ON THE EQUIVALENCE PROBLEM AND INTEGRATION
OF DIFFERENTIAL SYSTEMS

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Introduction

The purpose of the presnet paper is to study the relationship between the theory of Lie pseudogroups and the problem of integration of differential systems (cf. [6] pp. 30–47).

Let $\mathfrak{G}$ be a Lie pseudogroup on a manifold M and S a differential system on M. Let $\mathfrak{G}(S)$ denote the largest subpseudogroup of $\mathfrak{G}$ leaving S invariant. Then the problems to be considered may be stated as follows.

A) Classify differential systems on M under the action of $\mathfrak{G}$.

B) For each differential system S on M, determine the structure of $\mathfrak{G}(S)$.

C) Using the structure of $\mathfrak{G}(S)$, reduce the problem of integration of S to that of some auxiliary differential systems, each of which is invariant under the action of a Lie pseudogroup and irreducible in a sense.

To study these problems, we use the theory of Lie pseudogroups which is developed in [7]. The problems A) and B) are subordinate to the so-called general equivalence problem (see [2] §§ 11–13). The problem C) is motivated by the classical scheme of S. Lie for the problem of integration (see [8] and [9] Introduction).

In Section 1, we recall briefly the theory of Lie pseudogroups. A Cartan system is a pair (P, C) consisting of a manifold P and an "invariant system" C on P. We can define an effective action of (P, C) on a manifold M. Then the action yields a Lie pseudogroup $\mathfrak{G}$ on M. (P, C) is called a defining Cartan system of $\mathfrak{G}$.

In Section 2, we shall study the equivalence problem of Pfaffian (differential) systems. Let (P, C) and $\mathfrak{G}$ be as above. For each Pfaffian system S on M, we construct a Cartan system $(P, C(S))$ in such a way that $(P, C(S))$ is a defining Cartan system of $\mathfrak{G}(S)$ (Theorem 2.3). Then, using $(P, C(S))$, we can study the structure of $\mathfrak{G}(S)$. Moreover, we prove the

following theorem.

THEOREM 2.2. *Let S_1 and S_2 be Pfaffian systems on M. Then S_1 and S_2 are locallly equivalent under the action of $\mathfrak{G}$ if and only if $(P, C(S_1))$ and $(P, C(S_2))$ are locally isomorphic.*

This theorem together with the isomorphic reduction theorem of a Cartan system which is established in [7] gives us a principle to study the problem A). In particular, in Section 4.4, we carry out the classification of second order ordinary differential equations $y'' = f(x, y, y')$ under the action of certain Lie pseudogroup.

In Section 3, we shall define a normal Cartan subsystem $(P, C^\sharp)$ of (P, C) and its quotient Cartan system $(P, C)/(P, C^\sharp)$. Then we prove an existence theorem (Theorem 3.3).

In Section 4, we shall study the problem of integration of a Pfaffian system S on P in the case when S is generated by invariant 1–forms of a Cartan system (P, C). Such a system will be denoted by $(S, (P, C))$. Then, by using C, we define a special subsystem of S, which is called a normal covariant system of (S, C) (Definition 4.3). S is said to be irreducible if there is no normal covariant system of (S, C). If S admits a normal covariant system, we can use the results in Section 3. After describing our method of reduction, we obtain the following theorems.

THEOREM 4.4. *The problem of integration of $(S, (P, C))$ is reduced to that of the following two types of systems:*
    1) *$(S_\sharp, (P_\sharp, C_\sharp))$, where $S_\sharp$ is completely integrable;*
    2) *$(S_*, (P_*, C_*))$, where S_* is irreducible.*

THEOREM 4.5. *Assume that $k + 1 = \operatorname{rank} S \geqq 2$ and that there are normal covariant systems S_i of (S, C), $1 \leqq i \leqq k$, satisfying the following conditions:*
    1) $\operatorname{rank} S_i = i, 1 \leqq i \leqq k$;
    2) $S_1 \subset S_2 \subset \cdots \subset S_k \subset S.$
Then the integration of S is generically reduced to that of some completely integrable Pfaffian systems.

For the integration of an irreducible system $(S_*, (P_*, C_*))$, we can also use the Cartan system (P_*, C_*). In many cases, the structure equation of (P_*, C_*) enables us to find another kind of useful subsystems of S. We shall deal with such subsystems in a forthcoming paper.

Theorem 4.5 may be applicable to some special types of partial differential equations (see [4] §§ 34–37).

Throughout this paper, all manifolds and mappings (functions) are assumed to be differentiable of class C^∞ and the adjective "differentiable" is omitted unless otherwise stated. The theory is, however, also well applied to the real analytic case.

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§ 1. Cartan systems and Lie pseudogroups

1.1. Let M be a manifold. We denote by $\mathfrak{F}(M)$ the ring of all functions on M and by $\bigwedge^1(M)$ the $\mathfrak{F}(M)$-module of all 1-forms on M. An $\mathfrak{F}(M)$-submodule S of $\bigwedge^1(M)$ is called a *Pfaffian system* of rank s on M if there are s 1-forms $\theta^1, \cdots, \theta^s$ on M such that

1) S is generated by $\theta^1, \cdots, \theta^s$;
2) $\theta^1, \cdots, \theta^s$ are linearly independent at each point of M.

We denote it by $S = \langle \theta^1, \cdots, \theta^s \rangle$. $(\theta^1, \cdots, \theta^s)$ is called a *generator* of S. A function f on M is called a *first integral* of S if $df \in S$. A submanifold N of M is called an *integral manifold* of S if $\iota^*\theta = 0$ for any $\theta \in S$, where $\iota; N \to M$ is the inclusion. S is said to be *completely integrable* if $d\theta \equiv 0 \pmod{S}$ for any $\theta \in S$.

Let $\mathfrak{G}(M)$ denote the set of all local transformations of M. For each $f \in \mathfrak{G}(M)$, we denote by $U(f)$ the domain of f and by $V(f)$ the range of f. Let $\mathfrak{G}$ be a pseudogroup on M (for the precise definition, see [7] or [11]). For an open set U of M, we denote by $\mathfrak{G}|U$ the pseudogroup on U consisting of all elements f of $\mathfrak{G}$ such that $U(f)$ and $V(f)$ are contained in U.

Let P be a manifold and let $C = (u^1, \cdots, u^n; \omega^1, \cdots, \omega^r)$ be a system of functions u^i and 1-forms ω^j on P ($n \geqq 0$, $r \geqq 1$). For each $p \in P$, we denote by $D(C)_p$ the linear subspace of the tangent space $T_p(P)$ consisting of all vectors X such that $\omega^j(X) = 0$, $X \lrcorner d\omega^j = 0$, $1 \leqq j \leqq r$, and $Xu^i = 0$, $1 \leqq i \leqq n$, where $X \lrcorner d\omega^j$ is the inner derivative of $d\omega^j$ with respect to X. C is called an *invariant system* on P if it satisfies the following conditions:

1) $\omega^1, \cdots, \omega^r$ are linearly independent at each point of P;
2) the Pfaffian system $\langle \omega^1, \cdots, \omega^r \rangle$ is completely integrable;
3) $D(C)_p = 0$ for any $p \in P$.

Then the pair (P, C) is called a *Cartan system*. We denote by $\mathfrak{G}(P, C)$

the pseudogroup on P consisting of all elements f of $\mathfrak{G}(P)$ such that $f^*\theta = \theta$ for every θ in C, where $f^*\theta$ is the pull-back of θ by f. A function u (resp. a 1-form ω) on an open set U of P is called an *invariant* (resp. an *invariant* 1-*form*) of (P, C) if it satisfies $f^*u = u$ (resp. $f^*\omega = \omega$) for any $f \in \mathfrak{G}(P, C)|U$. A system of invariants of (P, C), $v^1, \cdots, v^m$, is called a *complete system of invariants* of (P, C) if $v^1, \cdots, v^m$ are independent functions on P and if, for any invariant u of (P, C), there is a function F of m variables such that $u = F(v^1, \cdots, v^m)$. Let U be an open set of P. Restricting every member of C to U, we get an invariant system $C|U$ on U.

Let $C_i = (u_i^1, \cdots, u_i^n; \omega_i^1, \cdots, \omega_i^r)$ be an invariant system on a manifold P_i $(i = 1, 2)$. A diffeomorphism F of P_1 onto P_2 is called an *isomorphism* of (P_1, C_1) onto (P_2, C_2) if it satisfies $F^*u_1^j = u_2^j$, $1 \leq j \leq n$, and $F^*\omega_2^k = \omega_1^k$, $1 \leq k \leq r$. Let $p_i \in P_i$ $(i = 1, 2)$. We say that (P_1, C_1) and (P_2, C_2) are *locally isomorphic* at (p_1, p_2) if there are open neighborhoods U_i of p_i, $i = 1, 2$, and an isomorphism of $(U_1, C_1|U_1)$ onto $(U_2, C_2|U_2)$ with $F(p_1) = p_2$. For the equivalence problem of Cartan systems, we refer to [10].

Let P and M be manifolds and let $\pi: P \to M$ be a fibering (a surjective submersion). Let $E(\pi)$ denote the vertical distribution given by $E(\pi)_p = \mathrm{Ker}\,(\pi_*)_p$, $p \in P$, where $(\pi_*)_p$ is the differential of π at p. An element F of $\mathfrak{G}(P)$ is called a *prolongation* (resp. a *local prolongation*) of an element f of $\mathfrak{G}(M)$ if it satisfies (1) $\pi(U(F)) = U(f)$ (resp. $\pi(U(F)) \subset U(f)$) and (2) $\pi \circ F = f \circ \pi$ on $U(F)$.

Let C be an invariant system on P and let $\mathfrak{G}_0(M, C)$ denote the set of all elements f of $\mathfrak{G}(M)$ such that f has a prolongation F in $\mathfrak{G}(P, C)$. We say that (P, C) *acts* on M with respect to π if the following conditions are satisfied:

1) $E(\pi)$ is $\mathfrak{G}(P, C)$-invariant, that is, $F_*(E(\pi)_p) = E(\pi)_{F(p)}$ for all $F \in \mathfrak{G}(P, C)$ and $p \in U(F)$;

2) for any $f \in \mathfrak{G}_0(M, C)$ and any $p \in \pi^{-1}(U(f))$, there is an element F of $\mathfrak{G}(P, C)$ such that $p \in U(F)$ and F is a local prolongation of f.
Moreover, we say that (P, C) *acts effectively* on M if π satisfies the third condition:

3) if $\pi \circ F = \pi$ on $U(F)$, $F \in \mathfrak{G}(P, C)$, then $F = $ id. on $U(F)$.

We remark that if the system (P, π, M) is a principal fibre bundle then, under a condition on C, we can verify the conditions 1), 2) and 3) without knowing $\mathfrak{G}(P, C)$ explicitly.

When (P, C) acts on M, we denote by $\mathfrak{G}(M, C)$ the pseudogroup on M generated by $\mathfrak{G}_0(M, C)$ (see [11] p. 8).

PROPOSITION 1.1. *Assume that (P, C) acts on M with respect to a fibering $\pi\colon P \to M$. Then, for any $f \in \mathfrak{G}(M, C)$ and any $p \in \pi^{-1}(U(f))$, there is an element F of $\mathfrak{G}(P, C)$ such that $p \in U(F)$ and F is a local prolongation of f.*

For the proof, see [7] Proposition 1.1.

Let $C = (u^1, \cdots, u^n; \omega^1, \cdots, \omega^r)$. Then the action of (P, C) on M is said to be *normal* if, for every open set U of M and every 1-form θ on U, there are functions a_j on $\pi^{-1}(U)$, $1 \leqq j \leqq r$, such that $\pi^*\theta = \sum_{j=1}^{r} a_j \omega^j$ on $\pi^{-1}(U)$.

