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A Generalised Kummer-Type Transformation for the ${}_pF_p(x)$ Hypergeometric Function

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Abstract. In a recent paper, Miller derived a Kummer-type transformation for the generalised hypergeometric function ${}_pF_p(x)$ when pairs of parameters differ by unity, by means of a reduction formula for a certain Kampé de Fériet function. An alternative and simpler derivation of this transformation is obtained here by application of the well-known Kummer transformation for the confluent hypergeometric function corresponding to $p = 1$.

1 Introduction

The generalised hypergeometric function ${}_pF_p(x)$ is defined for complex values of x by the series

$$(1.1)\qquad {}_pF_p\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix}\,\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_p)_k} \frac{x^k}{k!} \qquad (|x| < \infty),$$

where for nonnegative integer k the Pochhammer symbol or ascending factorial $(a)_k$ is defined by $(a)_0 = 1$ and for $k \geq 1$ by $(a)_k = a(a+1)\cdots(a+k-1)$. However, for all integers k we write simply

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

We shall adopt the usual convention of writing the sequence $(a_1, \dots, a_p)$ simply as (a_p) and the product of p Pochhammer symbols as

$$((a_p))_k \equiv (a_1)_k \cdots (a_p)_k,$$

with an empty product ($p = 0$) reducing to unity. The function ${}_pF_p(x)$, with an equal number of numeratorial and denominatorial parameters, is the higher order extension of the familiar confluent hypergeometric function ${}_1F_1(x)$. This latter function satisfies the well-known Kummer transformation given by

$$(1.2)\qquad {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}\,\middle|\, x\right) = e^x\, {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix}\,\middle|\, -x\right).$$

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In [7], a Kummer-type transformation for the ${}_2F_2(x)$ function with three independent parameters was given by

$$(1.3) \qquad {}_2F_2\left(\begin{matrix} a, & c+1 \\ b, & c \end{matrix} \,\middle|\, x\right) = e^x\,{}_2F_2\left(\begin{matrix} b-a-1, & \xi+1 \\ b, & \xi \end{matrix} \,\middle|\, -x\right),$$

where the parameter ξ depends on a nonlinear combination of the parameters a, b, and c in the form

$$(1.4) \qquad \xi = \frac{c(1+a-b)}{a-c} \quad (a \neq c, b-a-1 \neq 0).$$

If we let $c \to \infty$, or put $b = c+1$, then (1.3) reduces to Kummer's transformation (1.2). A more restrictive form of (1.3) when $c = \frac{1}{2}a$, corresponding to only two independent parameters with $\xi = 1+a-b$, had been obtained earlier in [2,4]. Alternative proofs of (1.3) have been given in [5] using a reduction formula for the Kampé de Fériet double hypergeometric function and in [1,9] using different methods. In the case of four independent parameters a, b, c, and d, the corresponding transformation no longer involves a single ${}_2F_2$ function but an infinite sum [7] given by

$${}_2F_2\left(\begin{matrix} a, & c \\ b, & d \end{matrix} \,\middle|\, x\right) = e^x \sum_{n=0}^{\infty} \frac{(d-c)_n}{(d)_n\, n!} (-x)^n\, {}_2F_2\left(\begin{matrix} b-a, & c \\ b, & d+n \end{matrix} \,\middle|\, -x\right)$$

valid for complex x provided $b,\ d \neq 0, -1, -2, \dots$.

Recently, Miller [6] obtained an extension of the transformation (1.3) to the higher-order confluent hypergeometric function ${}_{p+1}F_{p+1}(x)$ with $p \geq 1$ in the form

$$(1.5) \qquad {}_{p+1}F_{p+1}\left(\begin{matrix} a, & (c_p+1) \\ b, & (c_p) \end{matrix} \,\middle|\, x\right) = e^x\,{}_{p+1}F_{p+1}\left(\begin{matrix} b-a-p, & (\xi_p+1) \\ b, & (\xi_p) \end{matrix} \,\middle|\, -x\right),$$

where the $\xi_1, \dots, \xi_p$ are nonvanishing zeros of a certain associated parametric polynomial of degree p defined in Section 2. The transformation (1.5) was obtained from a summation formula for a ${}_{p+2}F_{p+1}$ hypergeometric function of unit argument combined with a reduction identity for a certain Kampé de Fériet double hypergeometric function. The purpose of this note is to provide a more direct proof of (1.5) and to show how it follows as a consequence of Kummer's transformation (1.2).

