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# THE UNTRACEABLE EVENTS METHOD FOR ABSORBING PROCESSES

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**Abstract**

In this paper we propose a new method of determining the stability of queueing systems. We attain it using the absorbing process and introduce the untraceable events method to show the existence of the absorbing process. The advantage of our method is that we are able to discuss the stability of various variables for both discrete and continuous parameters in a general framework with nonstationary input. An untraceable event has the property that the state loses the memory of its origin. In a concrete model, we use the boundedness of the state at an epoch in time with respect to the initial condition and choose the form of the untraceable event corresponding to the input distribution.

*Keywords:* Absorbing process; untraceable event; Borovkov event; continuous-time parameter; asymptotic mean stationarity; queueing system

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## 1. Introduction

In the fundamental definition, a model of a queueing system or related stochastic phenomenon is called stable if the probability distribution over the states, as a function of time, converges to a proper distribution which is independent of initial conditions. The term 'stability' is sometimes used in several wider senses (see, e.g. [17] and [7, p. 30]). The stability problem is concerned with determining whether or not a particular model is stable in such fundamental or wider senses.

Much has been written about the stability of queueing systems. Some authors have studied the regenerative process and many have applied Markov theory to it (see [18] and [8]). This paper belongs to the group stemming from Loynes' paper [13]. We select the state vector $\boldsymbol{x}_t$ at time epoch $t$ such that, from an arbitrary time $r$ on, the sample path $\boldsymbol{x}_t$, $t \geq r$, is completely determined by the initial state, $\boldsymbol{x}_r = \boldsymbol{a}$, and input, $\phi$. We denote this sample path by $\boldsymbol{x}_t^r(\phi, \boldsymbol{a})$. Before introducing the probability, a new state process,

$$\boldsymbol{x}_t^*(\phi) = \lim_{r \to -\infty} \boldsymbol{x}_t^r(\phi, \boldsymbol{a}),$$

is constructed, given that this limit does not depend on the initial state $\boldsymbol{a} \in \bar{X}$. Thus, the parameter space $Z$ on which the times $r$ and $t$ move must contain the past, i.e.

$$Z = \mathbb{N} = \{\dots, -1, 0, 1, \dots\} \quad \text{or} \quad Z = \mathbb{R} = (-\infty, \infty).$$

Loynes [13] dealt with customer-stationary input and Kalähne [11] dealt with time-stationary input. Their techniques are restricted to special models. Borovkov [2], [3, Chapter 7], [4]

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introduced the renewing event $A(t)$ with parameter $L$ as the set of input elements $\phi$ such that the state after $t + L$ is represented by the part of the input after $t$, that is, the future state is functionally independent of the past input. He showed that if $\mathrm{P}(A(0)) > 0$ then the stability holds under a customer-stationary input. Borovkov applied his method to only a few simple models, and related papers [1], [6], [9], [10], [14], and [15] followed.

Since the customer stationarity of the input is a substantial factor in Borovkov's proof, it seems difficult or awkward to extend his result directly to the model with multiple inputs or to the model with nonstationary or time-stationary input. In this paper we point out that we can overcome these difficulties by showing the existence of the absorbing process introduced in [16]. This implies a model setting different from Borovkov's. Moreover, from the viewpoint of the absorbing process, the important property of $A(t)$ is not that it be determined by the future input but that it have lost the memory of the initial epoch $r$ and the initial state $\boldsymbol{x}_r = \boldsymbol{a}$. We give $A(t)$ a new definition, in harmony with the absorbing process, and call it the untraceable event. The untraceable event and the renewing event are similar in that they both use the functional independence of the past in most applications. These events are different in that we can show an example of the untraceable event without such independence.

Discussing stability via the absorbing process $\boldsymbol{x}_t^*(\phi)$ has the following advantages. We can deal with complicated state spaces. We can give a simple proof of Borovkov's result. We can treat nonstationarity and continuous-time parameters. We need not specialize the distribution of the input a priori. We need only prove the existence of the absorbing process, because it inherits certain properties, such as ergodicity, stationarity, periodicity, asymptotic stationarity, and asymptotic mean stationarity (AMS) (see [16, Theorem 6.1 and Theorem 7.2]), from the input process. Conversely, we can give these properties to the absorbing process by assuming them of the input. Since the real process $\boldsymbol{x}_t^r(\phi, \boldsymbol{a})$ with fixed $r$ and $\boldsymbol{a}$ is absorbed into the absorbing process, its many properties are derived from the absorbing process. In particular, the existence of its limiting distribution is guaranteed if we add the assumption of the stationarity or asymptotic stationarity of the input to the existence condition of the absorbing process.

