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A structure theorem for operators with closed range

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A characterization has previously been given for linear transformations in Hilbert space whose first $N + 1$ powers are partial isometries. An analogous characterization is now obtained for transformations whose first $N + 1$ powers have closed ranges. A hypothesis (that transformations have no isometric part) is found to be unnecessary in previous work.

1. Introduction

A (closed) subspace M of a Hilbert space H is said to reduce a continuous linear transformation T on H if M is invariant under both T and T^*. The operator T is a partial isometry if $\|Tf\| = \|f\|$ for every vector f in H which is orthogonal to the kernel of T, or equivalently, if $T = TT^*T$, [7].

In [8], Halmos and Wallen showed that every partial isometry all of whose positive integral powers are partial isometries is a unique direct sum of unitary operators, pure isometries, pure co-isometries, and truncated shifts, with each type of summand (each index for a truncated shift) occurring at most once. Using [6] and the canonical model of de Branges, Rovnyak [2], an explicit description of the reducing subspaces of partial isometries T such that $T^2, T^3, \ldots, T^{N+1}$ are partial isometries was obtained in [5] under the assumption that T has no isometric part, that is, there is no nonzero vector f in H such that $\|T^n f\| = \|f\|$ for

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every $n = 1, 2, \ldots$. This resulted in a characterization of such partial isometries as unique direct sums of truncated shifts T_j of index j $(j = 1, 2, \ldots, N)$ and partial isometries V with no isometric part whose range includes the kernel of V^N, with each type of summand occurring at most once, thereby extending the Halmos-Wallen Theorem.

In the present paper the problem of obtaining similar results for operators T with closed range whose first $N+1$ powers have closed range is considered. Under the stronger assumptions that the kernels of T^j and T^{*j} are invariant under TT^* and T^*T, respectively, for every $j = 1, 2, \ldots, N$, the above structure theorem is shown to hold essentially for T. A slightly more general concept of truncated shift is necessary, although these generalized shifts enjoy many of the basic properties of truncated shifts.

If $\mathcal{F}$ is a family of subsets of H, then $\vee\{F : F \in \mathcal{F}\}$ will denote the closed span of the union $\cup\{F : F \in \mathcal{F}\}$. For subspaces M and N of H, if $M \perp N$, then $M \oplus N$ will denote the orthogonal direct sum of M and N; if $N \subseteq M$, then $M \ominus N$ will be the orthogonal complement of N in M.

2. Reducing subspaces of generalized truncated shifts

We recall that for a given Hilbert space H_0 and integer $N \geq 1$, a truncated shift of index N is an operator T_N on the direct sum $H_0 \oplus H_0 \oplus \ldots \oplus H_0$ of N copies of H_0 which is defined by $T_N(e_1, e_2, \ldots, e_N) = (e_2, e_3, \ldots, e_N, 0)$, [8]. If $A_1, A_2, \ldots, A_N$ are operators on H_0, the diagonal operator $D = (A_1, A_2, \ldots, A_N)$ on $H_0 \oplus H_0 \oplus \ldots \oplus H_0$ is given by

$$D(e_1, e_2, \ldots, e_N) = (A_1e_1, A_2e_2, \ldots, A_Ne_N) .$$

An operator T on a Hilbert space H will be called a generalized truncated shift of index N if the images $T^{*j} \ker T$ $(j = 0, 1, \ldots, N-1)$ form a family of pairwise orthogonal (closed) subspaces which spans H such that $TT^{*j} \ker T = T^{*j-1} \ker T$ for every $j = 1, 2, \ldots, N-1$.

THEOREM 2.1. *An operator* T *on a Hilbert space* H *is a generalized truncated shift of index* $N \geq 2$ *if and only if there exist invertible operators* $A_1, A_2, \ldots, A_{N-1}$ *defined on a Hilbert space* H_0 *such that* T *is unitarily equivalent to the weighted truncated shift* DT_N *on* $H_0 \oplus H_0 \oplus \ldots \oplus H_0$, *where* $D = (A_1, A_2, \ldots, A_{N-1}, 0)$ *and* T_N *is a truncated shift of index* N.

Proof. A direct computation shows that every weighted truncated shift with invertible weights is a generalized truncated shift of the same index.

Conversely, let T be a generalized truncated shift of index N. Since $\ker T = T^j T^{*j} \ker T$, there exists a unitary operator $U_j : \ker T \to T^{*j} \ker T$ for each $j = 1, 2, \ldots, N-1$. Define the unitary operator

$$U : \ker T \oplus \ker T \oplus \ldots \oplus \ker T \to H$$

by $U(e_1, e_2, \ldots, e_N) = (e_1, U_1 e_2, U_2 e_3, \ldots, U_{N-1} e_N)$. The operator $A_j = U_{j-1}^{-1} T U_j$, where $U_0 = I$, defined on the kernel of T for every $j = 1, 2, \ldots, N-1$, is invertible, and $U^{-1} T U = DT_N$ where $D = (A_1, A_2, \ldots, A_{N-1}, 0)$.

