g - a_3h' - b_1 \\ a_1 + b_1A - a_2g + b_2h' & -2b_1 & 0 \\ a_2f + b_2g - a_3h' - b_1 & 0 & -2b_2 \end{pmatrix}.$$

We show that

$$\dot{V}(\eta, t) = \int_0^1 \eta\cdot C\eta ds, \tag{2.5}$$

recalling that the matrix C is a function of both s and t.

$$V(\eta) = \eta\cdot Q\eta$$
$$= a_1\eta_1^2 - 2b_1\eta_1\eta_2 - 2a_2\eta_1\eta_3 - 2b_2\eta_2\eta_3 + a_2\eta_2^2 + a_3\eta_3^2 .$$

With respect to a solution $(\eta_1(t), \eta_2(t), \eta_3(t))$ of the system (2.2) we have

$$\dot{V}(\eta, t) = [2a_1 - 2a_2R(t) + 2b_1A + 2b_2S(t)]\eta_1\eta_2$$
$$+ [2a_2A - 2b_1 + 2b_2R(T) - 2a_3S(t)]\eta_1\eta_3$$
$$+ [2b_1R(t) - 2a_1A + 2a_2S(t)]\eta_1^2 - 2b_1\eta_2^1 - 2b_2\eta_3^2 .$$

Recalling (2.4), $\dot{V}(\eta, t)$ can be written as

$$\dot{V}(\eta, t) = \int_0^1 [2a_1 - 2a_2g(s\eta_1(t) + \theta_1(t)) + 2b_1A + 2b_2h'(s\eta_1(t) + \theta_1(t))]ds\,\eta_1\eta_2$$
$$+ \int_0^1 [2a_2A - 2b_1 + 2b_2g(s\eta_1(t) + \theta_1(t)) - 2a_3h'(s\eta_1(t) + \theta_1(t))]ds\,\eta_1\eta_3$$
$$+ \int_0^1 [2b_1g(s\eta_1(t) + \theta_1(t)) - 2a_1A + 2a_2h'(s\eta_1(t) + \theta_1(t))]ds\,\eta_1^2$$
$$- \int_0^1 2b_1ds\eta_2^2 - \int_0^1 2b_2ds\,\eta_3^2 ,$$

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or

$$(2.6) \qquad \dot{V}(\eta, t) = \int_0^1 \eta \cdot C\eta ds \; .$$

Assume that constants a_1, a_2, a_3, b_1, b_2 have been chosen such that Q is positive definite. Let $0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3$ be the eigenvalues of Q, then

$$(2.7) \qquad a(\|\eta\|) \equiv \lambda_1 \|\eta\|^2 \leqq V(\eta) \leqq \lambda_3 \|\eta\|^2 \equiv b(\|\eta\|).$$

Let $\lambda_1(t, s)$, $\lambda_2(t, s)$, $\lambda_3(t, s)$ be the eigenvalues of C, if $\lambda_i(t, s) \leqq -\varepsilon < 0$ for $i = 1, 2, 3$, all t and $0 \leqq s \leqq 1$, then $\dot{V}(\eta, t) \leqq -\varepsilon \|\eta\|^2$, and the solutions of (2,2) are uniform-asymptotically stable in the large.

We will use the following theorem to obtain conditions on the matrix C which will guarantee that all eigenvalues of C are bounded above by a negative constant.

LEMMA 2.1 [2, p. 131]. Let $B = (b_{ij})$ be a hermitian matrix with eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$. Let $c_1, \cdots, c_{n-1}$ and $d_1, \cdots, d_n$ be $2n-1$ real numbers such that $c_i > 0$, $i = 1, \cdots, n$, and $d_i - d_{i+1} \geqq \frac{1}{c_i}$, $i = 1, \cdots, n-1$, and assume that $b_{ii} + c_i \sum_{j>i} |b_{ij}|^2 \leqq d_i$, $i = 1, \cdots, n$. Then $\lambda_i \leqq d_i$, $i = 1, \cdots, n$.