2 Proof of the Transformation (1.5)

The notation $\left\{ {n \atop k} \right\}$ will be employed to denote the Stirling number of the second kind. These numbers represent the number of ways to partition n objects into k nonempty sets and arise for nonnegative integers n in the generating relation [3]

$$(2.1) \qquad x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)\cdots(x-k+1), \quad \left\{ {n \atop 0} \right\} = \delta_{0n},$$

where δ_{0n} is the Kronecker symbol and, when $k = 0$, the product

$$x(x-1)\cdots(x-k+1)$$

is to be interpreted as 1. We also introduce the coefficients A_k appearing in the descending factorial expansion of the product $(c_1 + n)\cdots(c_p + n)$ as follows. Let

$$(c_1+n)\cdots(c_p+n) = \sum_{j=0}^{p} s_{p-j} n^j,$$

where $s_0 = 1$ and the s_i $(1 \le i \le p)$ are sums of all possible products of i distinct elements from the set $\{c_1, \ldots, c_p\}$. Then from (2.1), we have

$$\begin{aligned}(c_1+n)\cdots(c_p+n) &= \sum_{j=0}^{p} s_{p-j} \sum_{k=0}^{j} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} n(n-1)\cdots(n-k+1) \\ &= \sum_{k=0}^{p} A_k n(n-1)\cdots(n-k+1)\end{aligned} \tag{2.2}$$

upon reversal of the order of summation, where

$$A_k = \sum_{j=k}^{p} s_{p-j} \left\{ \begin{matrix} j \\ k \end{matrix} \right\}, \quad A_0 = \prod_{j=1}^{p} c_j, \quad A_p = 1. \tag{2.3}$$

Defining

$$F \equiv {}_{p+1}F_{p+1}\left(\begin{matrix} a, & (c_p+1) \\ b, & (c_p) \end{matrix} \middle| x \right),$$

we now express F as a series in powers of x by (1.1). Since $(c+1)_n/(c)_n = (c+n)/c$ we can write, using (2.2) and (2.3),

$$\begin{aligned} F &= \sum_{n=0}^{\infty} \frac{(a)_n\, x^n}{(b)_n\, n!} \frac{c_1+n}{c_1} \cdots \frac{c_p+n}{c_p} \\ &= \frac{1}{A_0} \sum_{n=0}^{\infty} \frac{(a)_n\, x^n}{(b)_n\, n!} \sum_{k=0}^{p} A_k n(n-1)\cdots(n-k+1) \\ &= \frac{1}{A_0} \sum_{k=0}^{p} A_k \sum_{n=k}^{\infty} \frac{(a)_n\, x^n}{(b)_n\, (n-k)!}, \end{aligned}$$

upon reversal of the order of summation and where we have replaced the lower limit of summation in the inner series by $n = k$. With the change of summation index

$m = n - k$ and use of the identity $(a)_{m+k} = (a+k)_m\,(a)_k$, we then find

$$F = \frac{1}{A_0}\sum_{k=0}^{p} x^k A_k \frac{(a)_k}{(b)_k}\sum_{m=0}^{\infty}\frac{(a+k)_m}{(b+k)_m}\frac{x^m}{m!} \tag{2.4}$$

$$= \frac{1}{A_0}\sum_{k=0}^{p} x^k A_k \frac{(a)_k}{(b)_k}\,{}_1F_1\left(\begin{matrix} a+k \\ b+k\end{matrix}\middle|\, x\right).$$

This has expressed our ${}_{p+1}F_{p+1}(x)$ function as a finite sum of ${}_1F_1(x)$ functions.

