MONOTONE AND E-SCHAUDER BASES OF SUBSPACES

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1. Introduction. The notions of monotone bases and bases of subspaces are well known in a normed linear space setting and have obvious extensions to pseudo-metrizable linear topological spaces. In this paper, these notions are extended to arbitrary linear topological spaces. The principal result gives a list of properties that are equivalent to a sequence (M_i) of complete subspaces being an e-Schauder basis of subspaces for the closed linear span of $\bigcup_1^\infty M_i$. A corollary of this theorem is the fact that an e-Schauder basis for a dense subspace of a linear topological space is an e-Schauder basis for the whole space.

2. A sequence, (M_i), of non-trivial subspaces of a linear topological space $(X, \mathfrak{T})$ is called a *basis of subspaces* for $(X, \mathfrak{T})$ provided that for each $x \in X$, there exists a unique sequence of vectors (x_i) such that $x_i \in M_i$ and $\sum_1^n x_i \to x$. (Here the arrow denotes convergence in the vector topology $\mathfrak{T}$.) A *basis* for $(X, \mathfrak{T})$ is a sequence of vectors (x_i) in X such that for each $x \in X$ there is a unique sequence of scalars (a_i) such that $\sum_1^n a_i x_i \to x$. Note that if (x_i) is a basis for $(X, \mathfrak{T})$ and M_i denotes the one-dimensional subspace of X spanned by x_i, then (M_i) is a basis of subspaces for $(X, \mathfrak{T})$. Conversely, it can easily be shown that a space with a basis of one-dimensional subspaces has a basis. These facts and others which will be noted below imply that the theorems concerning bases of subspaces which follow have corollaries involving bases. These, however, will be stated only when they seem of particular interest. If (x_i) is a basis for $(X, \mathfrak{T})$ and M_i is the one-dimensional subspace spanned by x_i, we shall call (M_i) the *basis of subspaces associated with* (x_i). If (M_i) is a basis of subspaces for $(X, \mathfrak{T})$, then it is possible to define for each i a projection $E_i: X \to M_i$ as follows: $E_i(x) = x_i$, where (x_i) is the unique sequence of vectors such that $x_i \in M_i$ and $\sum_1^n x_i \to x$. If each E_i is continuous, then (M_i) is called a *Schauder basis of subspaces*. Let $S_n = \sum_1^n E_i$. Then S_n is a linear mapping from X into X. Moreover, $S_n(x) \to x$ and $S_m[S_n(x)] = S_r(x)$ where $r = \min(m, n)$. If O is a point of equicontinuity of the sequence (S_n), then (M_i) is called an *e-Schauder basis of subspaces* for $(X, \mathfrak{T})$. Throughout the remainder of this paper we shall adopt the con-

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vention that if (M_i) is a basis of subspaces, then E_i will denote the projection defined above and S_n will denote the projection $\sum_1^n E_i$. An e-Schauder basis of subspaces is a Schauder basis of subspaces, but in general, the converse is not true. The two notions are clearly equivalent in all spaces in which pointwise bounded families of continuous linear functions are equicontinuous, e.g. in second category and barrelled spaces.

Associated with a basis (x_i) is a sequence of linear functionals (f_i) defined by $f_i(x) = a_i$, where (a_i) is the unique sequence of scalars such that $\sum_1^n a_i x_i \to x$. If each of the coefficient functionals f_i is continuous, (x_i) is called a *Schauder basis*. A basis is a Schauder basis if and only if the basis of subspaces associated with it is a Schauder basis of subspaces. A basis is defined to be an *e-Schauder basis* if and only if the associated basis of subspaces is an e-Schauder basis of subspaces.

A real-valued function p on a linear space X is called a paranorm on X provided it has the following properties:

(2.1) $\quad p(x) \geqslant 0$ for all $x \in X$,

(2.2) $\quad p(0) = 0$,

(2.3) $\quad p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$,

(2.4) $\quad p(-x) = p(x)$ for all $x \in X$,

(2.5) $\quad$ if (a_n) is a sequence of scalars such that $a_n \to a$ and (x_n) is a sequence in X such that $p(x - x_n) \to 0$, then $p(ax - a_n x_n) \to 0$.