A pseudogroup $\mathfrak{G}$ on M is called a *Lie pseudogroup* if there are a Cartan system (P, C) and a fibering $\pi\colon P \to M$ such that (P, C) acts effectively on M with respect to π and $\mathfrak{G} = \mathfrak{G}(M, C)$. (P, C) is called a *defining Cartan system* of $\mathfrak{G}$. We can construct a defining Cartan system of $\mathfrak{G}$ in such a way that the action is normal. In many cases, the system $((P, C), \pi, M)$ can be considered as a geometric structure on M, such as a (higher order) G-structure or a Cartan connection.

1.2. Let $C = (u^0, \cdots, u^n; \omega^0, \cdots, \omega^r)$ be an invariant system on a manifold P and let W be an open set of P. Suppose $(du^0)_p \neq 0$ for any $p \in W$. For each $q \in W$, we define a submanifold of W by

$$Q(W, q) = \{p \in W; u^0(p) = u^0(q)\}.$$

Let $\iota\colon Q(W, q) \to P$ be the inclusion. Then we can define a system of functions and 1-forms on $Q(W, q)$ by

$$C(W, q) = (\iota^*u^1, \cdots, \iota^*u^n; \iota^*\omega^1, \cdots, \iota^*\omega^r).$$

It should be remarked that $C(W, q)$ is not necessarily an invariant system. In [7], we proved the following reduction theorems (for the terminologies "reducible" and "an R-system", see [7] § 2.1).

THEOREM 1.2. *Let (P, C) and W be as above. If u^0 is reducible on W, then, for any $q \in W$, $(Q(W, q), C(W, q))$ is a Cartan system.*

THEOREM 1.3. *Let $C_i = (u_i^0, \cdots, u_i^n; \omega_i^0, \cdots, \omega_i^r)$ be an invariant system on a manifold P_i $(i = 1, 2)$. Assume that $\dim P_1 = \dim P_2$ and that u_i^0 is reducible on an open set W_i of P_i $(i = 1, 2)$. Let $q_i \in W_i$ $(i = 1, 2)$. Then*

(P_1, C_1) and (P_2, C_2) are locally isomorphic at (q_1, q_2) if and only if the following conditions are satisfied:

    1) $u_1^j(q_1) = u_2^j(q_2)$ for all $i = 0, \cdots, n$;

    2)   there are R-systems $(W_i', U, \Omega_i^\alpha, a_j, b^k, c_m^i)$ with $q_i \in W_i'$, $i = 1, 2$;

    3)   the Cartan systems $(Q(W_1, q_1), C(W_1, q_1))$ and $(Q(W_2, q_2), C(W_2, q_2))$ are locally isomorphic at (q_1, q_2).

§ 2.   The equivalence problem of Pfaffian systems

2.1.   Let M be a manifold and let S be a Pfaffian system on an open set U of M (for the definition, see § 1.1).   Let f be an element of $\mathfrak{G}(M)$ satisfying $V(f) = U$.   We denote by f^*S the Pfaffian system on $U(f)$ consisting of all 1-forms $f^*\theta, \theta \in S$.   Let V be an open subset of U.   We denote by $S|V$ the Pfaffian system on V generated by all 1-forms $\theta|V, \theta \in S$, where $\theta|V$ is the restriction of θ to V.   Let $\mathfrak{G}$ be a pseudogroup on M. We denote by $\mathfrak{G}(S)$ the subset of $\mathfrak{G}|U$ consisting of all elements $f \in \mathfrak{G}|U$ such that $f^*(S|V(f)) = S|U(f)$. $\mathfrak{G}(S)$ is a pseudogroup on U.   We say that $\mathfrak{G}$ *leaves S invariant* if $\mathfrak{G}(S) = \mathfrak{G}|U$.   Let S_i be a Pfaffian system on an open set U_i of M and let $x_i \in U_i$ $(i = 1, 2)$.

DEFINITION 2.1.   We say that S_1 and S_2 are *locally $\mathfrak{G}$-equivalent* at (x_1, x_2) if there is an element f of $\mathfrak{G}$ such that

    1) $x_1 \in U(f) \subset U_1$, $x_2 \in V(f) \subset U_2$;

    2) $f(x_1) = x_2$;

    3) $f^*(S_2|V(f)) = S_1|U(f)$.

Assume now that $\mathfrak{G}$ is a Lie pseudogroup on M (cf. § 1.1).   Let (P, C), $C = (u^1, \cdots, u^n; \omega^1, \cdots, \omega^r)$, be a defining Cartan system of $\mathfrak{G}$ with respect to a fibering $\pi\colon P \to M$.   Let $S = \langle \theta^1, \cdots, \theta^s \rangle$ be a Pfaffian system of rank s on an open set U of M.   Then we can define a Pfaffian system S^* on $\pi^{-1}(U)$ by $S^* = \langle \pi^*\theta^1, \cdots, \pi^*\theta^s \rangle$.   From now on, we assume that $s < r$.

DEFINITION 2.2.   Let W be an open set of $\pi^{-1}(U)$ and let u_k^j, $1 \leq j \leq s$, $s + 1 \leq k \leq r$, be functions on W.   Then $(W; u_k^j)$ is called a *system of invariants* of (S, C) with respect to the ordering $\omega^1, \cdots, \omega^r$ if, putting

$$\Omega^j = \omega^j + \sum_{k=s+1}^r u_k^j \omega^k, \qquad 1 \leq j \leq s,$$

$(\Omega^1, \cdots, \Omega^s)$ forms a generator of $S^*|W$.

Adding the functions u_k^j, $1 \leq j \leq s$, $s + 1 \leq k \leq r$, to $C|W$, we get an

invariant system on W, which we shall denote by $C(W; u_k^j)$. It is immediate to prove the following lemma.

LEMMA 2.1. *Let $(W_i; u_{k;i}^j)$ be a system of invariants of (S, C) with respect to the ordering $\omega^1, \cdots, \omega^r$ $(i = 1, 2)$. If $W_1 \cap W_2 \neq \phi$, then $u_{k;1}^j = u_{k;2}^j$ on $W_1 \cap W_2$, $1 \leqq j \leqq s$, $s + 1 \leqq k \leqq r$.*

DEFINITION 2.1. A system of invariants $(P^*; u_k^j)$ of (S, C) with respect to the ordering $\omega^1, \cdots, \omega^r$ is said to be *maximal* if, for any system of invariants $(W; v_k^j)$ of (S, C) with respect to the same ordering $\omega^1, \cdots, \omega^r$, we have $W \subset P^*$ (and hence $v_k^j = u_k^j$ on W by Lemma 2.1).

PROPOSITION 2.1. *Let $\mathfrak{G}$, (P, C) and S be as above. Assume that the action of (P, C) on M is normal. Then, for any $q \in \pi^{-1}(U)$, there is a maximal system of invariants $(P^*; u_k^j)$ of (S, C) with $q \in P^*$.*

Proof. By assumption, there are functions a_k^j on $\pi^{-1}(U)$, $1 \leqq j \leqq s$, $1 \leqq k \leqq r$, such that $\pi^* \theta^j = \sum_{k=1}^r a_k^j \omega^k$, $1 \leqq j \leqq s$. Since $\operatorname{rank}(a_k^j) = s$ on $\pi^{-1}(U)$, we can assume that the matrix $(a_k^j(q))_{1 \leqq j, k \leqq s}$ is non-singular. Let P^* denote the open subset of $\pi^{-1}(U)$ consisting of all points $p \in \pi^{-1}(U)$ such that the matrix $(a_k^j(p))_{1 \leqq j, k \leqq s}$ is non-singular. Then we have $q \in P^*$. Let $(b_k^j(p))_{1 \leqq j, k \leqq s}$ denote the inverse matrix of the above matrix. Set $u_k^j = \sum_{n=1}^s b_n^j a_k^n$ on P^*, $1 \leqq j \leqq s$, $s + 1 \leqq k \leqq r$. Then it is easy to verify that $(P^*; u_k^j)$ is a maximal system of invariants of (S, C) with respect to the ordering $\omega^1, \cdots, \omega^r$.

Let $\mathfrak{G}$ be a Lie pseudogroup on a manifold M and let (P, C), $C = (u^1, \cdots, u^n; \omega^1, \cdots, \omega^r)$, be a defining Cartan system of $\mathfrak{G}$ with respect to a fibering $\pi: P \to M$.

THEOREM 2.2. *Let S_i be a Pfaffian system of rank s on an open set U_i of M $(i = 1, 2)$. Let $(P_i^*; u_{k;i}^j, 1 \leqq j \leqq s, s + 1 \leqq k \leqq r)$ be a maximal system of invariants of (S_i, C) with respect to the ordering $\omega^1, \cdots, \omega^r$ and let $x_i \in \pi(P_i^*)$ $(i = 1, 2)$. Set $C_i^* = C(P_i^*; u_{k;i}^j)$ $(i = 1, 2)$. Then S_1 and S_2 are locally $\mathfrak{G}$-equivalent at (x_1, x_2) if and only if there are points q_i of P_i^* with $x_i = \pi(q_i)$, $i = 1, 2$, such that (P_1^*, C_1^*) and (P_2^*, C_2^*) are locally isomorphic at (q_1, q_2).*

THEOREM 2.3. *Let S be a Pfaffian system of rank s on an open set U of M. Let $(P^*; u_k^j, 1 \leqq j \leqq s, s + 1 \leqq k \leqq r)$ be a maximal system of invariants of (S, C) with respect to the ordering $\omega^1, \cdots, \omega^r$. Suppose $\pi(P^*) =$*

U. Set $C^* = C(P^*; u_k^j)$.   Then:

1) $\mathfrak{G}(S)$ is a Lie pseudogroup on U and (P^*, C^*) is a defining Cartan system of $\mathfrak{G}(S)$ with respect to the fibering $\pi|P^*: P^* \to U$.

2) The Pfaffian system $S^*|P^*$ is generated by s invariant 1-forms of (P^*, C^*), $\omega^j + \sum_{k=s+1}^{r} u_k^j \omega^k$, $1 \leqq j \leqq s$.

3) If $\mathfrak{G}$ leaves S invariant, then the functions u_k^j, $1 \leqq j \leqq s$, $s + 1 \leqq k \leqq r$, are invariants of (P, C).

The converse of 3) of Theorem 2.3 holds if $P^* = \pi^{-1}(U)$.

2.2.   We shall prove the theorems.

LEMMA 2.2. *Let the situation and the notations be as in Theorem 2.2. Let f be an element of $\mathfrak{G}$ satisfying $U(f) \subset U_1$ and $V(f) \subset U_2$. Assume that f has a prolongation F in $\mathfrak{G}(P, C)$.*

1) *If* $U(F) \subset P_1^*$ *and* $f^*(S_2| V(f)) = S_1| U(f)$, *then* $V(F) \subset P_2^*$ *and* $F^* u_{k;2}^j = u_{k;1}^j$ *on* $U(F)$, $1 \leqq j \leqq s$, $s + 1 \leqq k \leqq r$.

2) *If* $U(F) \subset P_1^*$, $V(F) \subset P_2^*$ *and* $F^* u_{k;2}^j = u_{k;1}^j$ *on* $U(F)$, $1 \leqq j \leqq s$, $s + 1 \leqq k \leqq r$, *then, for any* $x \in U(f)$, *there is an open neighborhood V of x in $U(f)$ such that $(f|V)^*(S_2|W) = S_1|V$, where we put $W = f(V)$.*

Proof. Let us fix a generator $(\theta_i^1, \cdots, \theta_i^s)$ of S_i $(i = 1, 2)$. By definition, putting

$$\Omega_i^j = \omega^j + \sum_{k=s+1}^{r} u_{k;i}^j \omega^k, \qquad 1 \leqq j \leqq s,$$

$(\Omega_i^1, \cdots, \Omega_i^s)$ forms a generator of $S_i^*|P_i^*$ $(i = 1, 2)$.