Application of Kummer's theorem (1.2) to (2.4) then yields

$$F = \frac{e^x}{A_0}\sum_{k=0}^{p} x^k A_k \frac{(a)_k}{(b)_k}\,{}_1F_1\left(\begin{matrix} b-a \\ b+k\end{matrix}\middle|\, -x\right) \tag{2.5}$$

$$= \frac{e^x}{A_0}\sum_{k=0}^{p}(-1)^k A_k \frac{(a)_k}{(b)_k}\sum_{n=0}^{\infty}\frac{(b-a)_n}{(b+k)_n}\frac{(-x)^{n+k}}{n!}.$$

Noting the identities

$$\frac{1}{(n-k)!} = \frac{(-1)^k(-n)_k}{n!}, \quad (b+k)_{n-k} = \frac{(b)_n}{(b)_k}$$

and

$$(b-a)_{n-k} = \frac{(\lambda)_n(\lambda+n)_{p-k}}{(\lambda)_p}, \quad \lambda \equiv b-a-p,$$

we now make the change of index $n \mapsto n-k$ in (2.5). Then

$$F = \frac{e^x}{A_0(\lambda)_p}\sum_{k=0}^{p} A_k(a)_k \sum_{n=k}^{\infty}\frac{(-n)_k(-x)^n}{(b)_n\, n!}(\lambda)_n(\lambda+n)_{p-k} \tag{2.6}$$

$$= \frac{e^x}{A_0(\lambda)_p}\sum_{n=0}^{\infty}\frac{(\lambda)_n(-x)^n}{(b)_n\, n!}\sum_{k=0}^{p} A_k(a)_k(-n)_k(\lambda+n)_{p-k},$$

upon reversal of the order of summation and where we have replaced the lower summation limit $n = k$ by $n = 0$ on account of the factor $(-n)_k$, which vanishes for $n < k$.

The finite sum appearing in (2.6) can be expressed by means of (2.3) as

$$\sum_{k=0}^{p} A_k(a)_k(-n)_k(\lambda+n)_{p-k} = \sum_{k=0}^{p}\sum_{j=k}^{p} s_{p-j}\left\{\begin{matrix} j \\ k\end{matrix}\right\}(a)_k(-n)_k(\lambda+n)_{p-k}$$

$$= \sum_{j=0}^{p} s_{p-j}\sum_{k=0}^{j}\left\{\begin{matrix} j \\ k\end{matrix}\right\}(a)_k(-n)_k(\lambda+n)_{p-k} \equiv Q_p(-n),$$

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where $Q_p(t)$ is the associated parametric polynomial defined in [6, Corollary 1] and we have carried out a reversal of the order of summation. The function $Q_p(-n)$ is a polynomial in n of degree p. Some straightforward algebra shows that

$$Q_p(-n) = \alpha_0 n^p + \alpha_1 n^{p-1} + \cdots + \alpha_{n-1} n + \alpha_n,$$

where, in particular, the coefficients

$$\alpha_n = A_0(\lambda)_p, \quad \alpha_0 = \sum_{k=0}^{p} (-1)^k A_k(a)_k = (c_1 - a)\cdots(c_p - a)$$

by (2.2). If we let the nonvanishing zeros (which requires the condition $(\lambda)_p \neq 0$) of $Q_p(t)$ be $\xi_1, \dots, \xi_p$, then $\alpha_n = \alpha_0 \xi_1 \cdots \xi_p$. Assuming $c_j \neq a$ $(1 \leq j \leq p)$ so that $\alpha_0 \neq 0$, we can then write, following [6, Lemma 4],

$$\begin{aligned} Q_p(-n) &= \alpha_0 (n + \xi_1) \cdots (n + \xi_p) \\ &= \alpha_n \frac{n + \xi_1}{\xi_1} \cdots \frac{n + \xi_p}{\xi_p} \\ &= \alpha_n \frac{(1 + \xi_1)_n}{(\xi_1)_n} \cdots \frac{(1 + \xi_p)_n}{(\xi_p)_n}. \end{aligned}$$

Hence, (2.6) can be expressed in the form

$$F = e^x \sum_{n=0}^{\infty} \frac{(\lambda)_n \, ((\xi_p + 1))_n}{(b)_n \, ((\xi_p))_n} \frac{(-x)^n}{n!}.$$

This then finally yields the desired transformation, which we record in the following theorem.