If p is a paranorm on a linear space X and for $x, y \in X$ we define

$$d(x, y) = p(x - y),$$

then it is easy to show that d is an invariant pseudo-metric which generates a vector topology for X. Conversely, if $(X, \mathfrak{T})$ is a pseudo-metrizable linear topological space and d is an invariant pseudo-metric (one always exists; see, for example, (**2**, p. 48)) which generates the vector topology $\mathfrak{T}$, then p, defined by $p(x) = d(x, 0)$ is a paranorm on X.

Let P be a family of paranorms on a linear space X. For $p \in P$ and $\epsilon > 0$, define $V_{p\epsilon} = \{x \in X\colon p(x) \leqslant \epsilon\}$. It is not difficult to show that the family of sets $\{\bigcap_{p \in F} V_{p\epsilon}\colon \epsilon > 0$ and F is a finite subset of $P\}$ is a local base for a vector topology for X. This topology will be called the *topology generated by* P. Note that properties (2.1)–(2.4) are sufficient to guarantee that P generates a topology and (2.5) ensures that scalar multiplication will be a continuous operation in this topology.

It is known (**2**, p. 50) that any linear topological space can be embedded in the product of pseudo-metrizable linear topological spaces. Using this and the definition of the product topology, it is not difficult to show that any vector topology for a linear space X can be generated by a family of paranorms on X (see (**2**, 6C, p. 51)). In view of the remarks above, it is evident that if $(X, \mathfrak{T})$ is pseudo-metrizable, it is possible to choose a single paranorm which generates $\mathfrak{T}$. It is well known that if $(X, \mathfrak{T})$ is locally convex, then there exists a family P of paranorms on X which generates $\mathfrak{T}$ and has the additional property that its members are pseudo-norms. Also, if $\mathfrak{T}_w$ is the weak topology and we define for each $f \in X^*$, $p_f(x) = |f(x)|$, then $\{p_f: f \in X^*\}$ generates $\mathfrak{T}_w$.

Let (M_i) be a sequence of subsets of a linear topological space $(X, \mathfrak{T})$ and let P be a family of paranorms on X. Then (M_i) is defined to be *monotone relative to* P if and only if $\sum_1^n x_i \to x$ and $x_i \in M_i$ implies that for each $p \in P$, $p(\sum_1^n x_i)$ is a non-decreasing function of n.

A sequence (M_i) of non-trivial subspaces of a linear topological space $(X, \mathfrak{T})$ is called *ω-independent* provided that $\sum_1^n x_i \to 0$ and $x_i \in M_i$ implies that $x_i = 0$ for each i. Note that a basis of subspaces is ω-independent.

In general, we shall use the notation of **(2)**.

3. The following lemma is a slight improvement of a result found in **(4)**. The proof of part (i) is essentially the same as the one given there and so it will be omitted.

3.1. LEMMA. *Let (M_i) be an ω-independent sequence of subspaces of a linear topological space $(X, \mathfrak{T})$ and let $X_c = \{x$: there exists a sequence (x_i) such that $x_i \in M_i$ and $\sum_1^n x_i \to x\}$. Let $\mathfrak{B}$ be a local base for $\mathfrak{T}$ consisting of closed sets and for $V \in \mathfrak{B}$ define $V'_c = \{x \in X_c$: $\sum_1^n x_i \in V$ for each n, where $x_i \in M_i$ and $\sum_1^n x_i \to x\}$. Then the following hold:*

(i) *$\mathfrak{B}_c' = \{V'_c$: $V \in \mathfrak{B}\}$ is a local base for a vector topology $\mathfrak{T}'_c$ for X_c.*