1) If we set $\Omega^j = (F^{-1})^* \Omega_1^j$ on $V(F)$, $1 \leqq j \leqq s$, then

$$\Omega^j = \omega^j + \sum_{k=s+1}^{r} ((F^{-1})^* u_{k;1}^j), \qquad 1 \leqq j \leqq s.$$

Using the condition $f^*(S_2| V(f)) = S_1| U(f))$, we can prove easily that $(\Omega^1, \cdots, \Omega^s)$ forms a generator of $S_2^*| V(F)$. This means that $(V(F); (F^{-1})^* u_{k;1}^j)$ is a system of invariants of (S_2, C) with respect to the ordering $\omega^1, \cdots, \omega^r$. Since $(P_2^*; u_{k;2}^j)$ is maximal, we have $V(F) \subset P_2^*$ and hence $(F^{-1})^* u_{k;1}^j = u_{k;2}^j$ for all j and k.

2) By assumption, we have $F^* \Omega_2^j = \Omega_1^j$ on $U(F)$, $1 \leqq j \leqq s$. Since $(\Omega_i^1, \cdots, \Omega_i^s)$ is a generator of $S_i^*|P_i^*$ $(i = 1, 2)$, there are functions A_k^j on $U(F)$, $1 \leqq j, k \leqq s$, such that the matrix $(A_k^j(p))$ is non-singular for any $p \in U(F)$ and

$$F^*(\pi^* \theta_2^j) = \sum_{k=1}^{s} A_k^j(\pi^* \theta_1^k), \qquad 1 \leqq j \leqq s.$$

Hence,

$$\pi^*(f^*\theta_2^j) = \sum_{k=1}^s A_k^j(\pi^*\theta_1^k), \qquad 1 \leq j \leq s.$$

Take an open neighborhood V of x in $U(f)$ and a cross section $\rho\colon V \to U(F)$ of the fibering $\pi|U(F)\colon U(F) \to U(f)$. Then we have

$$f^*\theta_2^j = \sum_{k=1}^s (\rho^*A_k^j)\theta_1^k, \qquad 1 \leq j \leq s,$$

on V and hence $(f|V)^*(S_2|W) = S_1|V \ (W = f(V))$.

Proof of Theorem 2.2. Assume first that S_1 and S_2 are locally $\mathfrak{G}$-equivalent at (x_1, x_2). Then there is an element f of $\mathfrak{G}$ with the properties listed in Definition 2.1. Take a point q_1 of P_1^* with $x_1 = \pi(q_1)$. By Proposition 1.1, there is an element F of $\mathfrak{G}(P, C)$ such that $q_1 \in U(F)$ and F is a local prolongation of f. Shrinking $U(f)$ and $U(F)$ if necessary, we can assume that $U(F) \subset P_1^*$ and that F is a prolongation of f. If we set $q_2 = F(q_1)$, then $x_2 = \pi(q_2)$. By Lemma 2.2–1), we can conclude that F is an isomorphism of $(U(F), C_1^*|U(F))$ onto $(V(F), C_2^*|V(F))$.

Conversely, assume that there are open neighborhoods W_i of q_i in P_i^*, $i = 1, 2$, and an isomorphism F of $(W_1, C_1^*|W_1)$ onto $(W_2, C_2^*|W_2)$ with $F(q_1) = q_2$. Then F belongs to $\mathfrak{G}(P, C)$. We can assume that F is a prolongation of an element f of $\mathfrak{G}$. Using F, f and Lemma 2.2–2), we can verify that S_1 and S_2 are locally $\mathfrak{G}$-equivalent at (x_1, x_2).

Proof of Theorem 2.3. Let π_0 denote the restriction of π to P^*. Since $\mathfrak{G}(P^*, C^*) \subset \mathfrak{G}(P, C)|P^*$, the vertical distribution $E(\pi_0)$ is $\mathfrak{G}(P^*, C^*)$-invariant. Take any $f \in \mathfrak{G}_0(U, C^*)$ and any $x \in U(f)$. Then f has a prolongation F in $\mathfrak{G}(P^*, C^*)$. It follows from Lemma 2.2–2) that there is an open neighborhood V of x in $U(f)$ such that $f|V \in \mathfrak{G}(S)$. This implies that $\mathfrak{G}_0(U, C^*) \subset \mathfrak{G}(S)$. Next take any $f \in \mathfrak{G}(S)$ and any $p \in \pi_0^{-1}(U(f))$. Then there is an element F of $\mathfrak{G}(P, C)$ such that $p \in U(F)$ and F is a local prolongation of f. As before, we can assume that $U(F) \subset \pi_0^{-1}(U(f))$. Then, by Lemma 2.2–1), F belongs to $\mathfrak{G}(P^*, C^*)$. Since $\mathfrak{G}_0(U, C^*) \subset \mathfrak{G}(S)$, this implies that (P^*, C^*) acts on U with respect to π_0. The above argument also shows that $\mathfrak{G}(U, C^*) = \mathfrak{G}(S)$. The effectivity of this action is obvious. This proves 1). 2) may be obvious.

To show 3), it will be sufficient to prove that $\mathfrak{G}(P, C)|P^*$ is contained in $\mathfrak{G}(P^*, C^*)$. Take any $F \in \mathfrak{G}(P, C)|P^*$ and any $p \in U(F)$. Then there are an open neighborhood W of p in $U(F)$ and an element f of $\mathfrak{G}|U$ such

that $F|W$ is a prolongation of f. Since $\mathfrak{G}(S) = \mathfrak{G}|U$, it follows from Lemma 2.2–1) that $F|W$ belongs to $\mathfrak{G}(P^*, C^*)$. Since $\mathfrak{G}(P^*, C^*)$ is a pseudogroup, we finally have $F \in \mathfrak{G}(P^*, C^*)$. This completes the proof of Theorem 2.3.

§ 3.   Quotient Cartan systems

3.1. Let M be a manifold and let E be a subset of the tangent bundle TM of M. We say that E is a *v-family* on M if, for any $x \in M$, $E_x = E \cap T_x(M)$ is a linear subspace of $T_x(M)$. Therefore, a distribution on M is a v-family E with the following properties: (1) $\dim E_x = $ constant on M and (2) E is a differentiable subbundle of TM.

Let E be a v-family on M and let Q be a set of 1-forms on M. We define v-families on M, $A(Q)$, $D(Q)$, $H(Q)$, and a set of 1-forms on M, $P(E)$, as follows.

$$A(Q)_x = \{X \in T_x(M);\ \theta(X) = 0,\ \theta \in Q\},$$
$$D(Q)_x = \{X \in A(Q)_x;\ X \lrcorner d\theta = 0,\ \theta \in Q\},$$
$$H(Q)_x = \{X \in A(Q)_x;\ d\theta(X, Y) = 0,\ \theta \in Q,\ Y \in A(Q)_x\},$$
$$P(E) = \{\theta \in \textstyle\bigwedge^1(M);\ \theta(X) = 0,\ X \in E\},$$

$x \in M$. We have $D(Q)_x \subset H(Q)_x \subset A(Q)_x$, $x \in M$. Set $\mathrm{Ch}\,(Q) = P(H(Q))$. Then $\mathrm{Ch}\,(Q)$ is called the *characteristic system* of Q. It is easy to prove the following lemmas.

LEMMA 3.1. *Assume that $D(Q)$ (resp. $H(Q)$) is a distribution on M. Then $D(Q)$ (resp. $H(Q)$) is completely integrable. Moreover, let $x \in M$ and let $x^1, \cdots, x^m$ be independent first integrals of $D(Q)$ defined on a small open neighborhood V of x ($m = \dim M - \dim D(Q)_x$). Then, for every $\theta \in Q$, there are functions of m variables a_j, $1 \leqq j \leqq m$, such that*

$$\theta = \textstyle\sum_{j=1}^m a_j(x^1, \cdots, x^m)dx^j \qquad on\ V.$$

LEMMA 3.2. *Let P be a manifold and let $\pi \colon P \dashrightarrow M$ be a fibering. Let π^*Q denote the set of all 1-forms of the form $\pi^*\theta$, $\theta \in Q$. Then:*

1) $E(\pi)_p \subset D(\pi^*Q)_p$ $(E(\pi)_p = \mathrm{Ker}\,(\pi_*)_p,\ p \in P)$.
2) $\pi_*(H(\pi^*Q)_p) = H(Q)_{\pi(p)}$ $(p \in P)$.
3) $\pi_*(D(\pi^*Q)_p) = D(Q)_{\pi(p)}$ $(p \in P)$.

Let $C = (u^1, \cdots, u^n; \omega^1, \cdots, \omega^r)$ be an invariant system on a manifold P and let $S = \langle \omega^1, \cdots, \omega^{s_0} \rangle$ $(s_0 \leqq r)$ be a Pfaffian system on P. Set $H =$

$\{u^1, \cdots, u^n\}$, $Q = \{\omega^1, \cdots, \omega^r\}$, $Q_0 = \{\omega^1, \cdots, \omega^{s_0}\}$ and $E^0 = A(Q_0)$. We define v-families on P, E^i, $i \geq 0$, and subsets of Q, Q_i, $i \geq 0$, by induction. Suppose that E^{i-1} and Q_{i-1} are defined for an integer $i \geq 1$. Then we set $E^i = D(Q_{i-1})$ and $Q_i = Q \cap P(E^i)$. We have $E^i \supset E^{i+1}$ and $Q_i \subset Q_{i+1}$ ($i \geq 0$). Set $E_p^\infty = \bigcap_{i \geq 0} E_p^i$ ($p \in P$), $Q_\infty = \bigcup_{i \geq 0} Q_i$ and

$$H_\infty = \{u \in H; Xu = 0, X \in E^\infty\}.$$

Suppose $H_\infty = \{u^1, \cdots, u^m\}$ and $Q_\infty = \{\omega^1, \cdots, \omega^s\}$ ($m \leq n$, $s_0 \leq s \leq r$). Finally, we set $S_\infty = \langle \omega^1, \cdots, \omega^s \rangle$.

DEFINITION 3.1. S is said to be *closed* in C if, for each $i \geq 0$, E^i is a distribution on P and if S_∞ is completely integrable.

DEFINITION 3.2. Let $C_0 = (\bar{u}^1, \cdots, \bar{u}^m; \bar{\omega}^1, \cdots, \bar{\omega}^s)$ be an invariant system on a manifold P_0. Then (P_0, C_0) is called a *quotient Cartan system* of (P, C) by S if there is a fibering $\pi: P \to P_0$ such that
1) $E_p^\infty = E(\pi)_p$ for any $p \in P$;
2) $\pi^* \bar{u}^i = u^i$, $1 \leq i \leq m$, and $\pi^* \bar{\omega}^j = \omega^j$, $1 \leq j \leq s$.
In this case, we say that S admits a quotient Cartan system $(P_0, C_0) = (P, C)/S$.

As for the uniqueness, we can prove easily the following.

PROPOSITION 3.1. *Let (P_i, C_i), $C_i = (\bar{u}_i^1, \cdots, \bar{u}_i^m; \bar{\omega}_i^1, \cdots, \bar{\omega}_i^s)$, be a quotient Cartan system of (P, C) by S with fibering π_i ($i = 1, 2$). Then, for any $p \in P$, (P_1, C_1) and (P_2, C_2) are locally isomorphic at $(\pi_1(p), \pi_2(p))$.*

Let $v^1, \cdots, v^{s_0}$ be functions on P. Set

$$C^\# = (u^1, \cdots, u^n, v^1, \cdots, v^{s_0}; \omega^1, \cdots, \omega^r).$$

DEFINITION 3.3. $(P, C^\#)$ is called a *normal Cartan subsystem* of (P, C) if there are s_0 1-forms in C, say $\omega^1, \cdots, \omega^{s_0}$, such that
1) $S = \langle \omega^1, \cdots, \omega^{s_0} \rangle$ is completely integrable;
2) $v^1, \cdots, v^{s_0}$ are independent first integrals of S.
S is called the *defining Pfaffian system* of $(P, C^\#)$.

Assume that there exists a quotient Cartan system (P_0, C_0) of (P, C) by S with fibering $\pi: P \to P_0$.