COROLLARY 2.1. *The adjoint of a generalized truncated shift is a generalized truncated shift of the same index.*

Proof. The adjoint of DT_N where $D = (A_1, A_2, \ldots, A_{N-1}, 0)$ as in Theorem 2.1 is unitarily equivalent to $D'T_N$ where $D' = (A_{N-1}^*, A_{N-2}^*, \ldots, A_1^*, 0)$.

COROLLARY 2.2. *The only generalized truncated shifts of index* N *that are partial isometries are the truncated shifts of index* N.

Proof. Let T be a partially isometric generalized truncated shift of index N. For each $j = 1, 2, \ldots, N-1$, define the operator $U_j : \ker T \to T^{*j} \ker T$ by $U_j f = T^{*j} f$ for f in the kernel of T. Then

U_j is unitary for every j, since T^{*j-1} ker T is contained in the range of T, and hence, for every vector f in the kernel of T,

$$\|U_j f\| = \|T^*(T^{*j-1}f)\| = \|T^*(T^{*j-2}f)\| = \ldots = \|f\|$$

for every $j = 1, 2, \ldots, N-1$.

As in the proof of Theorem 2.1, T is unitarily equivalent to DT_N, where $D = (A_1, A_2, \ldots, A_{N-1}, 0)$ and $A_j = U_{j-1}^{-1}TU_j$ $(U_0 = I)$. Let f be in the kernel of T. Since $T^{*j-1}f$ is in the range of T, we have that

$$A_j f = U_{j-1}^{-1}TU_j f = U_{j-1}^{-1}TT^*(T^{*j-1}f) = U_{j-1}^{-1}T^{*j-1}f = U_{j-1}^{-1}U_{j-1}f = f$$

for every $j = 1, 2, \ldots, N-1$. Therefore $A_j = I$ for every j and $DT_N = T_N$.

The next result characterizes the reducing subspace structure of generalized truncated shifts.

THEOREM 2.2. *Let T be an operator on a Hilbert space H such that the kernel of T is invariant under T^kT^{*k} for every $k = 1, 2, \ldots, N-1$, and*

$$H = \vee\{T^{*n} \ker T : n = 0, 1, \ldots, N-1\} .$$

A subspace M of H reduces T if and only if

$$M = \vee\{T^{*n}S : n = 0, 1, \ldots, N-1\}$$

for some unique subspace S of the kernel of T which is invariant under T^kT^{*k} for every $k = 1, 2, \ldots, N-1$. In this case,

$$H \ominus M = \vee\{T^{*n}(\ker T \ominus S) : n = 0, 1, \ldots, N-1\} .$$

Proof. Suppose M reduces T. Let P be the (orthogonal) projection of H onto M. Then

$$M = PH = \vee\{T^{*n}P \ker T : n = 0, 1, \ldots, N-1\} ,$$

and $S = P \ker T \subseteq \ker T$ is closed and invariant under $T^k T^{*k}$ for every $k = 1, 2, \ldots, N-1$.

The form of $H \ominus M$ is obtained similarly, since $H \ominus M = (1-P)H$ and $\ker T = P \ker T \oplus (1-P) \ker T$.

Conversely, suppose M is of the above form. For every $k, n = 0, 1, \ldots, N-1$, $T^{*k}S$ is orthogonal to $T^{*n}(\ker T \ominus S)$. Since $T^{*N} = 0$ and

$$H = \vee\{T^{*n} \ker T : n = 0, 1, \ldots, N-1\} ,$$

it follows that

$$H \ominus M = \vee\{T^{*n}(\ker T \ominus S) : n = 0, 1, \ldots, N-1\} ,$$

and therefore M reduces T .

As above,

$$M = \vee\{T^{*n}P \ker T : n = 0, 1, \ldots, N-1\} ,$$

where P is the projection onto M . Since S is contained in both M and the kernel of T , we have that $S = PS \subseteq P \ker T \subseteq \ker T$. It follows that $P \ker T \ominus S$ is contained in both M and $H \ominus M$, and consequently $S = P \ker T$.