(ii) *(M_i) is an e-Schauder basis of subspaces for $(X_c, \mathfrak{T}'_c)$.*

Proof of (ii). Let $\mathfrak{T}_c$ denote the topology induced on X_c by $\mathfrak{T}$. Since (M_i) is ω-independent, it is evident that (M_i) is a basis of subspaces for $(X_c, \mathfrak{T}_c)$. $\mathfrak{B}_c = \{V_c$: $V_c = V \cap X_c$, $V \in \mathfrak{B}\}$ is a local base for $\mathfrak{T}_c$ which consists of $\mathfrak{T}_c$-closed sets. Since V_c is $\mathfrak{T}_c$-closed, $V'_c \subseteq V_c$, so $\mathfrak{T}'_c$ is stronger than $\mathfrak{T}_c$. If $x \in X_c$, then there exists a unique sequence (x_i), with $x_i \in M_i$, having the property that $\sum_1^n x_i \to x$ (relative to $\mathfrak{T}_c$). We show that this convergence also holds relative to $\mathfrak{T}'_c$. Since $\mathfrak{T}'_c$ is stronger than $\mathfrak{T}_c$, if the convergence holds, then the sequence (x_i) must be unique. Let V'_c be given. Since $\sum_1^n x_i \to x$ (relative to $\mathfrak{T}_c$), there exists an integer N such that $m, n \geqslant N$ imply that $S_m(x) - S_n(x) \in V_c$. (Recall that $S_n(x) = \sum^n x_i$). We show that if $m \geqslant N$, then $x - S_m(x) \in V'_c$, i.e. $S_n[x - S_m(x)] \in V_c$ for each n. Suppose $m \geqslant N$, If $n \leqslant N$, then

$$S_n[x - S_m(x)] = S_n(x) - S_n[S_m(x)] = S_n(x) - S_n(x) = 0 \in V_c.$$

On the other hand, if $n > N$, then $S_n[x - S_m(x)] = S_n(x) - S_r(x)$, where $r = \min(n, m)$. Since $n, r \geqslant N$, we have $S_n(x) - S_r(x) \in V_c$. Thus, it follows that $\sum_1^n x_i \to x$ (relative to $\mathfrak{T}'_c$) and so (M_i) is a basis of subspaces for $(X_c, \mathfrak{T}'_c)$.

We now show that for each m, $S_m(V'_c) \subseteq V'_c$, thus proving that (S_n) is $\mathfrak{T}'_c$-equicontinuous at O and completing the proof of (ii). If $x \in V'_c$, then $S_n(x) \in V_c$ for each n, so $S_n[S_m(x)] = S_{\min(n,m)}(x) \in V_c$ for each n. Therefore, if $x \in V'_c$, $S_m(x) \in V'_c$ for each m, i.e. $S_m(V'_c) \subseteq V'_c$.

3.2. LEMMA. *Let (M_i) be a basis of subspaces for a linear topological space $(X, \mathfrak{T})$ and let $\mathfrak{B}$ be a local base for $\mathfrak{T}$ consisting of closed sets. For $V \in \mathfrak{B}$ define $V' = \{x: S_n(x) \in V$ for each $n\}$. Then $\mathfrak{B}' = \{V': V \in \mathfrak{B}\}$ is a local base for a vector topology $\mathfrak{T}'$ for X. $\mathfrak{T}'$ is stronger than $\mathfrak{T}$ and (M_i) is an e-Schauder basis of subspaces for $(X, \mathfrak{T}')$.*

Proof. A basis of subspaces is ω-independent, so the lemma follows almost immediately from Lemma 3.1.

3.3. LEMMA. *Let (M_i), $(X, \mathfrak{T})$, $\mathfrak{T}_c$, $\mathfrak{T}'_c$, $\mathfrak{B}_c$, and $\mathfrak{B}'_c$ be defined as in* 3.1. *Then $\mathfrak{T}_c = \mathfrak{T}'_c$ if and only if (M_i) is an e-Schauder basis of subspaces for $(X_c, \mathfrak{T}_c)$.*

Proof. If $\mathfrak{T}_c = \mathfrak{T}'_c$, it is clear from Lemma 3.1 that (M_i) is an e-Schauder basis of subspaces for $(X_c, \mathfrak{T}_c)$.

Conversely, suppose (M_i) is an e-Schauder basis of subspaces for $(X_c, \mathfrak{T}_c)$. Then O is a point of equicontinuity of the sequence of projections (S_n). It follows that for each $V_c \in \mathfrak{B}_c$, there exists a neighbourhood $U_c \in \mathfrak{B}_c$ such that $S_n(U_c) \subseteq V_c$ for each n. This implies that $U_c \subseteq V'_c$ and hence that $\mathfrak{T}'_c \subseteq \mathfrak{T}_c$. It is always true that $\mathfrak{T}_c \subseteq \mathfrak{T}'_c$, so the proof is complete.