Theorem 1 *For nonnegative integer p and $\lambda \equiv b - a - p$,*

$$(2.7) \qquad {}_{p+1}F_{p+1}\left(\begin{matrix} a, & (c_p + 1) \\ b, & (c_p) \end{matrix} \,\middle|\, x \right) = e^x \, {}_{p+1}F_{p+1}\left(\begin{matrix} b - a - p, & (\xi_p + 1) \\ b, & (\xi_p) \end{matrix} \,\middle|\, -x \right),$$

provided $(\lambda)_p \neq 0$ and $c_j \neq a$ $(1 \leq j \leq p)$, where $\xi_1, \dots, \xi_p$ are nonvanishing zeros of the associated parametric polynomial $Q_p(t)$ of degree p given by

$$(2.8) \qquad Q_p(t) = \sum_{j=0}^{p} s_{p-j} \sum_{k=0}^{j} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (a)_k (t)_k (\lambda - t)_{p-k},$$

and the s_{p-j} $(0 \leq j \leq p)$ are determined by the generating relation

$$(c_1 + n) \cdots (c_p + n) = \sum_{j=0}^{p} s_{p-j} n^j.$$

Note that when all of the $c_j = c$, then $s_{p-j} = \binom{p}{j} c^{p-j}$.

3 Discussion

In the case of the hypergeometric function on the left-hand side of (2.7), with corresponding numeratorial and denominatorial parameters differing by unity, the exponential factor that appears in the transformation is e^x. That this is the correct exponential factor to extract, even in the most general case of

$$ {}_pF_p\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix}\,\middle|\, x\right), $$

can be seen from the asymptotic growth of the latter for large x. From [8, §2.3], we have exponential growth as $|x| \to \infty$ in the right half-plane given by

$$ {}_pF_p\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix}\,\middle|\, x\right) \sim \prod_{r=1}^{p} \frac{\Gamma(a_r)}{\Gamma(b_r)}\, x^{\vartheta} e^x \quad (|\arg x| < \tfrac{1}{2}\pi), $$

where the parameter $\vartheta = \sum_{r=1}^{p}(a_r - b_r)$, and algebraic growth (with possible terms in $\log x$ depending on the values of the a_r) in the left half-plane $|\arg(-x)| < \frac{1}{2}\pi$.

When $p = 1$ and $c_1 = c$, the polynomial $Q_1(t)$ in (2.8) is

$$ Q_1(t) = (a - c)t + c(b - a - 1) $$

and the zero $\xi_1 = \xi$ is given by (1.4). The transformation (2.7) in this case then correctly reduces to that in (1.3).

In the case $p = 2$, we have [6]

$$ Q_2(t) = \alpha t^2 - ((\alpha + \beta)\lambda + \beta)t + c_1 c_2 \lambda(\lambda + 1), \tag{3.1} $$

where $\lambda = b - a - 2$ and

$$ \alpha = (c_1 - a)(c_2 - a), \quad \beta = c_1 c_2 - a(a + 1). $$

For real parameters a, b, c_1, and c_2, we note that the zeros ξ_1, ξ_2 can be real or a complex conjugate pair. For example, if $a = \frac{1}{2}$, $b = 1$, $c_1 = \frac{3}{4}$, and $c_2 = \frac{5}{4}$ then

$$ Q_2(t) = \tfrac{1}{16}(3t^2 + 6t + \tfrac{45}{4}), $$

so that $\xi_{1,2} = -1 \pm \frac{1}{2} i\sqrt{11}$. We then find the Kummer-type transformation

$$ {}_3F_3\left(\begin{matrix} \frac{1}{2}, & \frac{7}{4}, & \frac{9}{4} \\ 1, & \frac{3}{4}, & \frac{5}{4} \end{matrix}\,\middle|\, x\right) = e^x\, {}_3F_3\left(\begin{matrix} -\frac{3}{2}, & \frac{1}{2}i\sqrt{11}, & -\frac{1}{2}i\sqrt{11} \\ 1, & -1 + \frac{1}{2}i\sqrt{11}, & -1 - \frac{1}{2}i\sqrt{11} \end{matrix}\,\middle|\, -x\right). $$