Applying this theorem to the matrix C we obtain the following set of inequalities:

$$(2.8) \qquad \begin{gathered} 2b_1 g + 2a_2 h' - 2a_1 A + c_1[(a_1 + b_1 A - a_2 g + b_2 h') \\ + (a_2 A + b_2 g - a_2 h' - b_1)^2] \leqq d_1 , \\ -2b_1 \leqq d_2, \qquad -2b_2 \leqq d_3, \end{gathered}$$

where $d_1 - d_2 \geqq \frac{1}{c_1} > 0$, $d_2 - d_3 \geqq \frac{1}{c_2}$.

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In particular, if we take $d_1 = -\varepsilon < 0$, $c_1 = 1$, $d_2 = -1 - \varepsilon$, $c_2 = \frac{1}{\varepsilon}$ and $d_3 = -1 \; -2\varepsilon$, (2.7) reduces to conditions (ii)-(iv) of Theorem 1, and by Lemma 2.1 the eigenvalues of C will all satisfy

$$(2.9) \qquad \lambda_i(t, s) \leqq -\varepsilon < 0$$

for $i = 1, 2, 3$, $t \geqq 0$ and $0 \leqq s \leqq 1$.

Let $(\eta_1(t), \eta_2(t), \eta_3(t))$ be a solution of the system (2.2) and suppose that constants a_1, a_2, a_3, b_1 and b_2 have been selected such that conditions (i)-(iv) of Theorem 1 have been satisfied. Then, by (2.7) and (2.9), this solution is asymptotically stable in the large; i.e., $\eta_1(t) \to 0$, $\eta_2(t) \to 0$, $\eta_3(t) \to 0$ as $t \to \infty$. It remains to be shown that the solutions of (1.1) are convergent.

Let $(x_1(t), x_2(t), x_3(t))$ and $(y_1(t), y_2(t), y_3(t))$ be arbitrary solutions of the system (2.1), then

$$\lim_{t \to \infty} [x_i(t) - \theta_i(t)] = 0 \text{ and } \lim_{t \to \infty} [y_i(t) - \theta_i(t)] = 0 \quad \text{for } i = 1, 2, 3,$$

and hence

$$(2.10) \qquad \lim_{t \to \infty} [x_i(t) - y_i(t)] = 0 \quad \text{for } i = 1, 2, 3.$$

Now let $x \equiv x(t)$ and $y \equiv y(t)$ be solutions of (1.1), then $\lim_{t \to \infty} [x(t) - y(t)] = 0$ and from (2.1), letting $x(t) = x_1(t)$ and $y(t) = y_1(t)$, $x_2(t) = \dot{x}_1(t) - Ax_1(t) = \dot{x}(t) - Ax(t)$ and $y_2(t) = \dot{y}(t) - Ay(t)$. From (2.10),

$$\begin{aligned} 0 &= \lim_{t \to \infty} [x_2(t) - y_2(t)] \\ &= \lim_{t \to \infty} [\dot{x}(t) - Ax(t) - \dot{y}(t) + Ay(t)] \\ &= \lim_{t \to \infty} [\dot{x}(t) - \dot{y}(t)] \end{aligned}$$

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since $\lim_{t \to \infty} [Ax(t) - Ay(t)] = 0$. Again from (2.1) and (2.10),

$$0 = \lim_{t \to \infty} [x_3(t) - y_3(t)]$$
$$= \lim_{t \to \infty} [\ddot{x}(t) - Ax(t) + G(x(t)) - \ddot{y}(t) + Ay(t) - G(y(t))]$$
$$= \lim_{t \to \infty} [\ddot{x}(t) - \ddot{y}(t)]$$

since $\lim_{t \to \infty} [G(x(t)) - G(y(t))] = 0$ which follows from $G(x)$ continuous. Thus, under the hypotheses of Theorem 1, all solutions of (1.1) are convergent.

3. The geometrical nature of inequality (iv) in Theorem 1 will now be examined so that inequality (iv) can be replaced by a linear inequality. If we set $b_1 = b_2 = 1$, then the conditions of Theorem 1 become

(i) $a_1 a_2 a_3 > a_1 + 2a_2 + a_3 + a_2^3$

(ii) $a_1 a_2 > 1$

(iii) $[2g(x) + 2a_2 h'(x) - 2a_1 A + (a_1 + A - a_2 g(x) + h'(x))^2 + (a_2 A + g(x) - a_3 h'(x) - 1)^2] \leq -\varepsilon$.