3.4. LEMMA. *Let $(X, \mathfrak{T})$ be a linear topological space with a basis of subspaces (M_i), and let P be a family of paranorms which generates $\mathfrak{T}$. For each $p \in P$, define $p'(x) = \sup\{p[S_n(x)]: n = 1, 2, \ldots\}$. Then $P' = \{p': p \in P\}$ is a family of paranorms which generates $\mathfrak{T}'$, where $\mathfrak{T}'$ is the topology defined in* 3.2.

Proof. It is easy to verify that p' possesses properties (2.1)–(2.4). Thus, letting $V_{p'\epsilon} = \{x: p'(x) \leqslant \epsilon\}$, we see that the family of sets $\{V_{p'\epsilon}: \epsilon > 0\}$ is a local base for a topology $\mathfrak{T}_{p'}$ for X. To establish the fact that p' satisfies (2.5), it suffices to show that $\mathfrak{T}_{p'}$ is a vector topology. Property (2.5) will then follow from the fact that p' is continuous relative to the topology it generates and the fact that scalar multiplication is always a continuous operation in a linear space with a vector topology. Each element p in P is a paranorm and so the family of sets $\{V_{p\epsilon}: \epsilon > 0\}$ is a local base for a vector topology $\mathfrak{T}_p$. (Here $V_{p\epsilon}$ is defined just as $V_{p'\epsilon}$ was above.) Recall that $S_n(x) = \sum_1^n x_n$, where (x_i) is the unique sequence of vectors such that $x_i \in M_i$ and $\sum_1^n x_i \rightarrow x$. The topology $\mathfrak{T}_p$ is weaker than $\mathfrak{T}$, so since $S_n(x) \rightarrow x$ (relative to $\mathfrak{T}$), we have $S_n(x) \rightarrow x$ (relative to $\mathfrak{T}_p$). Using this and the fact that each S_n is linear, it is an easy matter to verify that if $V'_{p\epsilon} = \{x: S_n(x) \in V_{p\epsilon}$ for each $n\}$, then $\{V'_{p\epsilon}: \epsilon > 0\}$ is a local base for a vector topology $\mathfrak{T}'_p$. Note that $V'_{p\epsilon} = V_{p'\epsilon}$ so that $\mathfrak{T}_{p'} = \mathfrak{T}_p{}'$ and hence p' is a paranorm.

Let $\mathfrak{T}'$ be the topology defined in Lemma 3.2. For each finite subset F of P and each $\epsilon > 0$, let $V_{F\epsilon} = \bigcap_{p\in F} V_{p\epsilon}$, where $V_{p\epsilon}$ is defined as above. Since P generates $\mathfrak{T}$, the family of sets $\mathfrak{B} = \{V_{F\epsilon}\colon \epsilon > 0$ and F is a finite subset of $P\}$ is a local base for $\mathfrak{T}$ which consists of closed sets. Then $\{V'_{F\epsilon}\colon V_{F\epsilon} \in \mathfrak{B}\}$ is a local base for $\mathfrak{T}'$. (Note that $\mathfrak{T}'$ does not depend upon the particular local base for $\mathfrak{T}$ chosen, but only upon $\mathfrak{T}$.) The proof is concluded by observing that $V'_{F\epsilon} = \bigcap_{p\in F} V'_{p\epsilon}$ and $V'_{p\epsilon} = \{x\colon p'(x) \leqslant \epsilon\}$ so that P' generates $\mathfrak{T}'$.

4.1. LEMMA. *Let (M_i) denote an ω-independent sequence of subspaces and let M denote the closure of $\{O\}$. Then $M_i \cap M = \{O\}$ for each i.*

Proof. If for some k, $m \in M_k \cap M$, then $\sum_1^n x_i \to 0$, where $x_k = m$ and $x_i = 0$ for $i \neq k$. This implies that $m = 0$.