The continuous-parameter case is particularly useful. The representation of the input by the point process on the real line is convenient for models with multiple inputs, like polling systems. Moreover, the departure point process is represented by the counting measure, which is the mapping $h(\{\boldsymbol{x}_t^*(\phi)\})$ of the total absorbing process $\{\boldsymbol{x}_t^*(\phi)\colon -\infty < t < \infty\}$. Therefore, it inherits the properties, such as ergodicity and stationarity, of the input.

In deriving the many kinds of stability, the distribution of the input must have probability 1 on $M^*$, the set of all inputs $\phi$ with absorbing process $\boldsymbol{x}_t^*(\phi)$. To find $M^*$ directly in each model seems difficult. The untraceable events method which this paper proposes finds a measurable subset of $M^*$ instead of $M^*$ itself and shows a condition under which the input distribution has probability 1 on this subset.

The outstanding advantage of the absorbing process is not only the generality of the distribution of the input, but also the generality of the model building. We must not lose these generalities in finding the subset of $M^*$ described above. Fortunately, we expect that the set of state vectors $\{\boldsymbol{x}_t^r(\phi, \boldsymbol{a})\colon r < t,\ \boldsymbol{a} \in \bar{X}\}$, for fixed $t$ and $\phi$, occupies only part – often the bounded part – of the state space $X$ with probability 1 in many models. In the previous papers [14] and [15], this fact was combined with the renewing event. Here we also assume this fact to hold, and from it obtain our main result, Corollary 7.1. Since the proof of this boundedness depends on the structure of each model and requires lengthy discussion, here we present only one simple example, to help the reader's understanding.

## 2. Untraceable events

We consider a system with state vector $\boldsymbol{x}_t$ for which, when the input $\phi$ is given, the following relation holds:

$$\boldsymbol{x}_t = f(\boldsymbol{x}_s, \phi, s, t) \quad \text{if } s < t \text{ and } (s, t) \in Z^2. \tag{2.1}$$

In this equation the parameter space $Z$ is $\mathbb{R} = (-\infty, \infty)$ or $\mathbb{N} = \{0, \pm 1, \pm 2, \dots\}$. Let $X$ be the space of states and $M$ be the space of inputs. Let $\boldsymbol{x}_t^*(\phi)$ be the absorbing process with initial state space $\bar{X}$, and let $M^* \subset M$ be the space of inputs $\phi$ with absorbing process $\boldsymbol{x}_t^*(\phi)$. We use $\phi$ to denote an element of $M$, not a random input. We let $\boldsymbol{x}_t^r(\phi, \boldsymbol{a}) = f(\boldsymbol{a}, \phi, r, t)$. If $\phi \in M^*$ then $\boldsymbol{x}_t^r(\phi, \boldsymbol{a}) = \boldsymbol{x}_t^*(\phi)$ ($\boldsymbol{a} \in \bar{X}$) for large $t$. For details, see [16]; we make Assumption 2.1 thereof.

Let $\Phi \equiv \Phi(\omega)$ be a random input which is the measurable mapping from the basic probability space $(\Omega, \sigma(\Omega), \mathrm{P})$ to $(M, \sigma(M))$. If $\mathrm{P}(D) = 1$ for a measurable set $D$ in $M^*$, we say that there exists an absorbing process with probability 1. As stated in the introduction, if we know that the existence condition of the absorbing process is satisfied, many stability problems about $\boldsymbol{x}_t^r(\phi, \boldsymbol{a})$ are automatically solved, because the absorbing process inherits some properties of the random input $\Phi$. Therefore, the technical aim of this paper is to find the condition on P under which a measurable subset of $M^*$ has probability 1.

First of all we will find a subset of $M^*$. We show that the events with the following property generate a subset of $M^*$. The notation $\tilde{Z}$ is used for a subset of $Z$ which contains at least one doubly infinite sequence.

**Definition 2.1.** Suppose that a set $\bar{X}$ in $X$ and a class $\{A(t) \colon t \in \tilde{Z}\}$ of sets in $M$ are given. Assume that there is a positive function $L(t, \phi)$ satisfying $\lim_{t \to -\infty}\{t + L(t, \phi)\} = -\infty$. If

$$\boldsymbol{x}_{t+s}^r(\phi, \boldsymbol{a}) = f(\boldsymbol{b}, \phi, t, t+s)$$

holds for all $(r, t, \boldsymbol{a}, \boldsymbol{b}, s, \phi)$ satisfying

$$t \in \tilde{Z}, \qquad r \le t, \qquad \boldsymbol{a} \in \bar{X}, \qquad \boldsymbol{b} \in \bar{X}, \qquad L(t, \phi) \le s,$$
$$\phi \in A(t), \qquad (r, t+s) \in Z^2,$$

then we call $\{A(t)\}$ the class of untraceable events with interval length $L(t, \phi)$ and set of initial states $\bar{X}$.

