A GENERALISATION OF THE STIELTJES TRANSFORM

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(Received 31st January 1963)

1. Transforms of the type

$$f(p) = \int_0^\infty F(t, p)\phi(t)dt \quad \text{.............................(1.1)}$$

have long been known. If we take

$$F(t, p) = (t+p)^{-\sigma} e^{\frac{1}{2}(t+p)} W_{k,\mu}(t+p), \quad \text{........................(1.2)}$$

where $W_{k,\mu}(x)$ denotes the Whittaker function, we obtain the following transform:

$$f(p) = \int_0^\infty (t+p)^{-\sigma} e^{\frac{1}{2}(t+p)} W_{k,\mu}(t+p)\phi(t)dt. \quad \text{.................(1.3)}$$

When $k+\mu = \frac{1}{2}$ and $k-\sigma = -1$,

$$F(t, p) = \frac{1}{t+p}$$

and (1.3) reduces to the Stieltjes transform

$$f(p) = \int_0^\infty \frac{1}{t+p}\,\phi(t)dt.$$

We shall denote (1.3) symbolically as

$$f(p) \xrightarrow[k,\,\mu]{\sigma} \phi(t).$$

2. Inversion formula Let

$$\phi(t) = \begin{cases} 0(t^{\eta_1}), \ R(\eta_1) > 0 \text{ for small } t, \\ 0(t^{\eta_2}), \ R(\eta_2) < 0 \text{ for large } t, \end{cases}$$

where $|\arg t| < \pi$ and $\phi(t)$ is of bounded variation in $(0, \infty)$.

Multiplying by $p^{\rho-1}$ and then integrating from 0 to ∞ with respect to p and assuming that $\psi(\rho)$, the Mellin transform of $f(p)$, exists, we get

$$\int_0^\infty f(p)p^{\rho-1}dp = \psi(\rho),$$

$$= \int_0^\infty p^{\rho-1}dp \int_0^\infty (t+p)^{-\sigma} e^{\frac{1}{2}(t+p)} W_{k,\mu}(t+p)\phi(t)dt,$$

$$= \int_0^\infty \phi(t)dt \int_0^\infty p^{\rho-1}(t+p)^{-\sigma} e^{\frac{1}{2}(t+p)} W_{k,\mu}(t+p)dt,$$

https://doi.org/10.1017/S0013091500011111 Published online by Cambridge University Press

on changing the order of integration. Now, by (**1**), p. 412 (51),

$$\int_0^\infty x^{\rho-1}(a+x)^{-\rho}e^{\frac{1}{2}x}W_{k,\mu}(a+x)dx = \frac{\Gamma(\rho)a^\rho e^{-\frac{1}{2}a}}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-k-\mu)} \times G_{23}^{31}\left(a \,\middle|\, \begin{matrix} k-\sigma+1, & 0 \\ -e, \frac{1}{2}+\mu-\sigma, & \frac{1}{2}-\mu-\sigma \end{matrix}\right),$$

where $|\arg a| < \pi$, $0 < R(\rho) < R(\sigma - k)$. Hence we get

$$\psi(\rho) = \frac{\Gamma(\rho)}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-k-\mu)}\int_0^\infty t^\rho \phi(t) G_{23}^{31}\left(t \,\middle|\, \begin{matrix} t-\sigma+1, & 0 \\ -\rho, \frac{1}{2}+\mu-\sigma, & \frac{1}{2}-\mu-\sigma \end{matrix}\right) dt, \quad \text{.........(2.2)}$$

where $|\arg t| < \pi$, $0 < R(\rho) < R(\sigma - k)$.
Now, applying Mellin's inversion formula, we get

$$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi(\rho)t^{-\rho}d\rho.$$

3. Uniqueness theorem

Let ϕ_1 and ϕ_2 be continuous in $t \geqq 0$ and

$$f(p) \xrightarrow[k,\,\mu]{\sigma} \phi_1(t)$$

and also

$$f(p) \xrightarrow[k,\,\mu]{\sigma} \phi_2(t).$$

Then

$$\phi_1(t) \equiv \phi_2(t).$$

In future work the equation (2.2) will be called the SA-transform of $\phi(t)$.

The author is grateful to Prof. Brij Mohan for his help and guidance in the preparation of this note.

REFERENCE

(**1**) H. Bateman, *Tables of integral transforms*, Vol. II, 411-412.

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