An immediate consequence of this lemma is that a linear topological space is Hausdorff if it has a basis of closed subspaces, since each closed subspace contains the closure of $\{O\}$. Another application will follow later.

The proof of the following is routine and will be omitted.

4.2. LEMMA. *Let $(X, \mathfrak{T})$ be a linear topological space. Y a subspace of $(X, \mathfrak{T})$, and T a continuous linear mapping from Y into a complete subspace M of $(X, \mathfrak{T})$. Then :*

(i) *T can be extended to a continuous linear mapping $\hat{T}\colon \bar{Y} \to M$. ($\bar{Y}$ denotes the closure of Y.)*

(ii) *If F is a family of linear mappings from Y into M which is equicontinuous at O, and for each $T \in F$, $\hat{T}$ is a continuous extension of T to $\bar{Y}$, then the family $\hat{F} = \{\hat{T}\colon T \in F\}$ is also equicontinuous at O.*

We are now ready to state and prove the principal result of this paper.

4.3. THEOREM. *Let (M_i) be an ω-independent sequence of complete subspaces of a linear topological space $(X, \mathfrak{T})$. Then the following are equivalent:*

(A) *(M_i) is an e-Schauder basis of subspaces for the linear span of $\bigcup_1^\infty M_i$.*

(B) *(M_i) is an e-Schauder basis of subspaces for the closed linear span of $\bigcup_1^\infty M_i$.*

(C) *There exists a family P^* of paranorms generating $\mathfrak{T}$ such that (M_i) is monotone relative to P^*.*

(D) *There exists a family P of paranorms generating $\mathfrak{T}$ which has the property that for each $p \in P$, there exist a constant K_p and a $\mathfrak{T}$-continuous paranorm q_p on X such that $x_i \in M_i$ and $n < m$ imply that*

$$p(\textstyle\sum_1^n x_i) \leqslant K_p\, q_p(\sum_1^m x_i).$$

(E) *There exists a family P of paranorms generating $\mathfrak{T}$ which has the property that for each $p \in P$, there exist a constant K_p and a $\mathfrak{T}$-continuous paranorm q_p on X such that $\sum_1^n x_i \to x$ and $x_i \in M_i$ imply that*

$$p(\textstyle\sum_1^n x_i) \leqslant K_p\, q_p(x).$$

(F) *There exists a local base* $\mathfrak{V}$ *for* $\mathfrak{T}$ *such that if* $x_i \in M_i$, $V \in \mathfrak{V}$, $x \in V$, *and* $\sum_1^n x_i \to x$, *then* $\sum_1^n x_i \in V$ *for each* n.

(G) *There exists a local base* $\mathfrak{V}$ *for* $\mathfrak{T}$ *such that if* $x_i \in M_i$, $V \in \mathfrak{V}$, *and* $\sum_1^n x_i \in V$, *then* $\sum_1^{n-1} x_i \in V$.

Proof. (A) $\Rightarrow$ (B): Let Y denote the linear span of $\bigcup_1^\infty M_i$, and let (E_i) denote the sequence of linear operators associated with (M_i). For each i, E_i: $Y \to M_i$, and M_i is complete, so by (i) of Lemma 4.2, each E_i can be extended to a continuous linear mapping $\hat{E}_i$: $\bar{Y} \to M_i$.

Let $\hat{S}_n = \sum_1^n \hat{E}_i$ (recall that $S_n = \sum_1^n E_i$). The family $\{S_n: n = 1, 2, \ldots\}$ is equicontinuous at O and $\hat{S}_n$ is a continuous extension of S_n, so by (ii) of Lemma 4.2 it follows that $\{\hat{S}_n: n = 1, 2, \ldots\}$ is equicontinuous at O. We show that if $\bar{y} \in \bar{Y}$, then $\sum_1^n \hat{E}_i(\bar{y}) \to \bar{y}$. Since $\hat{E}_i(\bar{y}) \in M_i$ and (M_i) is ω-independent, it will then follow that (M_i) is a basis of subspaces for $\bar{Y}$. Let V (any neighbourhood of O) be given. Choose a neighbourhood U of O such that $U + U + U \subseteq V$. Since $\{\hat{S}_n\}$ is equicontinuous at O, there is a neighbourhood W of O such that for each n, $\hat{S}_n(W \cap \bar{Y}) \subseteq U \cap \bar{Y} \subseteq U$. Also, there is at least one point $y \in Y$ such that $\bar{y} - y \in W \cap U$. Then for each n, $\hat{S}_n(y - \bar{y}) = \hat{S}_n(y) - \hat{S}_n(\bar{y}) \subseteq U$. Recalling that $\hat{S}_n$ is an extension of S_n, we also note that $\hat{S}_n(y) = S_n(y)$ and so there exists an integer N such that if $n \geqslant N$, $y - \hat{S}_n(y) \in U$. Hence if $n \geqslant N$, we have