Assuming that positive constants a_1, a_2 and a_3 have been selected satisfying (i), (ii) and (iii) we examine the region of stability in the $g - h'$ plane for the inequality (iii).

Replacing g by $g + g_0$, h' by $h' + h_0$ and expanding, (iii) becomes

$$(3.2) \quad \begin{aligned} &(a_2^2 + 1)\, g^2(x) - 2(a_2 + a_3)\, g(x)\, h'(x) + (a_3^2 + 1)\, [h'(x)]^2 \\ &+ [2(a_2^2 + 1)\, g_0 - 2(a_2 + a_3)\, h_0 - 2a_1 a_2]\, g(x) \\ &+ [-2(a_2 + a_3)\, g_0 + 2(a_3^2 + 1)\, h_0 + 2(a_1 + a_2 + a_3 + A - a_2 a_3 A)] h'(x) \\ &< - a_1^2 - A^2 - a_2^2 A^2 - 1 + 2a_2 A - (a_2^2 + 1)\, g_0^2 + a(a_2 + a_3)\, g_0 h_0 \\ &- (a_3^2 + 1) h_0^2 + 2a_1 a_2 g_0 + 2(a_2 a_3 A - a_1 - a_2 - a_3 - A) h_0 - \varepsilon . \end{aligned}$$

The quadratic discriminant is $-4(a_2a_3 - 1)^2$, so the region is the interior of an ellipse or parabola. It is a parabola if $a_2a_3 = 1$, but in this case condition (i) is violated, hence the region is the interior of an ellipse.

Setting the coefficients of $g(x)$ and $h'(x)$ equal to zero in (3.2), we find that the ellipse is centered at

$$(3.3)\qquad h_0 = \frac{a_1a_2a_3 + (a_2a_3A - a_2 - a_3 - A)(a_2^2 + 1) - a_1}{(a_2a_3 - 1)^2}$$

$$g_0 = \frac{a_1a_2 + (a_2 + a_3)h_0}{a_2^2 + 1} .$$

Selecting $0 < \psi < \frac{\pi}{2}$ such that $\cot 2\psi = \frac{1}{2}(a_3 - a_2)$, (3.2) reduces to

$$(3.4)\qquad \alpha g^2(x) + \beta h'^2(x) < \gamma$$

where

$$(3.5)\qquad \begin{aligned} \alpha &= (a_2^2 + 1)\cos^2\psi - 2(a_2 + a_3)\sin\psi\cos\psi + (a_3^2 + 1)\sin^2\psi \\ \beta &= (a_2^2 + 1)\sin^2\psi + 2(a_2 + a_3)\sin\psi\cos\psi + (a_3^2 + 1)\cos^2\psi \\ \gamma &= -a_1^2 - A^2 - a_2^2A^2 - 1 + 2a_2A - (a_2^2 + 1)g_0^2 + 2(a_2 + a_3)g_0h_0 \\ &\quad - (a_3^2 + 1)h_0^2 + 2a_1a_2g_0 + 2(a_2a_3A - a_1 - a_2 - a_3 - A)h_0 - \varepsilon . \end{aligned}$$

From (3.4) we see that the region of stability is an ellipse centered at (g_0, h_0) with major axis of length $\sqrt{\gamma/\alpha}$, and minor axis of length $\sqrt{\gamma/\beta}$. Noting that $0 < \alpha < \beta$, we use the preceding description to state the following.

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THEOREM 2. Assume that A is a positive constant, $g \varepsilon C^0(R^1)$, $h \varepsilon C^1(R^1)$ and $e(t)$ is such as to guarantee the existence of solutions of (1); then all solutions of (1) converge provided there exist positive constants a_1, a_2, a_3, ε, g_0, h_0, α, β and γ satisfying (3.1), (3.3) and (3.5) such that

(i) $g_0 - \sqrt{\frac{\gamma}{2\beta}} < g(x) < g_0 + \sqrt{\frac{\gamma}{2\beta}}$

(ii) $h_0 - \sqrt{\frac{\gamma}{2\beta}} < h'(x) < h_0 + \sqrt{\frac{\gamma}{2\beta}}$

for all $x \varepsilon R^1$.