COROLLARY 2.3. *Let* $T = \sum_{j=1}^{N} \oplus \hat{T}_j$ *, where* $\hat{T}_j$ *is a generalized truncated shift of index* j *. A subspace* M *reduces* T *if and only if* $M = \sum_{j=1}^{N} \oplus M_j$*, where* M_j *reduces* $\hat{T}_j$ *.*

Proof. Clearly every subspace of the form $\sum_{j=1}^{N} \oplus M_j$, where M_j reduces $\hat{T}_j$, reduces T .

Let M reduce T and let P denote the projection onto M . Fix j $(1 \leq j \leq N)$. By the representation of T and the definition of generalized truncated shift, it follows that

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$$\ker \hat{T}_j = \ker T \cap \ker T^{*j} \cap \text{range } T^{j-1} .$$

Since P commutes with T^i and T^{*i} for every $i = 1, 2, \ldots, j$, we have that $S = P \ker \hat{T}_j$ is contained in $\ker \hat{T}_j$ and is invariant under $\hat{T}_j^k \hat{T}_j^{*k}$ for every $k = 1, 2, \ldots, j-1$. Therefore by Theorem 2.2, if H_j is the domain of $\hat{T}_j$, then $M_j = \sum_{i=0}^{j-1} \oplus \hat{T}_j^{*i} S$ reduces $\hat{T}_j$, and

$$H_j \ominus M_j = \sum_{i=0}^{j-1} \oplus \hat{T}_j^{*i} \left(\ker \hat{T}_j \ominus S\right) .$$

Since j was arbitrary $(1 \le j \le N)$, we thus conclude that $M = \sum_{j=1}^{N} \oplus M_j$, where M_j reduces $\hat{T}_j$.

COROLLARY 2.4. *Let $T = DT_N$ be a weighted truncated shift of index N defined on $H = H_0 \oplus H_0 \oplus \ldots \oplus H_0$ for some Hilbert space H_0, where $D = \left(A_1, A_2, \ldots, A_{N-1}, 0\right)$ and A_j is one-to-one for every $j = 1, 2, \ldots, N-1$. A subspace M reduces T if and only if*

$$M = S \oplus \sum_{n=1}^{N-1} \oplus \vee\left\{\left(\prod_{1}^{n} A_j\right)^* f : f \in S\right\}$$

for some unique subspace S of H_0 which is invariant under $\left(\prod_{1}^{k} A_j\right)\left(\prod_{1}^{k} A_j\right)^*$ for every $k = 1, 2, \ldots, N-1$. In this case

$$H \ominus M = \left(H_0 \ominus S\right) \oplus \sum_{n=1}^{N-1} \oplus \vee\left\{\left(\prod_{1}^{n} A_j\right)^* f : f \in H_0 \ominus S\right\} .$$

Proof. A direct computation using Theorem 2.2.

REMARK 2.1. In Corollary 2.4, if A_j is invertible for every $j = 1, 2, \ldots, N-1$, then $\left(\prod_{1}^{n} A_j\right)^* S$ and $\left(\prod_{1}^{n} A_j\right)^* \left(H_0 \ominus S\right)$ are closed for every $n = 1, 2, \ldots, N-1$.

REMARK 2.2. In Corollary 2.4, if A_j is one-to-one and hermitian for every $j = 1, 2, \ldots, N-1$, then an induction argument shows that

$$\left(\prod_1^n A_j\right)^* S = S \quad \text{and} \quad \left(\prod_1^n A_j\right)^* (H_0 \ominus S) = H_0 \ominus S$$

for every $n = 1, 2, \ldots, N-1$. In this case, the conditions $\left(\prod_1^k A_j\right)\left(\prod_1^k A_j\right)^* S \subseteq S$ for every $k = 1, 2, \ldots, N-1$ are equivalent to $A_j S \subseteq S$ for every $j = 1, 2, \ldots, N-1$.

Theorem 2.2 may be modified to hold for the case $N = \infty$. As in Corollary 2.4 this case includes the usual weighted shifts with one-to-one operator weights: if $\{A_1, A_2, \ldots\}$ is a uniformly bounded sequence of operators on a complex Hilbert space C, the weighted backward shift W with weights $A_1, A_2, \ldots$ on the Hilbert space $H^2(C) = C \oplus C \oplus \ldots$ of all square-summable sequences $\{a_j\}_{j=0}^{\infty}$, a_j in C, with norm $\|\{a_j\}\|^2 = \sum |a_j|^2$, is defined by $W(a_0, a_1, \ldots) = (A_1 a_1, A_2 a_2, \ldots)$ ([9], [10]). When $A_j = I$ for every $j = 1, 2, \ldots$, W is called the unilateral backward shift and will be denoted $W = U_+^*$.

By a natural extension of Corollary 2.4 and Remarks 2.1 and 2.2 we have the following consequences.