$$\bar{y} - \hat{S}_n(\bar{y}) = \bar{y} - y + y - \hat{S}_n(y) + \hat{S}_n(y) - \hat{S}_n(\bar{y}) \in U + U + U \subseteq V.$$

Therefore (M_i) is a basis of subspaces for $\bar{Y}$, the closed linear span of $\bigcup_1^\infty M_i$, and in fact, since $\{\hat{S}_n\}$ is equicontinuous at O, (M_i) is an e-Schauder basis of subspaces for $\bar{Y}$.

(B) $\Rightarrow$ (C): If (M_i) is an e-Schauder basis of subspaces for $\bar{Y}$, it is evident that $\bar{Y} = X_c = \{x \in X$: there exists a sequence (x_i), such that $x_i \in M_i$ and $\sum_1^n x_i \to x\}$. If P is a family of paranorms which generates $\mathfrak{T}$, then P also generates $\mathfrak{T}_c$, the relative topology for X_c. For $x \in X_c$, define

$$p'(x) = \sup\{p[S_n(x)]: n = 1, 2, \ldots\}.$$

Applying Lemmas 3.3 and 3.4 to the space $(X_c, \mathfrak{T}_c)$ we obtain the fact that P' is a family of paranorms on X_c which generates $\mathfrak{T}_c$. Using this and the fact that $p(x) \leqslant p'(x)$ it is a routine matter to verify that if for $p \in P$ we define

$$p^*(x) = \begin{cases} p'(x) & \text{if } x \in X_c, \\ p(x) & \text{if } x \notin X_c, \end{cases}$$

then $P^* = \{p^*: p \in P\}$ is a family of paranorms on X which generates $\mathfrak{T}$. We show that (M_i) is monotone relative to P^*. Suppose $n < m$ and $x \in X_c$. Then

$$\begin{aligned} p^*[S_n(x)] &= p'[S_n(x)] = \sup_k \{p[S_k(S_n(x))]\} \\ &= \sup_k\{p[S_r(x)]: r = \min(k, n)\} \leqslant \sup\{p[S_r(x)]: r = \min(k, m)\} \\ &= p^*[S_m(x)]. \end{aligned}$$

(C) $\Rightarrow$ (D): Let $P = P^*$, $K_p = 1$, and $q_p = p^*$.

(D) $\Rightarrow$ (E): This is evident from the fact that if q_p is $\mathfrak{T}$-continuous and $\sum_1^n x_i \rightarrow x$, then