4. The relatively simple nature of the conditions needed in Theorem 1 reflect the rather special nature of the matrix C used in that proof. If one attempts to use Lemma 2.1 to determine corresponding conditions for the matrix which arises from quation (1.2), it is found that the inequalities so determined are very unmanageable. In Theorem 3 Gersgorin circles are used to determine bounds for the eigenvalues of the matrix in question. Although the inequalities obtained are more easily manipulated they do not yield as sharp results as does the use of Lemma 2.1.

THEOREM 3. If $f, g \varepsilon C^0(R^1)$, $h \varepsilon C^1(R^1)$ and $e(t)$ is such as to guarantee the existence of solutions of (1.2), then all solutions of (1.2) converge provided there exist constants a_1, a_2, a_3, b_1, b_2 and b_3 such that the following inequalities are satisfied for some $\varepsilon > 0$ and for all x and y:

(i) $a_1 > 0$, $a_1 a_2 > b_1^2$, $b_1 > 0$;

(ii) $a_1 a_2 a_3 > a_1 b_3^2 + a_2 b_2^2 + a_3 b_1^2 + 2b_1 b_2 b_3$;

(iii) $\frac{b_3}{a_1} - \frac{b_1}{a_1} \leqq f(y)$;

(iv) $b_3 \leqq -b_2 h'(x) - b_1 g(x)$;

(v) $\frac{\varepsilon}{2b_3} - \frac{b_2}{b_3} - \frac{a_2}{b_3} - 1 \leqq f(y) \leqq \frac{b_2}{b_3} - \frac{a_2}{b_3} - \frac{\varepsilon}{2b_3} + 1$;

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(vi) $b_1 - b_3 + \frac{\varepsilon}{2} \leqq b_3 g(x) - a_3 h'(x) \leqq b_1 + b_3 - \frac{\varepsilon}{2}$;

(vii) $\frac{\varepsilon}{2} - a_1 - b_3 \leqq b_1 f(y) + b_3 h'(x) - a_2 g(x) \leqq b_3 - a_1 - \frac{\varepsilon}{2}$.

Proof of Theorem 3. Equation (1.2) is equivalent to the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 - F(x_2) - G(x_1) \\ \dot{x}_3 &= e(t) - h(x_1) \end{aligned} \tag{4.1}$$

where $F(x) = \int_0^x f(s)ds$ and $G(x) = \int_0^x g(s)ds$.

If $\theta(t)$ is an arbitrary but fixed solution of (4.1), then any other solution of (4.1) can be written as $x(t) = \theta(t) + \eta(t)$; and for each fixed $x(t)$, $\eta(t)$ is a solution of the system

$$\dot{\eta} = \begin{pmatrix} 0 & 1 & 0 \\ -R(t) & -P(t) & 1 \\ -S(t) & 0 & 0 \end{pmatrix} \eta \tag{4.2}$$

where

$$P(t) = \begin{cases} \dfrac{F(x_2(t)) - F(\theta_2(t))}{x_2(t) - \theta_2(t)} & \text{if } x_2(t) \neq \theta_2(t) \\ f(\theta_2(t)) & \text{if } x_2(t) = \theta_1(t) \end{cases}$$

(4.3)

$$R(t) = \begin{cases} \dfrac{G(x_1(t)) - G(\theta_1(t))}{x_1(t) - \theta_1(t)} & \text{if } x_1(t) \neq \theta_1(t) \\ g(\theta_1(t)) & \text{if } x_1(t) = \theta_1(t) \end{cases}$$

$$S(t) = \begin{cases} \dfrac{h(x_1(t)) - h(\theta_1(t))}{x_1(t) - \theta_1(t)} & \text{if } x_1(t) \neq \theta_1(t) \\ h'(\theta_1(t)) & \text{if } x_1(t) = \theta_1(t). \end{cases}$$

The proof of Theorem 3 will depend upon the same function $V(\eta)$ as defined in the proof of Theorem 1.