COROLLARY 2.5 (Lambert [9]). *Let W be a weighted backward shift on $H^2(C)$ with invertible weights $A_1, A_2, \ldots$. A subspace M of $H^2(C)$ reduces W if and only if $M = S \oplus \sum_{n=1}^{\infty} \oplus \left(\prod_{j=1}^n A_j\right)^* S$ for some unique subspace S of C which is invariant under $\left(\prod_1^k A_j\right)\left(\prod_1^k A_j\right)^*$ for every $k = 1, 2, \ldots$. In this case*

$$H \ominus M = (H_0 \ominus S) \oplus \sum_{n=1}^{\infty} \oplus \left(\prod_{j=1}^n A_j\right)^* (H_0 \ominus S) .$$

COROLLARY 2.6 (Nikol'skiĭ [10]). *Let W be a weighted backward*

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shift on $H^2(C)$ *with one-to-one, hermitian weights* $A_1, A_2, \ldots$. *A subspace* M *of* $H^2(C)$ *reduces* W *if and only if* $M = \sum_0^\infty \oplus S$ *for some unique subspace* S *of* C *which is invariant under* $A_1, A_2, \ldots$.

3. Operators with closed range

In this section the structure of partial isometries with no isometric part whose first $N + 1$ positive integral powers are partial isometries as obtained in [5] will be extended to certain operators whose first $N + 1$ powers have closed range. The relationship of these results to partial isometries will be determined in the next section. We begin by establishing a technical lemma for these operators.

LEMMA 3.1. *The following are equivalent for an operator* T *with closed range:*

(1) *the kernel of* T *is invariant under* T^jT^{*j} *for every* $j = 1, 2, \ldots, N$;

(2) *the kernel of* T^j *is invariant under* TT^* *for every* $j = 1, 2, \ldots, N$;

(3) *the image* $T^{*j-1} \ker T$ *is invariant under* TT^* *for every* $j = 1, 2, \ldots, N$.

In this case $T^2, T^3, \ldots, T^{N+1}$ *have closed ranges, and the kernel of* T^{j+1} *, for every* $j = 1, 2, \ldots, N$ *, is the orthogonal direct sum of the subspaces* $T^{*i} \ker T$ $(i = 0, 1, \ldots, j)$.

Proof. (1) implies (2). By induction assume that $T^jT^{*j} \ker T \subseteq \ker T$, T^j has closed range, $T^{*j-1} \ker T$ is closed, $\ker T^j = \sum_{i=0}^{j-1} \oplus T^{*i} \ker T$, and $TT^* \ker T^{j-1} \subseteq \ker T^{j-1}$ for every $j = 1, 2, \ldots, N$. It suffices to show that T^{N+1} has closed range, $T^{*N} \ker T$ is closed, $\ker T^{N+1} = \sum_{i=0}^{N} \oplus T^{*i} \ker T$ and

$TT^* \ker T^N \subseteq \ker T^N$.

Let f be in the closure of the range of T^{*N+1} . Since T^{*N} has closed range, $f = T^{*N}g$ for some g in H . Write $g = T^*h + k$ where h is in H and k is in the kernel of T . Then, by (1), $T^{N+1}f = T^{N+1}T^{*N+1}h$, so that $f - T^{*N+1}h$ is in both the kernel of T^{N+1} and the closure of the range of T^{*N+1} . Therefore f is in the range of T^{*N+1} . It follows that T^{*N+1} , and consequently T^{N+1} , have closed ranges.

Similarly, let f be in the closure of $T^{*N} \ker T$. As above, $f = T^{*N}g$ where g is in H , and if $g = T^*h + k$ where h is in H and k is in the kernel of T , then $T^{N+1}f = T^{N+1}T^{*N+1}h$. By (1), $T^{N+1}f = 0$ and hence $T^{*N+1}h = 0$. Therefore $f = T^{*N}k$ is in $T^{*N} \ker T$.

Next note that $T^{*i} \ker T$ is orthogonal to $T^{*j} \ker T$ for all $0 \le i \ne j \le N$, and $\sum_{i=0}^{N} \oplus T^{*i} \ker T$ is contained in the kernel of T^{N+1} by (1). Let f be in $\ker T^{N+1} \ominus \sum_{i=0}^{N} \oplus T^{*i} \ker T$. By assumption f is orthogonal to the kernel of T^N , so that $f = T^{*N}g$, where g is in H , since T^{*N} has closed range. As above, if $g = T^*h + k$, where h is in H and k is in the kernel of T , then $f = T^{*N}k$. Therefore $f = 0$, and $\ker T^{N+1} = \sum_{i=0}^{N} \oplus T^{*i} \ker T$.