**Remark 2.1.** The function $L(t, \phi)$ is chosen to be constant, $L(t, \phi) = L$, in most applications. We suppress $\phi$ and write $L(t) \equiv L(t, \phi)$ if no confusion will occur.

**Remark 2.2.** If the condition in Definition 2.1 is satisfied then, when $t$ and $\phi$ are fixed, $\boldsymbol{x}_{t+s}^r(\phi, \boldsymbol{a})$ does not depend on the initial condition $(r, \boldsymbol{a})$ for any $s$ larger than $L(t)$. This means that we cannot trace the origin from the current state after $t + L(t)$; in other words, the state loses the memory of its initial condition. We thus call the set defined in Definition 2.1 the untraceable event.

**Remark 2.3.** If $\bar{X} = X$, Definition 2.1 becomes meaningless in most models. In such cases it is useful to select a sequence of sets $\bar{X}_1, \bar{X}_2, \dots,$ such that $\bar{X}_1 \subset \bar{X}_2 \subset \cdots$ and $\bigcup_{i=1}^{\infty} \bar{X}_i = X$. Let $M_i^*$ be the set of all elements in $M$ which have an absorbing process with initial state space $\bar{X}_i$. From Theorem 2.2 of [16], the set $M^*$ with $\bar{X} = X$ is given by $M^* = \bigcap_{i=1}^{\infty} M_i^*$. If, for each $i$, we can find the measurable subset $D_i^*$ of $M_i^*$ which has probability 1 for a given distribution of the input, the set $\bigcap_{i=1}^{\infty} D_i^*$ is a subset of $M^*$ and has probability 1. Thus, we achieve our aim by finding the untraceable events for each $\bar{X}_i$.

Loss of memory of the initial condition is also the essential property of the absorbing process. This common property generates the following theorem.

**Theorem 2.1.** *If an input $\phi$ has a class $\{A(t)\colon t \in \tilde{Z}\}$ of untraceable events, with interval length $L(t)$ and a set of initial states $\bar{X}$, such that $\phi \in A(t_j)$ for a doubly infinite sequence $\{t_j\} \subset \tilde{Z}$ with $\lim_{j\to-\infty} t_j = -\infty$ and $\lim_{j\to\infty} t_j = \infty$, then $\phi$ is contained in the set $M^*$ with the same initial state space $\bar{X}$.*

*Proof.* Without loss of generality we assume that $t_j + L(t_j) < t_{j+1} + L(t_{j+1})$ for every $j \in \mathbb{N}$. We define

$$\boldsymbol{x}_t^*(\phi) := f(\boldsymbol{b}, \phi, t_j, t), \qquad t_j + L(t_j) \le t < t_{j+1} + L(t_{j+1}),\ \boldsymbol{b} \in \bar{X}.$$

Definition 2.1 shows that this $\boldsymbol{x}_t^*(\phi)$ is independent of the selection of $\boldsymbol{b}$ and that $\boldsymbol{x}_t^r(\phi, \boldsymbol{a}) = \boldsymbol{x}_t^*(\phi)$ for any $(r, t, \boldsymbol{a})$ such that $r \le t_j < t_j + L(t_j) \le t$ and $\boldsymbol{a} \in \bar{X}$. This means that the function $h(r, \boldsymbol{a}, \phi)$ of [16, Equation (2.2)] is smaller than or equal to $t_j + L(t_j)$. Hence, $\lim_{r\to-\infty} h(r, \boldsymbol{a}, \phi) = -\infty$ and $\boldsymbol{x}_t^*(\phi)$ is the absorbing process.

For a class $\{B(t)\colon t \in \tilde{Z}\}$ of subsets in $M$, we will often use the set of the form

$$\Theta(B) \equiv \Theta(B(t)\colon t \in \tilde{Z}) := \left\{\bigcap_{s=-\infty}^{\infty} \bigcup_{t\le s} B(t)\right\} \cap \left\{\bigcap_{s=-\infty}^{\infty} \bigcup_{s\le t} B(t)\right\},$$

where $s \in \tilde{Z}$ and $t \in \tilde{Z}$. Using this notation, we can write Theorem 2.1 as $\Theta(A) \subset M^*$.