Finally, we comment on the situation when the difference Δ_j between corresponding pairs of numeratorial and denominatorial parameters c_j exceeds unity. For example, if $p = 1$ and $\Delta_1 = 2$, then

$$ \begin{aligned} {}_2F_2\left(\begin{matrix} a, & c + 2 \\ b, & c \end{matrix}\,\middle|\, x\right) &= {}_3F_3\left(\begin{matrix} a, & c + 1, & c + 2 \\ b, & c, & c + 1 \end{matrix}\,\middle|\, x\right) \\ &= e^x\, {}_3F_3\left(\begin{matrix} b - a - 2, & \xi_1 + 1, & \xi_2 + 1 \\ b, & \xi_1, & \xi_2 \end{matrix}\,\middle|\, -x\right), \end{aligned} $$

where ξ_1, ξ_2 are the zeros of the quadratic $Q_2(t)$ in (3.1) with $c_1 = c$ and $c_2 = c + 1$. If $\Delta_1 = m$, where m is a positive integer, then we have

$$(3.2)\qquad {}_2F_2\left(\begin{matrix} a, & c+m \\ b, & c \end{matrix}\,\middle|\, x\right) = {}_{m+1}F_{m+1}\left(\begin{matrix} a, & c+1, c+2, \dots, c+m \\ b, & c, c+1, \dots, c+m-1 \end{matrix}\,\middle|\, x\right) = e^x\,{}_{m+1}F_{m+1}\left(\begin{matrix} b-a-1, & (\xi_m+1) \\ b, & (\xi_m) \end{matrix}\,\middle|\, -x\right),$$

where $\xi_1, \dots, \xi_m$ are the zeros of the polynomial $Q_m(t)$ with $c_r = c + r - 1$ $(1 \le r \le m)$. Similarly, if the difference associated with the parameters c_j is $\Delta_j = m_j$, where the m_j are positive integers, then we find in the case $p = 2$, for example, that

$$(3.3)\qquad {}_3F_3\left(\begin{matrix} a, & d_1+m_1, & d_2+m_2 \\ b, & d_1, & d_2 \end{matrix}\,\middle|\, x\right)$$
$$= {}_{\mu+1}F_{\mu+1}\left(\begin{matrix} a, & d_1+1, \dots, d_1+m_1, & d_2+1, \dots, d_2+m_2 \\ b, & d_1, \dots, d_1+m_1-1, & d_2, \dots, d_2+m_2-1 \end{matrix}\,\middle|\, x\right)$$
$$= e^x\,{}_{\mu+1}F_{\mu+1}\left(\begin{matrix} b-a-\mu, & (\xi_\mu+1) \\ b, & (\xi_\mu) \end{matrix}\,\middle|\, -x\right),$$

where $\mu = m_1 + m_2$ and $\xi_1, \dots, \xi_\mu$ are the zeros of the polynomial $Q_\mu(t)$ in (2.8) with

$$c_r = d_1 + r - 1 \quad (1 \le r \le m_1), \qquad c_{m_1+r} = d_2 + r - 1 \quad (1 \le r \le m_2).$$

Extension to higher order ${}_pF_p(x)$ is straightforward.

The results in (3.2) and (3.3) express a ${}_pF_p(x)$ function, when corresponding parameters differ by more than unity, in terms of higher-order hypergeometric functions with argument $-x$. In the case of ${}_2F_2(x)$, however, an alternative representation for the left-hand side of (3.2) can be given in terms of a finite number of ${}_2F_2(-x)$ functions as [7]

$${}_2F_2\left(\begin{matrix} a, & c+m \\ b, & c \end{matrix}\,\middle|\, x\right) = e^x \sum_{k=0}^{m} \binom{m}{k} \frac{x^k}{(c)_k}\,{}_2F_2\left(\begin{matrix} b-a, & c+m \\ b, & c+k \end{matrix}\,\middle|\, -x\right).$$

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