DEFINITION 3.4. (P_0, C_0) is called a *quotient Cartan system* of (P, C) by $(P, C^\#)$ if the following conditions are satisfied:

   1) $G(P, C^\sharp) = \{F \in G(P, C);\ \pi \circ F = \pi \text{ on } U(F)\}$;

   2) (P, C) acts on P_0 with respect to π and $\mathfrak{G}(P_0, C_0) = \mathfrak{G}(P_0, C)$.

In this case, we say that $(P, C^\sharp)$ admits a quotient Cartan system $(P_0, C_0) = (P, C)/(P, C^\sharp)$.

3.2. We shall prove the following existence theorems. Let (P, C), $C = (u^1, \cdots, u^n; \omega^1, \cdots, \omega^r)$, be a Cartan system and let $S = \langle \omega^1, \cdots, \omega^{s_0} \rangle$. We use the notations in Section 3.1.

THEOREM 3.2.   *Assume that S is closed in C.   Then, for any $p \in P$, there is an open neighborhood P' of p such that $S|P'$ admits a quotient Cartan system $(P', C|P')/(S|P')$.*

THEOREM 3.3.   *Let $(P, C^\sharp)$, $C^\sharp = (u^1, \cdots, u^n, v^1, \cdots, v^{s_0}; \omega^1, \cdots, \omega^r)$, be a normal Cartan subsystem of (P, C) with defining Pfaffian system S.   Assume that the following conditions are satisfied*:

   1) *S is closed in C;*

   2) *$du^i \equiv 0$ (mod. $\omega^1, \cdots, \omega^s$), $1 \leqq i \leqq n$;*

   3) *Let $t = \dim E_p^\infty - (r - s)$ $(p \in P)$.   Then $t \geqq 0$ and there are t 1-forms $\omega^{r+1}, \cdots, \omega^{r+t}$ on P and functions C_{jk}^i of n variables, $s + 1 \leqq i \leqq r + t$, $1 \leqq j, k \leqq r + t$, such that*

   a) *If $X \in E^\infty$ satisfies $\omega^i(X) = 0$, $s + 1 \leqq i \leqq r + t$, then $X = 0$;*

   b) *$C_{jk}^i + C_{kj}^i = 0$ for all i, j, k and*

$$d\omega^i = \frac{1}{2} \sum_{j,k=1}^{r+t} C_{jk}^i(u^1, \cdots, u^n)\omega^j \wedge \omega^k, \qquad s + 1 \leqq i \leqq r + t.$$

Then, for any $p \in P$, there is an open neighborhood P' of p such that $(P', C^\sharp|P')$ admits a quotient Cartan system $(P', C|P')/(P', C^\sharp|P')$.

   Proof of the theorems.   Suppose first that S is closed in C.   Let $d \geqq 0$ be the smallest integer satisfying $Q_\infty = Q_d$.   Then we have $E^i = E^{d+1}$ for all $i \geqq d + 1$ and hence $D(Q_\infty) = E^\infty$.   Let us set $h = \dim P - \dim E_p^\infty$. Since E^∞ is completely integrable, we can take independent first integrals $x^1, \cdots, x^h$ of E^∞ defined on a small open neighborhood P' of p.   Let $\boldsymbol{R}^h$ be the standard euclidean space of dimension h.   Let $\pi: P' \to \boldsymbol{R}^h$ denote the mapping given by $\pi(q) = (x^1(q), \cdots, x^h(q))$ $(q \in P')$.   We can assume that π is a fibering of P' onto an open set P_0 of $\boldsymbol{R}^h$.   Then, $E^\infty = E(\pi)$ on P'.   By Lemma 3.1, there are functions $\bar{u}^i$, $1 \leqq i \leqq m$, and 1-forms $\bar{\omega}^j$, $1 \leqq j \leqq s$, on P_0 such that $\pi^* \bar{u}^i = u^i$, $1 \leqq i \leqq m$, and $\pi^* \bar{\omega}^j = \omega^j$, $1 \leqq j \leqq s$.

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By assumption, $\langle \overline{\omega}^1, \cdots, \overline{\omega}^s \rangle$ is completely integrable. Set $C' = (u^1, \cdots, u^m; \omega^1, \cdots, \omega^s)$ and $C_0 = (\overline{u}^1, \cdots, \overline{u}^m; \overline{\omega}^1, \cdots, \overline{\omega}^s)$. Then, from Lemma 3.2, we have $\pi_*(D(C')_q) = D(C_0)_{\pi(q)}$ for any $q \in P'$ (see § 1.1). Since $D(C') = D(Q_\infty)$, we finally have $D(C_0)_x = 0$ for any $x \in P_0$. Therefore (P_0, C_0) is a Cartan system. This completes the proof of Theorem 3.2.

Next suppose that the conditions of Theorem 3.3 are satisfied. We shall prove 1) of Definition 3.4. For this purpose, we first show how to construct $x^1, \cdots, x^h$. Notice that $v^1, \cdots, v^{s_0}$ are first integrals of E^∞. Set $v_0^i = v^i$, $1 \leq i \leq s_0$, and $s_j = \dim(E_p^{j-1}/E_p^j)$, $j \geq 1$. Let k be an integer with $0 \leq k \leq d$. Assume that we obtain independent first integrals of E^k on P', $v_0^1, \cdots, v_0^{s_0}, \cdots, v_k^1, \cdots, v_k^{s_k}$. Set $Q_k = \{\omega^1, \cdots, \omega^{t_k}\}$ $(t_k \leq s)$. Since $Q_k \subset P(E^k)$, there are functions A_l^{ij} on P', $0 \leq i \leq k$, $1 \leq j \leq t_k$, $1 \leq l \leq s_i$, such that

$$\omega^j = \sum_{i=0}^k \sum_{l=1}^{s_i} A_l^{ij} dv_i^l, \qquad 1 \leq j \leq t_k.$$

Then it is not hard to see that A_l^{ij}'s are first integrals of E^{k+1}. If $k < d$ or $E^\infty \subsetneq E^d$, we can choose independent functions v_{k+1}^j on P', $1 \leq j \leq s_{k+1}$, in such a way that

a) v_i^j, $0 \leq i \leq k+1$, $1 \leq j \leq s_i$, are independent on P';

b) for each j, $1 \leq j \leq s_{k+1}$, v_{k+1}^j coincides with some A_l^{ij}.

Then v_i^j, $0 \leq i \leq k+1$, $1 \leq j \leq s_i$, are first integrals of E^{k+1}. We can repeat this procedure. In this way, renumbering v_i^j, we obtain $x^1, \cdots, x^h$.

Now take any $F \in \mathfrak{G}(P', C^\sharp | P')$. Then, $F^* v_0^j = v_0^j$, $1 \leq j \leq s_0$. Assume that F satisfies $F^* v_i^j = v_i^j$, $0 \leq i \leq k$, $1 \leq j \leq s_i$. Then we have $F^* A_l^{ij} = A_l^{ij}$ for all i, j and l and hence $F^* v_{k+1}^j = v_{k+1}^j$, $1 \leq j \leq s_{k+1}$. Thus, by induction, we can prove $F^* x^j = x^j$, $1 \leq j \leq h$. This implies that $\pi \circ F = \pi$ on $U(F)$. The converse relation may be obvious.

Finally, we prove 2) of Definition 3.4. Since $E(\pi) = E^\infty$, it follows from Lemma 3.2 that $E(\pi)$ is $\mathfrak{G}(P', C | P')$-invariant. It is easy to verify that $\mathfrak{G}_0(P_0, C | P') \subset \mathfrak{G}(P_0, C_0)$ (see § 1.1). In view of the proof of Theorem 2.3–1), it suffices to prove that, for any $f \in \mathfrak{G}(P_0, C_0)$ and any $q_1 \in \pi^{-1}(U(f))$, there is a local prolongation $F \in \mathfrak{G}(P', C | P')$ of f with $q_1 \in U(F)$. Set $W_1 = \pi^{-1}(U(f))$ and $W_2 = \pi^{-1}(V(f))$. Let $\pi_i: W_1 \times W_2 \to W_i$ be the natural projection $(i = 1, 2)$. Set $U^i = \pi_1^* u^i - \pi_2^* u^i$, $1 \leq i \leq n$, and, $\theta^j = \pi_1^* \omega^j - \pi_2^* \omega^j$, $1 \leq j \leq r + t$. We should integrate the exterior differential system Σ on $W_1 \times W_2$ generated by $U^1, \cdots, U^n, \theta^1, \cdots, \theta^r$ (cf. [5]). Let us consider the submanifold N of $U(f) \times V(f)$ given by

$$N_0 = \{(x, f(x)) \in U(f) \times V(f); \qquad x \in U(f)\}$$

and the submanifold $N = (\pi \times \pi)^{-1}(N_0)$ of $W_1 \times W_2$. Then we have $\dim N = \dim P + (r - s + t)$. Let S' denote the Pfaffian system on N given by $S' = \langle \iota^* \theta^{s+1}, \cdots, \iota^* \theta^{r+t} \rangle$, where $\iota : N \to W_1 \times W_2$ is the inclusion. Since $U^i = 0$, $1 \leq i \leq n$, and $\theta^j = 0$, $1 \leq j \leq s$, on N, the condition 3)–b) of Theorem 3.3 implies that S' is completely integrable. For the given point q_1, there is a point q_2 of W_2 such that $(q_1, q_2) \in N$. Let L be the leaf of S' with $(q_1, q_2) \in L$. L is an integral manifold of Σ with $\dim L = \dim P$. Moreover, the condition 3)–a) implies that the differential $(\pi_i | L)_* : T_z(L) \to T_{q_i}(P)$ $(z = (q_1, q_2))$ is non-singular $(i = 1, 2)$. So we can define a local transformation F of P' by $F = (\pi_2 | L) \circ (\pi_1 | L)^{-1}$ on a small open neighborhood V of q_1 in W_1. Clearly, we have $F^* u^i = u^i$, $1 \leq i \leq n$, $F^* \omega^j = \omega^j$, $1 \leq j \leq r$, and $\pi \circ F = f \circ \pi$ on V. Thus F is a desired element of $\mathfrak{G}(P', C | P')$. This completes the proof of Theorem 3.3.

§ 4. Integration of Pfaffian systems

4.1. By the integration of a Pfaffian system S, we mean to find integral manifolds of S. Let M be a manifold and let θ be a closed 1-form on M. A function f on an open set U of M is called an *integral* of θ if $df = \theta$ on U. By the language "quadrature", we mean to construct an integral of a closed 1-form. We first consider a completely integrable Pfaffian system S of rank s on M.

DEFINITION 4.1. Let $(\theta^1, \cdots, \theta^s)$ be a generator of S and let $\omega^1, \cdots, \omega^s$ be 1-forms on M. We say that $(\theta^1, \cdots, \theta^s)$ is a *solvable generator* of S with auxiliary 1-forms $\omega^1, \cdots, \omega^s$ if the following conditions are satisfied:
   1) $d\theta^1 = \omega^1 \wedge \theta^1$, $d\omega^1 = 0$;
   2) (if $s \geq 2$) for all $k = 2, \cdots, s$,

$$d\theta^k \equiv \omega^k \wedge \theta^k, \; d\omega^k \equiv 0 \qquad (\mathrm{mod} \; \theta^1, \cdots, \theta^{k-1}).$$

Take an integral g of ω^1. Then we get a closed 1-form $\theta = e^{-g} \theta^1$. Take an integral of θ, f, defined on an open set U of M and consider the submanifold N of U defined by $f = $ constant. If $s = 1$, then N is an integral manifold of S. If $s \geq 2$, then $\theta^1 = 0$, $d\theta^2 = \omega^2 \wedge \theta^2$ and $d\omega^2 = 0$ on N. So we can repeat the above procedure. Thus we have the following proposition.

PROPOSITION 4.1. *Let $m = \dim M$. Assume that we find a solvable*

generator of S. Then every $(m - s)$-dimensional integral manifold of S is locally constructed by quadratures.