In applications the measurability of $A(t)$ is usually clear for any fixed $t$. We can choose the countable set $\tilde{Z}$ in later sections in such a way that $\Theta(A)$ is measurable.

## 3. Method of untraceable events and main results

Our ultimate aim is to find the sufficient condition for the existence of a subset of $M^*$ with probability 1. To do so we will use the fact that $\Theta(A) \subset M^*$.

First, how do we construct the untraceable events? Let us consider the case in which, for a fixed $\phi$, the state $\boldsymbol{x}_t^r(\phi, \boldsymbol{a})$ at an arbitrarily fixed $t$ exists in a certain set in $X$ for any $(r, \boldsymbol{a})$, $r \le t, \boldsymbol{a} \in \bar{X}$. That is, while the strict proof of this assertion is necessary, in many models there is a measurable set $M_1(t)$ with positive probability satisfying

$$M_1(t) \subset \{\phi\colon \boldsymbol{x}_t^r(\phi, \boldsymbol{a}) \in Y \text{ for all } r \le t \text{ and all } \boldsymbol{a} \in \bar{X}\} \tag{3.1}$$

for a sufficiently wide and usually bounded set $Y$ in $X$. Using this boundedness we will restrict the state on the interval $[t, t + L)$. That is, we construct a measurable set $M_2(t)$ such that

$$M_2(t) \subset \{\phi\colon \boldsymbol{x}_{t+L}^t(\phi, \boldsymbol{b}) = \boldsymbol{x}_{t+L}^t(\phi, \boldsymbol{a}) \text{ for any } (\boldsymbol{b}, \boldsymbol{a}) \in Y \times \bar{X}\}.$$

If $\phi \in M_1(t) \cap M_2(t)$ then $\boldsymbol{x}_{t+L}^r(\phi, \boldsymbol{a})$ does not depend on $r \le t$ and $\boldsymbol{a} \in \bar{X}$. Hence, $A(t) = M_1(t) \cap M_2(t)$ is the untraceable event and $\Theta(A) \subset M^*$.

Second, we will consider the probability $\mathrm{P}(\Theta(A))$. In the case with customer-stationary and ergodic input on the nonnegative integer domain $\{0, 1, 2, \dots\}$, Borovkov [4] (or [5, Theorem 11.3]) gave a useful theorem. His proof is somewhat complicated. The untraceable events method gives a simple proof of his result, as follows. This theorem holds for both $Z = \mathbb{N}$ and $Z = \mathbb{R}$. It is applicable also for a periodic input distribution. We choose $\tilde{Z} = \{jd\colon j \in \mathbb{N}\} \subset Z$ for a positive number $d$, and let $\{T_t\colon t \in \tilde{Z}\}$ be a group of measurable transformations on $(M, \sigma(M))$.

**Theorem 3.1.** *Assume that $T_s A(t) = A(t-s)$ for arbitrary $t$ and $s$ in $\tilde{Z}$. Assume that the input $\Phi$ is stationary and ergodic with respect to $T_s$ on $\tilde{Z}$. Then there exists an absorbing process with probability 1, if* $\mathrm{P}(A(0)) > 0$.

*Proof.* From part (e1) of [12, Proposition 1.8, p. 6], or by Birkhoff's ergodic theorem applied to the set function of $\{\omega\colon \Phi(\omega) \in A(0)\}$, we have $T_{t_j}\Phi(\omega) \in A(0)$ with probability 1 for a certain doubly infinite sequence $\{t_j(\omega) \in \tilde{Z}\}$. This means that $\mathrm{P}(\Theta(A(t)\colon t \in \tilde{Z})) = 1$.

We assume that we can prove that $\mathrm{P}(M_1(0)) > 0$ for a certain model. Then, if we construct $M_2(t)$ satisfying $\mathrm{P}(M_2(0) \mid M_1(0)) > 0$, we clearly obtain $\mathrm{P}(A(0)) > 0$. Usually the transformation $T_t$ defined in the concrete model is the shift transformation satisfying

$$M_1(t) = T_{-t}M_1(0) \quad \text{and} \quad M_2(t) = T_{-t}M_2(0).$$

Therefore, from this theorem we obtain the existence of the absorbing process by showing the positivity of $\mathrm{P}(M_2(0) \mid M_1(0))$ and $\mathrm{P}(M_1(0))$ individually.