From (4.3) it is seen that $P(t)$, $R(t)$ and $S(t)$ can be written as:

$$P(t) = \int_0^1 f(s\eta_2(t) + \theta_2(t))ds$$

(4.4)
$$R(t) = \int_0^1 g(s\eta_1(t) + \theta_1(t))ds$$

$$S(t) = \int_0^1 h'(s\,\eta_1(t) + \theta_1(t))ds\ .$$

Now setting

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -g(s\eta_1(t) + \theta_1(t)) & -f(s\eta_2(t) + \theta_2(t)) & 1 \\ -h'(s\eta_1(t) + \theta_1(t)) & 0 & 0 \end{pmatrix}$$

and $C = B^TQ + QB$ we have

$$C = \begin{pmatrix} 2b_1g + 2b_2h' & a_1 + b_1f + b_3h' - a_2g & b_3g - a_3h' - b_1 \\ a_1 + b_1f + b_3h' - a_2g & -2b_1 - 2a_2f & a_2 + b_3f - b_2 \\ b_3g - a_3h' - b_1 & a_2 + b_3f - b_2 & -2b_3 \end{pmatrix}.$$

Now using (4.1), (4.4) and calculating as in the proof of Theorem 1,
$\dot{V}(\eta, t) = \int_0^1 \eta \cdot C\eta ds.$

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Hence, if we assume that $V(\eta)$ is positive definite, then the zero solution of the system (4.3) will be asymptotically stable in the large if the eigenvalues of C are uniformly bounded above by some negative number; i.e., there exists $\varepsilon > 0$ such that if $\lambda_i(t, s)$, $i = 1, 2, 3$, are the eigenvalues of C then $\lambda_i(t, s) < -\varepsilon$, $i = 1, 2, 3$, $0 \leq s \leq 1$ and $t \geq 0$.

The following theorem will be used to establish a sufficient condition that all the eigenvalues of C will be less than some fixed negative number.

LEMMA 4.1 [5, p. 196]. The characteristic roots of the $n \times n$ hermitian matrix $A = (a_{ij})$ lie in the closed region of the complex plane consisting of all the disks

$$|Z - a_{ii}| \leq P_i , \qquad i = 1, \cdots, n,$$

where

$$P_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| .$$

This result is due to S.A. Geršgorin.

From Lemma 4.1 we see that the eigenvalues of C will have the desired property if

(1) $-2b_1 g - 2b_2 h' - \varepsilon \geq |a_1 + b_1 f + b_3 h' - a_2 g| + |b_3 g - a_3 h' - b_1|$

(2) $2b_1 + 2a_2 f - \varepsilon \geq |a_1 + b_1 f + b_3 h' - a_2 g| + |a_2 + b_3 f - b_2|$

(3) $2b_3 - \varepsilon \geq |b_3 g - a_3 h' - b_1| + |a_2 + b_3 g - b_2|$.

It is assumed that each inequality is valid for all values of the variables of the functions f, g and h' although the variables are not displayed.

If we assume that constants a_1, a_2, a_3, b_1, b_2 and b_3 exist such that conditions (i)-(vii) of Theorem 3 are satisfied, then inequalities (1)-(3) are satisfied which implies that the eigenvalues of C are bounded above by a negative constant $-\varepsilon$. Thus,

$\dot{V}_{(5.13)}(\eta, t) \leqq -\varepsilon \|\eta\|^2$, and all solutions of (4.1) converge to zero. From the definition of $\theta(t)$ and from the system (4.1) it is clear that, if $x \equiv x(t)$ and $y \equiv y(t)$ are solutions of (1.2), then, $\lim_{t\to\infty} [x^{(i)}(t) - y^{(i)}(t)] = 0$, $i = 1, 2, 3$.

5. When $F(x,t)$ is periodic of period T, La Salle [3] has shown that extreme stability and the existence of a bounded solution imply the existence of a periodic solution of period T.

Applying the results of Theorem 1 to this result, we get the following.

THEOREM 4. *If $e(t + T) = e(t)$ for all $t \geq 0$, $T > 0$, and (1.1) has a bounded solution and the conditions of Theorem 1 are satisfied, then (1.1) has a periodic solution of period T, and every other solution of (1.1) converges to that periodic solution.*

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https://doi.org/10.4153/CMB-1969-101-0 Published online by Cambridge University Press