This means that we can find s independent first integrals of S by quadratures. Let $(\theta^1, \cdots, \theta^s)$ be a generator of S. An element θ of S is called an *integral* 1-*form* of S if there are functions A_{jk} on M, $1 \leq j < k \leq s$, such that $d\theta = \sum_{j<k} A_{jk}\theta^j \wedge \theta^k$ (cf. [1]). Assume that the generator $(\theta^1, \cdots, \theta^s)$ consists of integral 1-forms θ^i of S, $1 \leq i \leq s$. Then we find some first integrals of S by differentiations. After reducing all known first integrals to constants, we get a Pfaffian system $\tilde{S} = \langle \tilde{\theta}^1, \cdots, \tilde{\theta}^t \rangle$ satisfying

$$d\tilde{\theta}^i = \frac{1}{2} \sum_{j,k=1}^{t} C_{jk}^i \tilde{\theta}^j \wedge \tilde{\theta}^k, \qquad 1 \leq i \leq t,$$

where C_{jk}^i, $1 \leq i, j, k \leq t$, are constants with $C_{jk}^i + C_{kj}^i = 0$ (see [1]). *If these constants define a solvable Lie algebra, we can find a solvable generator of $\tilde{S}$ by algebraic operations.*

4.2. Let S be a Pfaffian system on a manifold M. S is said to be *regular* if the characteristic system $\mathrm{Ch}\,(S)$ of S is a Pfaffian system on M. In this case, $\mathrm{Ch}\,(S)$ is completely integrable (see § 3.1). The following lemma is well-known (see [4] p. 1060 and [5] p. 52).

LEMMA 4.1. *Let* $s = \mathrm{rank}\,S$. *Assume that there exists an absolute parallelism* $\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^t$ *on* M *satisfying* $S = \langle \theta^1, \cdots, \theta^s \rangle$ $(s + t = \dim M)$. *Set*

$$d\theta^i \equiv \frac{1}{2} \sum_{j,k=1}^{t} F_{jk}^i \omega^j \wedge \omega^k \qquad (\mathrm{mod}\ \theta^1, \cdots, \theta^s),\ 1 \leq i \leq s,$$

where F_{jk}^i, $1 \leq i \leq s$, $1 \leq j, k \leq t$, *are functions on* M *with* $F_{jk}^i + F_{kj}^i = 0$. *Then*:

1) *As an* $\mathfrak{F}(M)$-*submodule of* $\bigwedge^1(M)$, $\mathrm{Ch}\,(S)$ *is generated by* $\theta^1, \cdots, \theta^s$ *and* $\sum_{k=1}^{t} F_{jk}^i \omega^k$, $1 \leq i \leq s$, $1 \leq j \leq t$.

2) *If* S *is regular, then there are integral 1-forms* $\bar{\theta}^1, \cdots, \bar{\theta}^s$ *of* $\mathrm{Ch}\,(S)$ *such that* $(\bar{\theta}^1, \cdots, \bar{\theta}^s)$ *forms a generator of* S.

DEFINITION 4.2. Let $\mathfrak{G}$ be a pseudogroup on M. A regular Pfaffian system S' on M is called a *covariant system* of S with respect to $\mathfrak{G}$ if $\mathfrak{G}(S)$ leaves S' invariant.

In this case, $\mathrm{Ch}\,(S')$ is also a covariant system of S. In [3] and [4],

E. Cartan used such covariant systems for the problem of integration of second order partial differential equations. We want to use them more effectively.

Assume that $\mathfrak{G}$ is a Lie pseudogroup. Consider a defining Cartan system (P^*, C^*) of $\mathfrak{G}(S)$ with respect to a fibering π of P^* onto M (for example, we can take the Cartan system (P^*, C^*) which appeared in Theorem 2.3). As before, for a Pfaffian system $\tilde{S}$ on M, we denote by $\tilde{S}^*$ the Pfaffian system on P^* generated by all 1-forms $\pi^*\theta$, $\theta \in \tilde{S}$. If $\tilde{S}$ is regular, then $\mathrm{Ch}\,(\tilde{S})^* = \mathrm{Ch}\,(\tilde{S}^*)$ by Lemma 4.2–1). Now we can see that the Pfaffian system to be considered is given by S^*. Let S' be a covariant system of S with respect to $\mathfrak{G}$. Then $(S')^*$ is a covariant system of S^* with respect to $\mathfrak{G}(P^*, C^*)$. Assume that the action of (P^*, C^*) on M is normal. Then, by Proposition 2.1 and Theorem 2.3, the systems S^*, $(S')^*$ and $\mathrm{Ch}\,(S')^*$ are generated by invariant 1-forms of (P^*, C^*). Changing C^* if necessary, we can assume that $\mathrm{Ch}\,(S')^*$ is generated by some 1-forms in C^*.

By this consideration, we are led to the following problem. *Being given a Cartan system (P, C), $C = (u^1, \cdots, u^n; \omega^1, \cdots, \omega^r)$, study the problem of integration of a regular Pfaffian system S on P generated by invariant 1-forms of (P, C).* Such a system will be denoted by $(S, (P, C))$. As $\mathfrak{G}(P, C)$ leaves S invariant, we can consider covariant systems of S. However, we should restrict ourselves to the study of those covariant systems of S which are constructed by the so-called rational operations.

DEFINITION 4.3. A regular Pfaffian system S' of rank t on P is called a *normal covariant system* of (S, C) if it satisfies the following conditions:
1) $S' \subset S$ and $0 < t < \mathrm{rank}\,S$;
2) S' is generated by t 1-forms $\sum_{j=1}^{r} a_j^i \omega^j$, $1 \leq i \leq t$, where a_j^i, $1 \leq i \leq t$, $1 \leq j \leq r$, are real constants;
3) $\mathrm{Ch}\,(S')$ is generated by some 1-forms in C;
4) $\mathrm{Ch}\,(S')$ is closed in C (see Definition 3.1);
5) $S' = \mathrm{Ch}\,(S') \cap S$.

S is said to be *irreducible* with respect to (P, C) if there is no normal covariant system of (S, C).

The structure equation of (P, C) enables us to find normal covariant systems of (S, C). Let P' be an open set of P. Then it is clear that $S'|P'$ is a normal covariant system of $(S|P', C|P')$.

Theorem 4.2. *Let (P, C), S and S' be as above. Then, for any $p \in P$, there are Cartan systems (P_0, C_0) and (P_1, C_1) and a regular Pfaffian system S'_0 on P such that*

    1) *P_1 is an open neighborhood of p in P and $C_1 = C|P_1$;*

    2) *(P_0, C_0) is a quotient Cartan system of (P_1, C_1) by $\mathrm{Ch}\,(S')|P_1$ with fibering $\pi: P_1 \to P_0$;*

    3) *$(S'_0)^* = S'|P_1$ with respect to the fibering π;*

    4) *S'_0 is generated by invariant 1-forms of (P_0, C_0);*

    5) *for a submanifold N_0 of P_0, $N = \pi^{-1}(N_0)$ is an integral manifold of S' if and only if N_0 is an integral manifold of S'_0;*

    6) *for a normal covariant system S'' of $(S|P_1, C_1)$ satisfying $S'' \subset S'|P_1$ and $0 < \mathrm{rank}\, S'' < \mathrm{rank}\, S'$, there is a unique normal covariant system S''_0 of (S'_0, C_0) with $(S''_0)^* = S''$;*

    7) *for a normal covariant system S''_0 of (S'_0, C_0), $S'' = (S''_0)^*$ is a normal covariant system of $(S|P_1, C_1)$ satisfying $S'' \subset S'|P_1$ and $0 < \mathrm{rank}\, S'' < \mathrm{rank}\, S'$.*

It should be remarked that once $\mathrm{Ch}\,(S')$ is integrated we can construct the objects in Theorem 4.2 by the rational operations. However, we can determine $\dim P_0$ and the structure equation of (P_0, C_0) without integration. From 6) of Theorem 4.2, we have the following corollary.

Corollary 4.3. *Let (P, C) and S be as before. Assume that there are normal covariant systems S_i of (S, C), $1 \leqq i \leqq k$, satisfying the following conditions:*

    a) $0 < \mathrm{rank}\, S_i < \mathrm{rank}\, S_{i+1}$, $1 \leqq i \leqq k - 1$ *(if $k \geqq 2$);*

    b) $S_1 \subset S_2 \subset \cdots \subset S_k \subset S.$

Then, for any $p \in P$, there are a sequence of Cartan systems

$$(P_{k+1}, C_{k+1}) \xrightarrow{\pi_k} (P_k, C_k) \longrightarrow \cdots \xrightarrow{\pi_2} (P_2, C_2) \xrightarrow{\pi_1} (P_1, C_1)$$

and regular Pfaffian systems $\tilde{S}_i$ and $\tilde{S}'_i$ on P_i, $1 \leqq i \leqq k + 1$, such that

    1) *P_{k+1} is an open neighborhood of p in P and $C_{k+1} = C|P_{k+1}$;*

    2) *$\tilde{S}_{k+1} = S|P_{k+1}$, $\tilde{S}'_{k+1} = S_k|P_{k+1}$ and $\tilde{S}'_1 = \tilde{S}_1$;*

    3) *$\mathrm{rank}\, \tilde{S}_i = \mathrm{rank}\, S_i$, $1 \leqq i \leqq k$;*

    4) *$\tilde{S}_i$ is generated by invariant 1-forms of (P_i, C_i), $1 \leqq i \leqq k + 1$;*

    5) *$\tilde{S}'_i$ is a normal covariant system of $(\tilde{S}_i, C_i)$, $2 \leqq i \leqq k + 1$;*

    6) *(P_i, C_i) is a quotient Cartan system of (P_{i+1}, C_{i+1}) by $\mathrm{Ch}(\tilde{S}'_{i+1})$ with fibering $\pi_i: P_{i+1} \to P_i$, $1 \leqq i \leqq k$;*

7) $(\tilde{S}_i)^* = \tilde{S}'_{i+1}$ *with respect to the fibering* π_i, $1 \leq i \leq k$.

Proof of Theorem 4.2.   We use the notations in Section 3.1.   We set
$\mathrm{Ch}\,(S') = \langle \omega^1, \cdots, \omega^{s_0} \rangle$ and $\mathrm{Ch}\,(S')_\infty = \langle \omega^1, \cdots, \omega^s \rangle$.   Moreover,

$$Q = \{\omega^1, \cdots, \omega^r\},\ Q_\infty = \{\omega^1, \cdots, \omega^s\} \quad \text{and} \quad E^\infty = D(Q_\infty).$$

By Theorem 3.2, there is an open neighborhood P_1 of p in P such that
$\mathrm{Ch}\,(S')|P_1$ admits a quotient system (P_0, C_0) with fibering $\pi: P_1 \to P_0$.   Let
$C_0 = (\bar{u}^1, \cdots, \bar{u}^m; \bar{\omega}^1, \cdots, \bar{\omega}^s)$ $(m \leq n)$.   Without loss of generality, we can
assume that $P_1 = P$.   Before going further, we prove the following lemma.

LEMMA 4.2.   *Let* $\tilde{S}_0$ *be a Pfaffian system on* P_0 *and let* $\tilde{S} = (\tilde{S}_0)^*$ *Then*
$\mathrm{Ch}\,(\tilde{S}) = \langle \omega^1, \cdots, \omega^k \rangle$ *if and only if* $k \leq s$ *and* $\mathrm{Ch}\,(\tilde{S}_0) = \langle \bar{\omega}^1, \cdots, \bar{\omega}^k \rangle$.