Finally since $TT^*\left(T^{*N-1} \ker T\right) \subseteq \ker T^N$ by (1), and $\ker T^N = \ker T^{N-1} \oplus T^{*N-1} \ker T$, it follows that $TT^* \ker T^N \subseteq \ker T^N$.

(2) implies (1). By induction assume (2) and $T^jT^{*j} \ker T \subseteq \ker T$ for every $j = 1, 2, \ldots, N-1$. Then $T^{*N-1} \ker T$ is contained in the kernel of T^N , and therefore

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$$T\left(T^N T^{*N} \ker T\right) = T^N (TT^*) T^{*N-1} \ker T \subseteq T^N \ker T^N = \{0\} .$$

(2) implies (3). By induction assume (2) and

$$TT^*\left(T^{*j-1} \ker T\right) \subseteq T^{*j-1} \ker T$$

for every $j = 1, 2, \ldots, N-1$. Since (2) is equivalent to (1), the above shows that $T^{*i} \ker T$ is closed for every $i = 0, 1, \ldots, N-1$, and $\ker T^N = \sum_{i=0}^{N-1} \oplus T^{*i} \ker T$. Moreover by (2), $TT^*\left(T^{*N-1} \ker T\right) \subseteq \ker T^N$. Thus since $TT^*\left(T^{*N-1} \ker T\right)$ is orthogonal to $T^{*i} \ker T$ for every $i = 0, 1, \ldots, N-2$, it follows that $TT^*\left(T^{*N-1} \ker T\right)$ is contained in $T^{*N-1} \ker T$.

(3) implies (1). An immediate consequence of (3) and the identity $T^j T^{*j} = T^{j-1}(TT^*)T^{*j-1}$.

THEOREM 3.1. *A necessary and sufficient condition that* T *be an operator on Hilbert space with closed range such that the kernels of* T^j *and* T^{*j} *are invariant under* TT^* *and* T^*T *respectively for every* $j = 1, 2, \ldots, N$ *is that* $T = \hat{T}_1 \oplus \hat{T}_2 \oplus \ldots \oplus \hat{T}_N \oplus V$ *where* $\hat{T}_j$ *is a generalized truncated shift of index* j *and* V *is an operator with closed range such that* $VV^*\left(\ker V^j\right) = \ker V^j$ *and* $V^*V\left(\ker V^{*j}\right) = \ker V^{*j}$ *for every* $j = 1, 2, \ldots, N$ *. Moreover, the representation so expressed is unique, and a projection* P *commutes with* T *if and only if* $P = P_1 \oplus P_2 \oplus \ldots \oplus P_N \oplus Q$ *where* P_j *and* Q *are projections which commute with* $\hat{T}_j$ *and* V *respectively* $(j = 1, 2, \ldots, N)$.

Proof. Sufficiency follows directly from Corollary 2.1, Lemma 3.1, and the definition of generalized truncated shift.

To show necessity let $C_j = \ker T \cap T^{j-1} \ker T^*$ for every $j = 1, 2, \ldots, N$. Since the kernel of T^* is invariant under $T^{*j-1}T^{j-1}$ by Lemma 3.1, we have that $C_j = \ker T \cap \ker T^{*j} \cap \text{range } T^{j-1}$ for every $j = 1, 2, \ldots, N$.

The linear manifold $T^{*^{i-1}}C_j$ is invariant under TT^* for all $0 < i < j \le N$: to verify this, fix i and j $(0 < i < j \le N)$ and let $f = T^{j-1}g$ be in C_j where g is in the kernel of T^* . Since $T^k \ker T^*$ is invariant under T^*T for every $k = 1, 2, \ldots, j-2$ by Lemma 3.1, $TT^{*^i}f = TT^{*^{i-1}}(T^*T)T^{j-2}g$ is in the kernel of $T^{*^{j-i+1}}$. Furthermore since $T^{*^{i-1}} \ker T$ is invariant under TT^* by Lemma 3.1, $TT^{*^i}f$ is in $T^{*^{i-1}}\left(\ker T \cap \ker T^{*^j}\right)$. Therefore $TT^{*^i}f$ is in $T^{*^{i-1}}C_j$, since the kernel of T^{*^j} is the orthogonal direct sum of the subspaces $T^k \ker T^*$ $(k = 0, 1, \ldots, j-1)$ and $TT^{*^i}f$ is orthogonal to $T^{*^{i-1}}T^k \ker T^*$ for all $k < j-1$.