$$q_p(\sum_1^n x_i) \rightarrow q_p(x).$$

(E) $\Rightarrow$ (F): Let P be a family of paranorms which satisfies the condition given in (E). Since (M_i) is ω-independent, (M_i) is a basis of subspaces for $(X_c, \mathfrak{T}_c)$, where $X_c = \{x$: there exists a sequence (x_i) with $x_i \in M_i$ such that $\sum_1^n x_i \rightarrow x\}$ and $\mathfrak{T}_c$ is the relative topology for X_c. The equicontinuity of the linear mappings follows easily from the fact that given $p \in P$ there exists a constant K_p and a $\mathfrak{T}$-continuous paranorm q_p such that $p[S_n(x)] \leqslant K_p\, q_p(x)$. Hence (M_i) is an e-Schauder basis of subspaces for $(X_c, \mathfrak{T}_c)$. Let $\mathfrak{B}_c$ be any local base for $\mathfrak{T}_c$ and define $\mathfrak{B}'_c$ as in Lemma 3.1. Then by Lemma 3.3, $\mathfrak{B}'_c$ is also a local base for $\mathfrak{T}_c$. Recall now that in the proof of Lemma 3.1 it was shown that if $V'_c \in \mathfrak{B}'_c$, then $S_n(V'_c) \subseteq V'_c$. In other words, if $x_i \in M_i$, $V'_c \in \mathfrak{B}'_c$, $x \in V'_c$ and $\sum_1^n x_i \rightarrow x$, then $\sum_1^n x_i = S_n(x) \in V'_c$. It is now clear from the definition of X_c that if $\mathfrak{B}$ is any local base for $\mathfrak{T}$ such that $\mathfrak{B}'_c = \{V \cap X_c: V \in \mathfrak{B}\}$, then $\mathfrak{B}$ will have the desired property.

(G) $\Rightarrow$ (A): Let $(X_c, \mathfrak{T}_c)$ be as defined in the proof of (F) $\Rightarrow$ (G). Since X_c contains the linear span of $\cup_1^\infty M_i$ and (M_i) is a basis of subspaces for $(X_c, \mathfrak{T}_c)$, it suffices to show that the family $\{S_n: n = 1, 2, \ldots\}$ is equicontinuous. We are given that there exists a local base $\mathfrak{B}$ for $\mathfrak{T}$ such that if $x \in X_c$, $S_n(x) \in V$, and $V \in \mathfrak{B}$, then $S_{n-1}(x) \in V$. If we let $V_c = V \cap X_c$, then $\mathfrak{B}_c = \{V_c: V \in \mathfrak{B}\}$ is a local base for $\mathfrak{T}_c$. We complete the proof of the theorem by showing that if V_c^0 is the interior of V_c, then $S_n(V_c^0) \subseteq V_c$ for each n. If $x \in V_c^0$, then since $S_n(x) \rightarrow x$, there exists an integer N such that $n \geqslant N$ implies $S_n(x) \in V_c^0 \subseteq V_c$. Hence, using the given condition $N - 1$ times, we conclude that $S_n(x) \in V_c$ for all n.

The above theorem demonstrates the close connection between monotone and e-Schauder bases of subspaces. It also shows that if X is an infinite-dimensional Banach space, then X does not possess a basis of subspaces which is monotone relative to any family of paranorms generating the weak topology, since McArthur and Retherford (**3**, Remark 3, p. 208) have proved that such a space does not possess a weak e-Schauder basis of subspaces.

The requirement that (M_i) be ω-independent can be dropped if $(X, \mathfrak{T})$ is assumed to be a Hausdorff space. This is because the conditions in (A) and (B) clearly imply ω-independence (in any space) and the conditions in (C)–(G) guarantee that if $x_i \in M_i$ and $\sum_1^n x_i \rightarrow 0$, then x_i belongs to the closure of $\{O\}$. The proof of this is easy and will not be given here.

The implication (D) $\Rightarrow$ (B) has been proved by Retherford and McArthur (**4**) for complete locally convex Hausdorff spaces.

If (b_i) is a basis for a linear topological space $(X, \mathfrak{T})$ and (M_i) is the basis of one-dimensional subspaces associated with (b_i), then (M_i) is clearly ω-independent and so by Lemma 4.1, each M_i is Hausdorff. From this it follows that M_i is linearly homeomorphic to the scalar field of X and therefore complete. Thus, the corollary below follows immediately from Theorem 4.3 and the remarks made above.

4.4. COROLLARY. *Let (b_i) denote a sequence of vectors in a Hausdorff linear topological space $(X, \mathfrak{T})$. Then the following are equivalent:*

(a) *(b_i) is an e-Schauder basis for its linear span.*

(b) *(b_i) is an e-Schauder basis for its closed linear span.*

(c) *There exists a family P^* of paranorms generating $\mathfrak{T}$ which has the following property: if $p^* \in P^*$ and (t_i) is a sequence of scalars such that*

$$\sum_1^\infty t_i b_i = x,$$

then $p^*(\sum_1^n t_i b_i)$ is a non-decreasing function of n.