Proof of Lemma 4.2.   Suppose first $\mathrm{Ch}\,(\tilde{S}) = \langle \omega^1, \cdots, \omega^k \rangle$.   From Lemma
3.2, we have $E^\infty \subset H(\tilde{S})$ and hence $Q \cap \mathrm{Ch}\,(\tilde{S}) \subset Q_\infty$.   This implies that
ω^i belongs to Q_∞, $1 \leq i \leq k$.   Thus, $k \leq s$ and $\pi^*\bar{\omega}^i = \omega^i$, $1 \leq i \leq k$.   Take
any $X \in H(\tilde{S})$.   Then, by Lemma 1.2, we can write $X = \pi_* Y$ for some
$Y \in H(\tilde{S})$.   Then, $\bar{\omega}^i(X) = \omega^i(Y) = 0$, $1 \leq i \leq k$.   Thus, $\bar{\omega}^i \in \mathrm{Ch}\,(\tilde{S}_0)$, $1 \leq i \leq$
k.   Next, for any $\bar{\omega} \in \mathrm{Ch}\,(\tilde{S}_0)$ and any $Y \in H(\tilde{S})$, we have $(\pi^*\bar{\omega})(Y) = 0$ and
hence $\pi^*\bar{\omega} \in \mathrm{Ch}\,(\tilde{S})$.   This means that $\bar{\omega}$ is an $\mathfrak{F}(P_0)$-linear combination of
$\bar{\omega}^1, \cdots, \bar{\omega}^k$.   This proves $\mathrm{Ch}\,(\tilde{S}_0) = \langle \bar{\omega}^1, \cdots, \bar{\omega}^k \rangle$.

The converse follows easily from the formula $\mathrm{Ch}\,(\tilde{S}) = \mathrm{Ch}\,(\tilde{S}_0)^*$ (cf.
Lemma 4.1–1)).

We return to the proof of Theorem 4.2.   Since $S' \subset \mathrm{Ch}(S')$, it follows
from the condition 2) of Definition 4.3 that S' is generated by t 1-forms
$\theta^i = \sum_{j=1}^{s_0} b^i_j \omega^j$, $1 \leq i \leq t$, where b^i_j's are real constants.   If we set

$$\bar{\theta}^i = \sum_{j=1}^{s_0} b^i_j \bar{\omega}^j, \qquad 1 \leq i \leq t,$$

then $\pi^*\bar{\theta}^i = \theta^i$, $1 \leq i \leq t$.   We can define a Pfaffian system on P_0 by
$S'_0 = \langle \bar{\theta}^1, \cdots, \bar{\theta}^t \rangle$.   Then we have $(S'_0)^* = S'$ and hence $\mathrm{Ch}\,(S'_0) = \langle \bar{\omega}^1, \cdots,$
$\bar{\omega}^{s_0} \rangle$ by Lemma 4.2.   This proves 3) and 4) of Theorem 4.2.   5) may be
obvious.

Let S'' be a normal covariant system of (S, C) with $S'' \subset S'$.   Then,
by the same argument as above, there is a unique Pfaffian system S''_0 on
P_0 such that $(S''_0)^* = S''$ and S''_0 is generated by 1-forms $\sum_{j=1}^{s} c^i_j \bar{\omega}^j$, $1 \leq$
$i \leq \mathrm{rank}\, S''$, where c^i_j's are real constants.   Now we prove 6) and 7).   In
view of the above remark and Lemma 4.2, it will be sufficient to show
that $\mathrm{Ch}\,(S'') = \langle \omega^1, \cdots, \omega^k \rangle$ $(k \leq s)$ is closed in C if and only if $\mathrm{Ch}\,(S''_0) =$

$\langle \bar{\omega}^1, \cdots, \bar{\omega}^k \rangle$ is closed in C_0. Set $\tilde{Q}_0 = \{\omega^1, \cdots, \omega^k\}$ and $\bar{Q}_0 = \{\bar{\omega}^1, \cdots, \bar{\omega}^k\}$. Let $(\tilde{E}^i, \tilde{Q}_i)$ (resp. $(\bar{E}^i, \bar{Q}_i)$) be the system defined by $\mathrm{Ch}\,(S'')$ (resp. $\mathrm{Ch}\,(S_0'')$). Assume that $\pi^* \bar{Q}_{i-1} = \tilde{Q}_{i-1}$ for an integer $i \geqq 1$. Then, from Lemma 3.2, we have

$$E^\infty \subset D(\pi^* \bar{Q}_{i-1}) = D(\tilde{Q}_{i-1}) = \tilde{E}^i$$

and

$$\pi_* \tilde{E}^i = \pi_*(D(\pi^* \bar{Q}_{i-1})) = D(\bar{Q}_{i-1}) = \bar{E}^i.$$

Take any $\bar{\omega} \in \bar{Q}_i$. For any $X \in \tilde{E}^i$, we have $(\pi^* \bar{\omega})(X) = 0$ and hence $\pi^* \bar{\omega} \in \tilde{Q}_i$. Next, take any $\omega \in \tilde{Q}_i$. Since $\tilde{Q}_i \subset Q_\infty$, there is a 1-form $\bar{\omega}$ in C_0 such that $\omega = \pi^* \bar{\omega}$. Take any $Y \in \bar{E}^i$. We can write $Y = \pi_* X$ for some $X \in \tilde{E}^i$. Then we have

$$\bar{\omega}(Y) = \bar{\omega}(\pi_* X) = \omega(X) = 0$$

and hence $\bar{\omega} \in \bar{Q}_i$. We have thereby proved $\pi^* \bar{Q}_i = \tilde{Q}_i$. Since $\pi^* \bar{Q}_0 = \tilde{Q}_0$, we can prove, by induction, the following relations:

$$E^\infty \subset \tilde{E}^i, \pi_* \tilde{E}^i = \bar{E}^i, \pi^* \bar{Q}_i = \tilde{Q}_i \qquad (i \geqq 0).$$

Hence,

$$E^\infty \subset \tilde{E}^\infty, \pi_* \tilde{E}^\infty = \bar{E}^\infty, \pi^* \bar{Q}_\infty = \tilde{Q}_\infty.$$

Using these relations, we can easily verify that $\mathrm{Ch}\,(S'')$ is closed if and only if so is $\mathrm{Ch}\,(S_0'')$. This completes the proof of Theorem 4.2.

4.3. We are now in a position to study the problem of integration of a system $(S, (P, C))$, where $C = (u^1, \cdots, u^n; \omega^1, \cdots, \omega^r)$ is an invariant system on a manifold P and S is a regular Pfaffian system on P generated by invariant 1-forms of (P, C). A normal covariant system S' of (S, C) is said to be *minimal* if there is no normal covariant system S'' of (S, C) satisfying $S'' \subset S'$ and rank $S'' <$ rank S'. From now on, we suppose that S is neither completely integrable nor irreducible with respect to (P, C).

Lemma 4.3. *For each $p \in P$, there are an open neighborhood P' of p and a minimal normal covariant system S' of $(S|P', C|P')$ such that, for any open neighborhood U of p in P', $S'|U$ is also a minimal normal covariant system of $(S|U, C|U)$.*

Proof. Let S_1 be a minimal normal covariant system of (S, C). Assume that S_1 does not satisfy the property in Lemma 4.3. Then we can find an

open neighborhood P_1 of p and a minimal normal covariant system S_2 of $(S|P_1, C|P_1)$ satisfying $S_2 \subset (S_1|P_1)$ and rank $S_2 <$ rank S_1. Assume further that S_2 does not satisfy the property. Then we can proceed as above, and so on. Since the rank of S is finite, we can repeat the procedure only finite number of times. Now Lemma 4.3 follows from this fact.

Let us fix a point p of P and take an open neighborhood P' of p and a minimal normal covariant system S' of $(S|P', C|P')$ with the property stated in Lemma 4.3. Let E^∞ be the distribution defined by $\mathrm{Ch}\,(S')$ (cf. § 3.1). Set $s_0 = \mathrm{rank}\,\mathrm{Ch}\,(S')$ and $h = \dim P - \dim E_p^\infty$. Assume that we can integrate $\mathrm{Ch}\,(S')$. Then we can choose h independent first integrals $v^1, \cdots, v^h$ of E^∞ defined on a small open neighborhood P_1 of p in P' in such a way that $v^1, \cdots, v^{s_0}$ are first integrals of $\mathrm{Ch}\,(S')$ (see the proof of Theorem 3.3). Let $\pi \colon P_1 \dashrightarrow R^h$ denote the mapping given by

$$\pi(q) = (v^1(q), \cdots, v^h(q)) \qquad (q \in P_1).$$

Then we can assume that π is a fibering of P_1 onto a small open set P_0 of R^h. Moreover, we can construct Cartan systems (P_0, C_0) and (P_1, C_1) and a regular Pfaffian system S_0' satisfying the properties listed in Theorem 4.2. By Theorem 4.2–7) and Lemma 4.3, S_0' is irreducible with respect to (P_0, C_0).

Let $(x^1, \cdots, x^h)$ be the canonical coordinate system on R^h (and hence on P_0). Assume now that we can find an integral manifold N_0 of S_0' with defining equations $f^i(x^1, \cdots, x^h) = 0$, $1 \leqq i \leqq m$. Let $N = \pi^{-1}(N_0)$ and let $S|N$ denote the $\mathfrak{F}(N)$-submodule of $\bigwedge^1(N)$ generated by all 1-forms $\iota^*\theta$, $\theta \in S|P_1$, where $\iota \colon N \to P_1$ is the inclusion. Then it is clear that N is defined by the equations $f^i(v^1, \cdots, v^h) = 0$, $1 \leqq i \leqq m$, and that $S|N$ is finitely generated. Let us set

$$C_1^\sharp = (u^1, \cdots, u^n, v^1, \cdots, v^{s_0}; \omega^1, \cdots, \omega^r).$$

Then $(P_1, C_1^\sharp)$ is a normal Cartan subsystem of (P_1, C_1) with defining Pfaffian system $\mathrm{Ch}\,(S')|P_1$. From the first part of the proof of Theorem 3.3, it can be seen that $v^1, \cdots, v^h$ are invariants of $(P_1, C_1^\sharp)$. So each function $f^i = f^i(v^1, \cdots, v^h)$ is an invariant of $(P_1, C_1^\sharp)$. We want to get a Cartan system $(N, \tilde{C})$ so that $S|N$ is generated by invariant 1-forms of $(N, \tilde{C})$. For this purpose, we use Theorem 1.2. Carrying out the isomorphic reduction of $(P_1, C_1^\sharp)$ by the invariants f^i, $1 \leqq i \leqq m$, we may get a Cartan system $(N, \tilde{C})$ (we reduce each f^i to zero). N_0 (or N) is said to be *generic*

if the above procedure really yields a Cartan system $(N, \tilde{C})$ and if $S|N$ is a regular Pfaffian system on N. In this case, $S|N$ is generated by invariant 1-forms of $(N, \tilde{C})$ because, for every invariant 1-form ω of $(P_1, C_1^{\sharp})$, $\iota^{*}\omega$ is an invariant 1-form of $(N, \tilde{C})$. Thus, for a generic integral manifold N_0 of S_0', we get a system $(S|N, (N|\tilde{C}))$, where we put $N = \pi^{-\prime}(N_0)$. Let us arrange the above procedures as follows.

Step 1)   For a point p of P, find an open neighborhood P' of p and a minimal normal covariant system S' of $(S|P', C|P')$ with the property stated in Lemma 4.3.

Step 2)   Find s_0 independent first integrals of $\mathrm{Ch}\,(S')$.

Step 3)   Construct Cartan systems (P_0, C_0) and (P_1, C_1), a fibering π: $P_1 \to P_0$ and a regular Pfaffian system S_0' on P_0 satisfying the properties listed in Theorem 4.2.

Step 4)   Find an integral manifold N_0 of S_0' with defining equations $f^i(x^1, \cdots, x^h) = 0$, $1 \leqq i \leqq m$.

Step 5)   Carry out the isomorphic reduction of $(P_1, C_1^{\sharp})$ by the invariants $f^i(v^1, \cdots, v^h)$, $1 \leqq i \leqq m$ (we reduce f^i to zero).

Step 6)   For a generic integral manifold N_0 of S_0', describe the system $(S|N, (N, \tilde{C}))$, where $N = \pi^{-1}(N_0)$.