Let H_j be the closed span of the images $T^{*^i}C_j$ $(i = 0, 1, \ldots, j-1)$ for every $j = 1, 2, \ldots, N$. Fix j $(1 \le j \le N)$. Then H_j reduces T by the above, and C_j is the kernel of T restricted to H_j . Hence, by Lemma 3.1, $T^{*^i}C_j$ is closed, and since TT^* has closed range and has $T^{*^{i-1}}C_j$ as an invariant subspace, it follows that $TT^*\left(T^{*^{i-1}}C_j\right)$ is closed for every $i = 1, 2, \ldots, j-1$. Since $TT^{*^i}C_j$ is dense in $T^{*^{i-1}}C_j$, we have that $TT^{*^i}C_j = T^{*^{i-1}}C_j$ for every $i = 1, 2, \ldots, j-1$. Therefore, since j was arbitrary, the restriction of T to H_j is a generalized truncated shift $\hat{T}_j$ of index j for every $j = 1, 2, \ldots, N$.

Since the kernel of T is invariant under $T^{j-1}T^{*^{j-1}}$ by Lemma 3.1, it follows that

$$C_j = \left(\ker T \cap \ker T^{*^j}\right) \ominus \left(\ker T \cap \ker T^{*^{j-1}}\right)$$

for every $j = 1, 2, \ldots, N$. Consider the restriction V of T to the orthogonal complement of $\sum_{j=1}^{N} \oplus H_j$. Clearly $V^*V \ker V^{*^j} \subseteq \ker V^{*^j}$ and

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$VV^* \ker V^j \subseteq \ker V^j$ for every $j = 1, 2, \ldots, N$. Thus, since $\ker V = \ker T \cap \operatorname{range} T^N$, the image $V^{*j} \ker V$ is contained in the range of V for every $j = 0, 1, \ldots, N-1$. Therefore $\ker V^j \ominus VV^* \ker V^j$ is contained in both the kernel of V^* and the kernel of V^j, and consequently $\ker V^j = VV^* \ker V^j$ for every $j = 1, 2, \ldots, N$. Similarly $\ker V^{*j} = V^*V \ker V^{*j}$ for every $j = 1, 2, \ldots, N$, since the images $V^j \ker V^*$ $(j = 0, 1, \ldots, N-1)$ are contained in the range of V^*.

Next, let M reduce $T = \sum_{j=1}^{N} \oplus \hat{T}_j \oplus V$ and let P be the projection onto M. By the above construction,

$$\sum_{j=1}^{N} \oplus H_j = \sum_{j=0}^{N-1} \oplus T^{*j}\left(\ker T \cap \ker T^{*N}\right) .$$

Since P commutes with T and T^{*N}, we have that $\ker T \cap \ker T^{*N}$ is invariant under P. Therefore

$$P \sum_{j=1}^{N} \oplus H_j = \sum_{j=0}^{N-1} \oplus T^{*j} P\left(\ker T \cap \ker T^{*N}\right)$$

is contained in $\sum_{j=1}^{N} \oplus H_j$, and thus $M = \hat{M} \oplus N$, where $\hat{M}$ reduces $\sum_{j=1}^{N} \oplus \hat{T}_j$ and N reduces V. The desired form of P now follows from Corollary 2.3.

Finally, uniqueness is a direct consequence of the explicit nature of the above construction.

REMARK 3.1. In the above theorem, it follows from Lemma 3.1 that $T = V$ if and only if, in addition to the invariance conditions on T, the kernel of T^* is orthogonal to the kernel of T^j for every $j = 1, 2, \ldots, N$.

REMARK 3.2. For an operator V with closed range, the conditions $VV^* \ker V^j = \ker V^j$ and $V^*V \ker V^{*j} = \ker V^{*j}$ for every

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$j = 1, 2, \ldots, N$ are equivalent to $V^j V^{*j} \ker V = \ker V$ and $V^{*j} V^j \ker V^* = \ker V^*$ for every $j = 1, 2, \ldots, N$. In the next section, these conditions will be simplified if V is a partial isometry.

Theorem 2.1 and the following result relate the decomposition in Theorem 3.1 to partial isometries. We recall that every operator V on Hilbert space has the polar decomposition $V = AW$ where $A = (VV^*)^{\frac{1}{2}}$ and W is a partial isometry with initial set the orthogonal complement of the kernel of V and final set the closure of the range of V [7].