(d) *There exists a family of paranorms P which generates $\mathfrak{T}$ and has the property that if $p \in P$, then there exists a constant K_p and a $\mathfrak{T}$-continuous paranorm q_p on X such that $n < m$ implies that*

$$p(\sum_1^n t_i b_i) \leqslant K_p\, q_p(\sum_1^m t_i b_i)$$

for each sequence of scalars (t_i).

(e) *There exists a local base $\mathfrak{V}$ for $\mathfrak{T}$ such that if $V \in \mathfrak{V}$ and $\sum_1^\infty t_i b_i \in V$, then $\sum_1^n t_i b_i \in V$ for each n.*

The implication (a) $\Rightarrow$ (b) is not true if the "e" is dropped from e-Schauder, even if $(X, \mathfrak{T})$ is a Banach space (see (**5**, Example 5, p. 209)). However, it is well known that a basis for a complete metrizable linear topological space $(X, \mathfrak{T})$ must be an e-Schauder basis for $(X, \mathfrak{T})$. Thus, in such a space (b) is equivalent to (c), (d), and (e), even if the assumption that the basis (b_i) is e-Schauder is dropped.

4.5. COROLLARY. *Let $(X, \mathfrak{T})$ be a metrizable linear topological space. Then a sequence (b_i) in X is an e-Schauder basis for its closed linear span if and only if there exists a paranorm p on X which generates $\mathfrak{T}$ and has the property that for some constant K, if $n < m$, then*

$$p(\sum_1^n t_i b_i) \leqslant Kp(\sum_1^m t_i b_i)$$

for each sequence of scalars (t_i).

The following is similar to a well-known result due to M. M. Grinblyum (**1**).

4.6. COROLLARY. *If $(X, \|\ \|)$ is a normed linear space, and (b_i) is a sequence of vectors in X, then the following are equivalent:*

(i) *(b_i) is an e-Schauder basis for its linear span.*

(ii) *(b_i) is an e-Schauder basis for its closed linear span.*

(iii) *There exists a constant K such that if (t_i) is any sequence of scalars and $n < m$, then*

$$\|\sum_i^n t_i b_i\| \leqslant K \|\sum_1^m t_i b_i\|.$$

Proof. That (i) $\Leftrightarrow$ (ii) and that (iii) $\Rightarrow$ (ii) follow immediately from Corollary 4.4.

To prove (ii) $\Rightarrow$ (iii) we note that the equicontinuity of the family $\{S_n\}$ (in this case, $S_n(\sum_1^\infty t_i b_i) = \sum_1^n t_i b_i$) implies that for some K, $||S_n|| \leqslant K$ for each n. Hence, if $x = \sum_1^m t_i b_i$, and $n < m$, then

$$||\textstyle\sum_1^n t_i b_i|| = ||S_n(x)|| \leqslant ||S_n||\,||x|| \leqslant K\,||\textstyle\sum_1^m t_i b_i||.$$

Requirement (ii) can be weakened by deleting the words "e-Schauder," but only at the expense of requiring X to be a Banach space. This, with (i) deleted, is essentially Grinblyum's theorem. A readily accessible statement and proof of this theorem is found in (**5**, p. 211). It is interesting to note that the only role completeness plays in the proof of this theorem is to ensure the equicontinuity of the family $\{S_n\}$.

The strength and usefulness of this equicontinuity will be again demonstrated in one final corollary (which incidentally follows directly from implication (A) $\Rightarrow$ (B) of Theorem 4.3 and the previously mentioned fact that the one-dimensional subspaces spanned by basis vectors are complete).

4.7. COROLLARY. *An e-Schauder basis for a dense subspace of a linear topological space $(X, \mathfrak{T})$ is an e-Schauder basis for $(X, \mathfrak{T})$.*

The equicontinuity requirement cannot be dropped even if $(X, \mathfrak{T})$ is a Banach space, since $(b_n) = (\cos n\pi)$ is a Schauder basis for the dense subspace of $C[0, 1]$ consisting of all polynomial functions, but (b_n) is not a basis for $C[0, 1]$. (See (**5**, Example 5, p. 209).)

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