If $S|N$ is neither completely integrable nor irreducible with respect to $(N, \tilde{C})$, then we go back to Step 1). Notice that we can achieve Steps 1), 3), 5) and 6) by rational operations. Moreover, S_0' is irreducible with respect to (P_0, C_0). As a general result, we have the following theorem. By the word "generically", we mean to consider, in Step 6), only generic integral manifolds of S_0'.

THEOREM 4.4.   *Let (P, C) and S be as before.   Then the integration of $(S, (P, C))$ is generically reduced to that of the following two types of systems*:
   1) $(S_{\sharp}, (P_{\sharp}, C_{\sharp}))$, *where $S_{\sharp}$ is completely integrable*:
   2) $(S_{*}, (P_{*}, C_{*}))$, *where S_{*} is irreducible with respect to (P_{*}, C_{*}).*

If all systems $(S_{*}, (P_{*}, C_{*}))$ of the second type satisfy the condition rank $S^{*} = 1$, then the integration of S is reduced to that of completely integrable Pfaffian systems.

THEOREM 4.5.   *Let (P, C) and S be as before.   Assume that*

$$k + 1 = \mathrm{rank}\,S \geqq 2$$

and that there are normal covariant systems S_i of (S, C), $1 \leqq i \leqq k$, satis-

fying the following conditions:

    1) rank $S_i = i$, $1 \leqq i \leqq k$;

    2) $S_1 \subset S_2 \subset \cdots \subset S_k \subset S$.

Then the integration of S is generically reduced to that of finite number of completely integrable Pfaffian systems.

The meaning of the word "generically" will be clarified in the following proof.

Proof of Theorem 4.5. Suppose that S is not completely integrable. First of all, we integrate completely integrable Pfaffian systems $\mathrm{Ch}\,(S_i)$, $1 \leqq i \leqq k$. Then, for each point p of P, we can construct a sequence of Cartan systems

$$(P_{k+1}, C_{k+1}) \xrightarrow{\pi_k} (P_k, C_k) \longrightarrow \cdots \xrightarrow{\pi_2} (P_2, C_2) \xrightarrow{\pi_1} (P_1, C_1)$$

and regular Pfaffian systems $\tilde{S}_i$ and $\tilde{S}_i'$ on P_i, $1 \leqq i \leqq k+1$, satisfying the properties listed in Corollary 4.3. Then, rank $\tilde{S}_i = i$, $1 \leqq i \leqq k+1$. Assume that we can find a *generic* integral manifold N_i of $\tilde{S}_i$ for an integer i $(1 \leqq i \leqq k)$. Set $N_{i+1} = \pi_i^{-1}(N_i)$ and consider the Pfaffian system $S_i = \tilde{S}_{i+1}|N_{i+1}$. By assumption, we have rank $S_i \leqq 1$. If rank $S_i = 0$, then N_{i+1} is an integral manifold of $\tilde{S}_{i+1}$. So we have only to consider the case rank $S_i = 1$. Since rank $\tilde{S}_1 = 1$, we are led to the integration of a regular Pfaffian system of rank 1, S_*, on a manifold P_*. In this case, we can use a theorem of Frobenius and Darboux (see [12] pp. 137–141).

Take a generator θ of S_*. A point x of P_* is said to be *generic* if there is an open neighborhood V of x such that $P(D(\theta|V))$ is a Pfaffian system on V, where $\theta|V$ is the resriction of θ to V (for the notation $P(D(\theta|V))$, see § 3.1). Consider a generic point x of P_* and an open neighborhood V of x satisfying the above condition. Notice that $\mathrm{Ch}\,(S_*)$ and $P(D(\theta|V))$ are completely integrable. Let $2k+1 = \mathrm{rank}\,\mathrm{Ch}\,(S_*)$. Then, by the theorem of Frobenius and Darboux, we can find independent first integrals of $\mathrm{Ch}\,(S_*)$, $x^1, \cdots, x^k, z, p_1, \cdots, p_k$, and a first integral u of $P(D(\theta|V))$ such that

$$\theta = dz - \textstyle\sum_{j=1}^k p_j dx^j \qquad \text{for rank } P(D(\theta|V)) = 2k+1,$$

$$\theta = u(dz - \textstyle\sum_{j=1}^k p_j dx^j) \qquad \text{for rank } P(D(\theta|V)) = 2k+2.$$

Using $x^1, \cdots, x^k, z, p_1, \cdots, p_k$, we can construct integral manifolds of S_*. This completes the proof of Theorem 4.5.

The above 1-form $dz - \sum_{j=1}^{k} p_j dx^j$ is a standard form of generators of S_*. This can be generalized as follows. Let S be a regular Pfaffian system of rank s on a manifold P. Then a generator $(\theta^1, \cdots, \theta^s)$ of S is called an *integral generator* of S if θ^i is an integral 1-form of Ch (S), $1 \leq i \leq s$. By Lemma 4.1–2), such a generator exists at least locally. Let $t = \text{rank Ch}(S)$. Assume that $s < t$ and that we can find an integral generator $(\theta^1, \cdots, \theta^s)$ of S and t independent first integrals $y^1, \cdots, y^t$ of Ch (S). Then each θ^i must be of the form:

$$\theta^i = \sum_{j=1}^{t} A_j^i(y^1, \cdots, y^t)dy^j \qquad (1 \leq i \leq s).$$

From this expression, we can get another integral generator $(\tilde{\theta}^1, \cdots, \tilde{\theta}^s)$ of S, where $\tilde{\theta}^i$ is given by

$$\tilde{\theta}^i = dy^i - \sum_{j=s+1}^{t} B_j^i(y^1, \cdots, y^t)dy, \qquad 1 \leq i \leq s.$$

This can be considered as a standard form of generators of S. Using such a standard form, we can often construct integral manifolds of S (see [3] p. 159 and pp. 169–171).

Consider a system $(S, (P, C))$, where S is irreducible with yespect to (P, C). Then we should first try to find an integral generator of S by integrating only completely integrable Pfaffian systems. If it is impossible, then we try to find a Cartan system (P, C') so that (S, C') admits a normal covariant system. Of course, this is not an effective method of integration of S.

Remark. Let us consider a general system $(S, (P, C))$. When we find integral manifolds of S on which some fixed linearly independent 1-forms on P are still linearly independent, there may appear in Step 4) and Step 6) certain integrability conditions. These are main obstructions for solving partial differential equations.

Let us illustrate it by a simple example. Consider a Cartan system (P, C), where $\dim P = 7$ and $C = (\omega^1, \cdots, \omega^7)$ with

$$\begin{cases} d\omega^3 = \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^5, \\ d\omega^4 = \omega^1 \wedge \omega^6 + \omega^2 \wedge \omega^7, \\ d\omega^5 = \omega^1 \wedge \omega^7 - \omega^2 \wedge \omega^6, \\ d\omega^i = 0, \qquad i = 1, 2, 6, 7. \end{cases}$$

Set $S = \langle \omega^3, \omega^4, \omega^5 \rangle$. Consider two-dimensional integral manifolds of S on which ω^1 and ω^2 are still linearly independent. Let $S' = \langle \omega^5 \rangle$. Then,

$\mathrm{Ch}\,(S') = \langle \omega^1, \omega^2, \omega^5, \omega^6, \omega^7 \rangle$ and $(\omega^1, \omega^2, \omega^6, \omega^7, \omega^5)$ is a solvable generator of $\mathrm{Ch}\,(S')$. We can see that S' is a normal covariant system of (S, C). Let (P_0, C_0) be a quotient Cartan system of (P, C) by $\mathrm{Ch}\,(S')$ with fibering π: $P \to P_0$. Then, $\dim P_0 = 5$ and $C_0 = (\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^5, \bar{\omega}^6, \bar{\omega}^7)$ with

$$\begin{cases} d\bar{\omega}^5 = \bar{\omega}^1 \wedge \bar{\omega}^7 - \bar{\omega}^2 \wedge \bar{\omega}^6, \\ d\bar{\omega}^i = 0, \qquad i = 1, 2, 6, 7. \end{cases}$$

S_0' is given by $S_0' = \langle \bar{\omega}^5 \rangle$. Let N_0 be a two-dimensional integral manifold of S_0' on which $\bar{\omega}^1$ and $\bar{\omega}^2$ are still linearly independent and let $N = \pi^{-1}(N_0)$. Then N is generic. We get a Cartan system $(N, \tilde{C})$, where $\dim N = 4$ and $\tilde{C} = (u^1, u^2; \theta^1, \theta^2, \theta^3, \theta^4)$ with

$$\begin{cases} d\theta^1 = 0, \\ d\theta^2 = 0, \\ d\theta^3 = \theta^1 \wedge \theta^4, \\ d\theta^4 = u\theta^1 \wedge \theta^2. \end{cases}$$

The invariant u is a function of u^1 and u^2. We have $S|N = \langle \theta^3, \theta^4 \rangle$. In this case, the integrability condition is given by $u = 0$. If $u = 0$, then $S|N$ is completely integrable and (θ^4, θ^3) forms a solvable generator of $S|N$.

4.4.   Finally, we consider the problem of classification. Let $\mathfrak{G}$ be a Lie pseudogroup on a manifold M and let (P, C) be a defining Cartan system of $\mathfrak{G}$ with respect to a fibering π of P onto M. Assume that the action is normal (cf. § 1.1). Let S_i be a regular Pfaffian system on an open set U_i of M $(i = 1, 2)$. Then we can consider the Cartan system (P_i^*, C_i^*) which appeared in Theorem 2.2. S_i determines a regular Pfaffian system S_i^* on P_i^*, which is generated by invariant 1-forms of (P_i^*, C_i^*) (Theorem 2.3–2)). If S_1 and S_2 are locally $\mathfrak{G}$-equivalent, then (P_1^*, C_1^*) and (P_2^*, C_2^*) are locally isomorphic (Theorem 2.2). Shrinking U_i and P_i^* if necessary, we can assume that there is an isomorphism F of (P_1^*, C_1^*) onto (P_2^*, C_2^*) with $F^*S_2^* = S_1^*$. Then there is a one-to-one correspondence between the set of all normal covariant systems of $(S_1^*, (P_1^*, C_1^*))$ and that of $(S_2^*, (P_2^*, C_2^*))$. Therfore we can conclude that the integration of S_1^* is formally equivalent to that of S_2^*. Thus we get a class of Pfaffian systems to which the same method of integration can be well applied.

We are interested in the problem of classification of all Pfaffian

systems S such that $\mathfrak{G}(S)$ is finite dimensional. In this case, using Theorem 1.3 finite number of times, we can reduce the problem to the following lemma.

LEMMA 4.4. *Let (P_i, C_i), $C_i = (\omega_i^1, \cdots, \omega_i^r)$, be a Cartan system with $r = \dim P_i$ $(i = 1, 2)$. Assume that there are constants $C_{kl;i}^j$, $1 \leqq j, k, l \leqq r$. satisfying $C_{kl,i}^j + C_{lk;i}^j = 0$ for all j, k, l and*

$$d\omega_i^j = \frac{1}{2} \sum_{k,l=1}^r C_{kl;i}^j \omega_i^k \wedge \omega_i^l, \qquad 1 \leqq j \leqq r \ (i = 1, 2).$$

Then (P_1, C_1) and (P_2, C_2) are locally isomorphic at every point of $P_1 \times P_2$ if and only if $C_{kl;1}^j = C_{kl;2}^j$, $1 \leqq j, k, l \leqq r$.

For the proof, see [12] p. 221.