PROPOSITION 3.1. *Let V be an operator with closed range such that $VV^*(\ker V^j) = \ker V^j$ for every $j = 1, 2, \ldots, N$. Then the partial isometry W in the polar decomposition $V = AW$ of V satisfies $WW^* \ker W^N = \ker W^N$ and therefore $W^2, W^3, \ldots, W^{N+1}$ are partial isometries.*

Proof. By induction assume that $VV^*(\ker V^j) = \ker V^j$ for every $j = 1, 2, \ldots, N$, $WW^* \ker W^{N-1} = \ker W^{N-1}$, and $\ker W^{N-1} = \ker V^{N-1}$. By Lemma 3.1, $\ker V^j = \ker V^{j-1} \oplus V^{*j-1} \ker V$ and

$$\ker W^j = \ker W^{j-1} \oplus W^{*j-1} \ker W$$

for every $j = 1, 2, \ldots, N$. Now $\ker V = \ker W$ and

$$\begin{aligned} W^N\left(V^{*N-1} \ker V\right) &= W^{N-1}(WV^*)V^{*N-2} \ker V \\ &\subseteq W^{N-1}A \ker V^{N-1} \subseteq W^{N-1} \ker V^{N-1} = W^{N-1} \ker W^{N-1} = \{0\} , \end{aligned}$$

since A is the strong limit of a sequence of polynomials in VV^*. Thus we have that $\ker V^N \subseteq \ker W^N$. Similarly $\ker W^N \subseteq \ker V^N$. Therefore $WW^* \ker W^N = WW^* \ker V^N = WW^*\left(VV^* \ker V^N\right) = VV^* \ker V^N = \ker V^N = \ker W^N$.

Finally $W^2, W^3, \ldots, W^{N+1}$ are partial isometries by [3, Theorem 2].

REMARK 3.3. In Theorem 3.1, if, in addition to the invariance conditions on T, the kernel of T^* is contained in the kernel of T^N, then $V = AW$ where $A = (VV^*)^{\frac{1}{2}}$ and W^* is an isometry.

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Let T be an operator on a Hilbert space H and suppose that T has closed range. We recall that the generalized inverse of T, denoted by T^+, is the operator on H defined as follows: if $h = Tf + g$ is the unique decomposition of a vector h in H, where f is orthogonal to the kernel of T and g is in the kernel of T^*, then $T^+h = f$ [1]. By a straightforward induction argument it follows that condition (1) of Lemma 3.1 implies that the identity $(T^{j+1})^+ = (T^+)^{j+1}$ holds on the range of T^{j+1} for every $j = 1, 2, \ldots, N$. The next result characterizes those operators satisfying Theorem 3.1 for which this identity holds everywhere.

PROPOSITION 3.2. *Let T be an operator such that T^{N+1} has closed range for some positive integer N and let $E = T(T^{N+1})^+T^N$. If $\|E\| \le 1$, then $TT^* \ker T^N \subseteq \ker T^N$. Conversely, if $TT^* \ker T^N \subseteq \ker T^N$, T and T^N have closed range, $(T^{N+1})^+ = T^+(T^N)^+$, and $T^{*N}T^N \ker T^* \subseteq \ker T^*$, then $\|E\| \le 1$.*

Proof. Assume $\|E\| \le 1$. Since $E^2 = E$ it follows that E is hermitian. Let f be in the kernel of T^N. Then $T^{*N}(T^{*N+1})^+T^*f = 0$, and hence $(T^{*N+1})^+T^*f = 0$. Therefore T^*f is in the kernel of T^{N+1}. Since f was arbitrary, $TT^* \ker T^N \subseteq \ker T^N$.

For the converse, note that $E = E^* = 0$ on the kernel of T^N. Let f be orthogonal to the kernel of T^N. Then

$$Ef = (TT^+)(T^{N^+}T^N)f = TT^+f$$

and

$$E^*f = (T^{*N}(T^{*N})^+)(T^{*+}T^*)f = (T^{N^+}T^N)(TT^+)f .$$

Since $T^{*N}T^N \ker T^* \subseteq \ker T^*$ and TT^+f is the projection of f onto the range of T, it follows that TT^+f is orthogonal to the kernel of T^N, and consequently $E^*f = Ef$. Therefore E is hermitian and idempotent, and thus $\|E\| \le 1$.

PROPOSITION 3.3. *Let T be a contraction with closed range such*

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that $T^2, T^3, \ldots, T^{N+1}$ *are partial isometries. Then* T *satisfies the hypotheses of Theorem* 3.1.

Proof. Since $(T^{j+1})^+ = T^{*j+1}$ for every $j = 1, 2, \ldots, N$ and $\|T\| \leq 1$, this result follows immediately from Proposition 3.2.

4. Power partial isometries

Partial isometries on Hilbert space all of whose positive integral powers are partial isometries were introduced and characterized in [8]. These results were extended in [5] to partial isometries T whose first $N + 1$ powers are partial isometries under the assumption that T has no isometric part. A direct consequence of the previous sections and of the following lemma makes this assumption unnecessary.