Now we deal with the problem of classification of second order ordinary differential equations

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

under the action of the following Lie pseudogroup on $\boldsymbol{R}^2$.

$$\mathfrak{G}': \begin{cases} X = x + a, \\ Y = Y(x, y), \end{cases} \quad (a \in \boldsymbol{R}).$$

Let (P, C) denote the Cartan system given by

$$P = \{(x, y, p, q, r, s, t) \in \boldsymbol{R}^7; \qquad q \neq 0\},$$

$$C: \begin{cases} \omega_1 = dx, \\ \omega^2 = qdy - pqdx, \\ \omega^3 = -qdp + r\omega^1 + s\omega^2, \\ \omega^4 = dq/q + s\omega^1 + t\omega^2. \end{cases}$$

Let $\pi: P \to \boldsymbol{R}^3$ and $\pi': P \to \boldsymbol{R}^2$ denote the natural projections defined by $\pi(z) = (x, y, p)$ and $\pi'(z) = (x, y)$, respectively $(z \in P)$. Then (P, C) is a defining Cartan system of $\mathfrak{G}'$. Moreover, (P, C) acts effectively on $\boldsymbol{R}^3$ with respect to π. So it defines a Lie pseudogroup $\mathfrak{G}$ on $\boldsymbol{R}^3$. Let f be a function on an open set $U(f)$ of $\boldsymbol{R}^3$. Set $\theta = dy - pdx$, $\theta_f = dp - f(x, y, p)dx$ and $S_f = \langle \theta, \theta_f \rangle$. Then we are led to the problem of classification of the Pfaffian systems S_f under the action of $\mathfrak{G}$.

We have $q\pi^*\theta = \omega^2$ and $sq\pi^*\theta - q\pi^*\theta_f = \omega^3 - u^0\omega^1$ on $P_f^* = \pi^{-1}(U(f))$, where we put $u^0 = r - qf$. $(P_f^*; u^0)$ is a maximal system of invariants of (S_f, C). Set $C_f^* = (u^0; \omega^1, \cdots, \omega^4)$. Then we get a Cartan system (P_f^*, C_f^*). We can see that u^0 is reducible on P_f^*. Putting $r = qf$, we get a Cartan system $(P_f^1, \tilde{C}_f^1)$. We can verify that $u_1^0 = 2s - f_p$ is an invariant of $(P_f^1, \tilde{C}_f^1)$ and that u_1^0 is reducible on P_f^1. Here we use the notations $f_x = \partial f/\partial x$, $f_p = \partial f/\partial p$ etc. Putting $s = f_p/2$, we get a Cartan system $(P_f^2, \tilde{C}_f^2)$. We can verify that $u_2^0 = t - (f_{pp}/2q)$ is an invariant of $(P_f^2, \tilde{C}_f^2)$ and that u_2^0 is reducible on P_f^2. Putting $t = f_{pp}/2q$, we obtain a Cartan system (Q_f, C_f), where

$$Q_f = \{(x, y, p, q) \in \mathbf{R}^4; \quad q \neq 0, \ (x, y, p) \in U(f)\}$$

and

$$C_f : \begin{cases} \omega^1 = dx, \\ \omega^2 = qdy - pqdx, \\ \omega^3 = -qdp + qf\omega^1 + (f_p/2)\omega^2, \\ \omega^4 = dq/q + (f_p/2)\omega^1 + (f_{pp}/2q)\omega^2. \end{cases}$$

(Q_f, C_f) is a defining Cartan system of $\mathfrak{G}(S_f)$ (cf. [7] Theorem 2.3). The Pfaffian system to be considered is now given by $S_f^* = \langle \omega^2, \omega^3 \rangle$. Let g be another function on an open set of $\mathbf{R}^3$. Then S_f and S_g are locally $\mathfrak{G}$-equivalent if and only if (Q_f, C_f) and (Q_g, C_g) are locally isomorphic (Theorems 1.3 and 2.2.). Therefore we have only to classify the Cartan systems (Q_f, C_f). We remark that $\mathfrak{G}(S_f)$ is finite dimensional for any f.

Now we have the following structure equation of (Q_f, C_f):

Eq. 1)
$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^3 \wedge \omega^1 + \omega^4 \wedge \omega^2, \\ d\omega^3 = \omega^4 \wedge \omega^3 - u_3\omega^2 \wedge \omega^1, \\ d\omega^4 = u_1\omega^3 \wedge \omega^2 - u_2\omega^2 \wedge \omega^1, \end{cases}$$

where u_1, u_2 and u_3 are invariants of (Q_f, C_f). They determine other invariants u_i, $4 \leq i \leq 10$, as follows:

Eq. 2)
$$\begin{cases} du_1 = u_4\omega^1 + u_5\omega^2 + u_6\omega^3 - 2u_1\omega^4, \\ du_2 = u_7\omega^1 + u_8\omega^2 + u_4\omega^3 - u_2\omega^4, \\ du_3 = u_9\omega^1 + u_{10}\omega^2 - u_2\omega^3. \end{cases}$$

We have $u_1 = -f_{ppp}/(2q^2)$. If $u_1 = 0$ on Q_f, then f, u_2 and u_3 are given as follows:

Eq. 3)
$$\begin{cases} f = a(x, y)p^2 + 2b(x, y)p + c(x, y), \\ u_2 = (a_x - b_y)/q, \\ u_3 = (a_x - b_y)p + (ac - c_y + b_x - b^2). \end{cases}$$

Using the invariants u_1, u_2 and u_3, we shall consider four cases. Then we can study each case by using Theorem 1.3.

Before going further, we give the following complete list of classification. We denote by m the dimension of $\mathfrak{G}(S_f)$ and by f_s the standard form of each type. As usual, we use the notations $F'(x) = dF/dx$, $G_p(x, p) = \partial G/\partial p$ etc.

A) $m = 4$, $f_s = ay$ $(a \in \boldsymbol{R})$.

B) $m = 3$, $f_s = F(x)y$ $(F'(x) \neq 0)$.

C_1) $m = 2$, $f_s = (1/2a)p^2 + 2(a + b - y)p + 2a(y^2 - 2by + (a - b)^2)$
$$(a, b \in \boldsymbol{R}, \ a \neq 0).$$

C_2) $m = 2$, $f_s = ap + F(p - ay)$ $(a \in \boldsymbol{R}, \ F'''(p) \neq 0)$.

D_1) $m = 1$, $f_s = a(x)p^2 + (F(x) + b(x) - 2a(x)F(x)y)p$
$$+ a(x)F(x)^2y^2 + (F'(x) - b(x)F(x))y + c(x)$$
$$\begin{pmatrix} a'(x) + a(x)F(x) = 1, \\ 4a(x)c(x) = (F(x) - b(x))^2 + 2(F'(x) - b'(x)) \end{pmatrix}.$$

D_2) $m = 1$, $f_s = F(x)p + F'(x)y + G(x, p - F(x)y)$ $(G_{ppp}(x, p) \neq 0)$.

D_3) $m = 1$, $f_s = a(y)p^2 + 2bp + c(y)$ $(b \in \boldsymbol{R}, \ c'(y) = a(y)c(y) - (y + b^2))$.

D_4) $m = 1$, $f_s = a(y)p^2 - 2yp + c(y)$.

D_5) $m = 1$, $f_s = G(y, p)$ $(G_{ppp}(y, p) \neq 0)$.

E_1) $m = 0$, $f_s = a(x, y)p^2 + 2b(x, y)p + c(x, y)$
$$(a_x = b_y, \ b_x - c_y = y + b^2 - ac, \ a_x^2 + b_x^2 + c_x^2 > 0).$$

E_2) $m = 0$, $f_s = a(x, y)p^2 + 2b(x, y)p + c(x, y)$
$$(a_x - b_y = 1, \ a_x^2 + b_x^2 + c_x^2 > 0).$$

E_3) $m = 0$, $f_s = F_x(x, y) + F_y(x, y)p + G(x, y, p - F(x, y))$
$$\begin{pmatrix} G_{ppp}(x, y, p) \neq 0. \quad y'' = f_s(x, y, y') \text{ is decomposed as follows:} \\ \begin{cases} y' = F(x, y) + z(x), \\ z'(x) = G(x, y, z(x)). \end{cases} \end{pmatrix}.$$

We shall give an outline of the argument.

Case I) u_1, u_2 and u_3 are constants. Then, from Eq. 2), we have $u_1 = u_2 = 0$. Putting $a = -u_3$, we get the first type A). $(\omega^3 + \sqrt{a}\,\omega^2, \omega^2)$ is a solvable generator of S_f^* with auxiliary 1-forms $\omega^4 - \sqrt{a}\,\omega^1, \omega^4 + \sqrt{a}\,\omega^1$.

Case II) $du_i = 0$, $i = 1, 2$, and $du_3 \neq 0$ on Q_f. Then, $u_1 = u_2 = 0$ and

u_3 is a function of x and y.

1) $u_3 = u_3(x)$. Then u_3 forms a complete system of invariants of (Q_f, C_f). Putting $F(x) = -u_3$, we get the second type B). If we know a solution of Riccati's equation $u'(x) = u(x)^2 - F(x)$, then $(\omega^3 - u(x)\omega^2, \omega^2)$ is a solvable generator of S_f^* with auxiliary 1-forms $\omega^4 + u(x)\omega^1, \omega^4 - u(x)\omega^1$.

2) $\partial u_3 / \partial y \neq 0$ on Q_f. Taking an element $X = x$, $Y = u_3(x, y)$ of $\mathfrak{G}'$, we can assume that $u_3 = y$. Then, $u_9 = p$, $u_{10} = 1/q$ and hence $\dim \mathfrak{G}(S_f) \leq 1$. In this case, $f, a(x, y)$ and $b(x, y)$ are also invariants of (Q_f, C_f), where f is given by Eq. 3). From this fact, we get the types D_3) and E_1).

Case III) $du_1 = 0$ and $du_2 \neq 0$ on Q_f. Then, $u_1 = 0$. We can see that u_2 and u_3 are reducible on Q_f. Putting $u_2 = 1$ and $u_3 = 0$, we get a Cartan system (Q_f^1, C_f^1), where $C_f^1 = (u_7, u_8, u_9, u_{10}; \omega^1, \omega^2)$. If all invariants are constants, then we have $u_7 - u_{10} = 1/u_8$ and $u_9 = u_{10}^2/u_8$. We get the type C_1), which depends on two constants. If one of the invariants u_7, u_8, u_9 and u_{10} is not a constant, then we can proceed as in Case II). If $\dim \mathfrak{G}(S_f) = 1$, then (Q_f^1, C_f^1) is locally isomorphic to one of the following two cases:

1) x forms a complete system of invariants of (Q_f^1, C_f^1);

2) y forms a complete system of invariants of (Q_f^1, C_f^1).

Then we get the types D_1) and D_4). If $\dim \mathfrak{G}(S_f) = 0$, we get the type E_2).

Case IV) $du_1 \neq 0$ on Q_f. We can see that u_1 is reducible on Q_f. Putting $u_1 = 1/(2a)$ ($a \in \mathbf{R}$, $a \neq 0$), we get a Cartan system (Q_f^2, C_f^2), where $C_f^2 = (u_2, u_3, u_4, u_5, u_6; \omega^1, \omega^2, \omega^3)$. It is not hard to verify that $\dim \mathfrak{G}(S_f) \leq 2$ and that if $\dim \mathfrak{G}(S_f) = 2$ then (Q_f^2, C_f^2) does not admit any invariant of the form $u = u(x, y)$. Therefore one of the invariants u_i, $2 \leq i \leq 6$, is reducible on Q_f^2. We can carry out the isomorphic reduction and get a Cartan system (Q_f^3, C_f^3), where $C_f^3 = (u_2, u_3, u_4, u_5, u_6; \omega^1, \omega^2)$. If all invariants of (Q_f^3, C_f^3) are constants, then we get the type C_2). If $\dim \mathfrak{G}(S_f) = 1$, then an invariant $v(x, y)$ forms a complete system of invariants of (Q_f^3, C_f^3). As in Case II), we can assume that $v = x$ or $v = y$. Then we get the types D_2) and D_5). If $\dim \mathfrak{G}(S_f) = 0$, we get the type E_3). Assume that f and g belong to the type E_3). Let $\bar{u}_i$, $2 \leq i \leq 6$, denote the invariants of (Q_g^3, C_g^3). Then the equivalence of S_f and S_g is determined by the relations $u_i = \bar{u}_i$, $2 \leq i \leq 6$. If the relations are compatible, they define a unique element of $\mathfrak{G}'$ which makes S_f and S_g equivalent.

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