LEMMA 4.1. *The following are equivalent for a partial isometry* V :

(1) $VV^* \ker V^N = \ker V^N$ *for some positive integer* N ;

(2) *the image* $V^{*j-1}(\ker V)$ *is contained in the range of* V *for every* $j = 1, 2, \ldots, N$.

(3) $V^jV^{*j} \ker V = \ker V$ *and* $V^{*j}V^j \ker V^* = \ker V^*$ *for every* $j = 1, 2, \ldots, N$.

In this case, $V^2, V^3, \ldots, V^{N+1}$ *are partial isometries.*

Proof. (1) implies (2). By (1), since V is a partial isometry, $VV^* = I$ on the kernel of V^j for every $j = 1, 2, \ldots, N$, and hence by Lemma 3.1, $\ker V^N = \sum_{j=0}^{N-1} \oplus V^{*j} \ker V$. Therefore $V^{*j-1}(\ker V) \subseteq \text{range } V$ for every $j = 1, 2, \ldots, N$.

(2) implies (3). An immediate consequence of (2) and the identities $V^jV^{*j} = V^{j-1}(VV^*)V^{*j-1}$ and $V^{*j}V^j = V^{*j-1}(V^*V)V^{j-1}$.

(3) implies (1). By Lemma 3.1, $\ker V^N = \sum_{j=0}^{N-1} \oplus V^{*j} \ker V$, and by [3, Theorem 2], $V^2, V^3, \ldots, V^{N+1}$ are partial isometries. Fix j

$(0 \le j \le N-1)$. Let f be in V^{*j} ker V . Then since V^jV^{*j} is the projection onto the range of V^j , we have that $f = V^{*j}g$ where $g = V^jV^{*j}g = V^jf$ is in the kernel of V . Moreover, $V^j(VV^*)f = V^{j+1}V^{*j+1}g = g = V^jf$. Therefore $(1-VV^*)f$ is in both the kernel of V^j and the kernel of V^* , and consequently, by (3), $VV^*f = f$. Since f and j were arbitrary, we conclude that VV^* ker V^N = ker V^N .

The following theorem is a consequence of Proposition 3.3, Theorem 3.1, Corollary 2.2, and Lemma 4.1.

THEOREM 4.1. *A necessary and sufficient condition that* T *be a partial isometry on Hilbert space such that* $T^2, T^3, \ldots, T^{N+1}$ *are partial isometries is that* $T = T_1 \oplus T_2 \oplus \ldots \oplus T_N \oplus V$ *where* T_j *is a truncated shift of index* j *and* V *is a partial isometry such that the kernel of* V^N *is contained in the range of* V . *Moreover, this representation is unique, and a projection* P *commutes with* T *if and only if* $P = P_1 \oplus P_2 \oplus \ldots \oplus P_N \oplus Q$ *where* P_j *and* Q *are projections which commute with* T_j *and* V *respectively* $(j = 1, 2, \ldots, N)$.

Theorems 3.1 and 4.1 have natural extensions to the case $N = \infty$ as the following result indicates.

COROLLARY 4.1 (Halmos-Wallen). T^j *is a partial isometry on Hilbert space for every* $j = 1, 2, \ldots$ *if and only if* $T = \left(\sum_1^\infty \oplus T_j\right) \oplus U_{+1}^* \oplus U_{+2} \oplus U$, *where* T_j *is a truncated shift of index* j , U_+ *is a unilateral shift, and* U *is unitary. Moreover, the representation so expressed is unique.*

Proof. The proof follows from Theorem 4.1 as in the proof of [5, Corollary 3.2].

COROLLARY 4.2 (Fishel [4]). *Let* T *be a partial isometry on Hilbert space. Then* $T = U_{+1}^* \oplus U_{+2} \oplus U$ *uniquely, where* U_{+i} *is a unilateral shift and* U *is unitary, if and only if the kernel of* T^* *is orthogonal to the kernel of* T^j *for every* $j = 1, 2, \ldots$.

Proof. Lemma 4.1, Theorem 4.1, Remark 3.1, and Corollary 4.1.

COROLLARY 4.3. *Let T be a partial isometry on Hilbert space. Then $T = T_1 \oplus T_2 \oplus \ldots \oplus T_N \oplus U_+^* \oplus U$ uniquely, where T_j is a truncated shift of index j, U_+ is a unilateral shift, and U is unitary, if and only if $T^2, T^3, \ldots, T^{N+1}$ are partial isometries and the kernel of T^* is contained in the kernel of T^N.*

Proof. Theorem 4.1, Remark 3.3, and Corollary 4.1.

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