When it is difficult to prove $\mathrm{P}(M_1(0)) > 0$ directly, we usually consider members of an increasing sequence of sets $\{Y_\delta\}$ in place of $Y$ in (3.1), and construct the sets $M_1(t,\delta)$, $M_2(t,\delta)$, and $A_\delta(t)$ for each $Y_\delta$. Then we prove that

$$\lim_{\delta\to\infty} \mathrm{P}(M_1(0,\delta)) = 1, \tag{3.2}$$

in which case there is a $\delta_0$ such that $\mathrm{P}(M_1(0,\delta_0)) > 0$. Since we do not know its value, we choose $M_2(0,\delta)$ such that $\mathrm{P}(M_2(0,\delta) \mid M_1(0,\delta)) > 0$ for any $\delta$. Then we can obtain $\mathrm{P}(A_{\delta_0}(0)) > 0$ and use Theorem 3.1. This method was first used in [14].

Third, in Sections 6 and 7 we consider nonstationary or nonergodic input with respect to the transformation $T_t$. Let us consider the events $A_\delta(t) = M_1(t,\delta) \cap M_2(t,\delta)$, $\delta \in \Delta$, on a doubly infinite sequence $\dots, t_{-1\delta}, t_{0\delta}, t_{1\delta}, \dots, \cdots < t_{-1\delta} < t_{0\delta} < t_{1\delta} < \cdots$. Here $\Delta$ is a countable set. Theorem 3.1 shows the existence of one class of untraceable events such that $\mathrm{P}(\Theta(A_\delta)) = 1$. However, it seems difficult to show such existence for nonstationary input. We consider many classes such that $\lim_{\delta\to\infty}\mathrm{P}(\Theta(A_\delta)) = 1$ or, more generally,

$$\mathrm{P}\bigg(\bigcup_{\delta\in\Delta}\Theta(A_\delta)\bigg) = 1. \tag{3.3}$$

Let $\pi_{j\delta}$ be the class of all subsets of $M_1(t_{j\delta},\delta)$ which have positive probabilities and are represented as the intersections of the finite sets among $M_1(t_{i\delta},\delta)$, $i \le j$, and $M \setminus M_2(t_{i\delta},\delta)$, $i < j$. In Section 6 we prove that if

$$\inf_{-\infty<j<\infty}\ \inf_{D\in\pi_{j\delta}} \mathrm{P}(M_2(t_{j\delta},\delta) \mid D) > 0, \tag{3.4}$$

then $\mathrm{P}(\Theta(A_\delta(t_{j\delta},\delta))) = \mathrm{P}(\Theta(M_1(t_{j\delta},\delta)))$. Hence, (3.3) is reduced to

$$\mathrm{P}\bigg(\bigcup_{\delta\in\Delta}\Theta(M_1(t_{j\delta},\delta))\bigg) = 1. \tag{3.5}$$

We summarize our technique in the following theorem.

**Theorem 3.2.** *Assume that, for fixed initial state space $\bar{X}$, there are measurable sets $M_1(t,\delta)$ and $M_2(t,\delta)$ and a doubly infinite sequence $\tilde{Z}_\delta = \{t_{j\delta} \colon j \in \mathbb{N}\}$ for each $\delta \in \Delta$ which satisfy (3.4) and (3.5). Moreover, assume that, for every $\delta$, $\{A_\delta(t) = M_1(t,\delta) \cap M_2(t,\delta) \colon t \in \tilde{Z}_\delta\}$ is the class of untraceable events. Then with probability 1 there exists an absorbing process $\boldsymbol{x}_t^*(\phi)$ with the initial state space $\bar{X}$.*

Fourth, in the concrete model we choose $M_2(t,\delta)$ to satisfy (3.4) according to the distribution of the input. However, (3.5) must be proved. We will use (3.2), because its proof is obtained generally under a mild condition and needs neither stationarity nor ergodicity. Since (3.2) does not imply (3.5), we need the additional assumption that we can use our main result Corollary 7.1 (see below) with $C(t,\delta) = M_1(t,\delta)$. We can then achieve our aim under a wide range of distributions of the input.

We often prove (3.2), even if we use Theorem 3.1. Therefore, though we must modify $M_1(t,\delta)$ a little, the result of Theorem 3.1 is extended under Corollary 7.1, which is then more useful in application.

## 4. Borovkov event

In this section, which is independent of the others, we will clarify the relation between Borovkov's renewing event and the untraceable event. In Borovkov's event the future state is determined by the future input. We use this functional independence in applications. His model setting is largely different from ours, so we will define the renewing event under our setting. His definition uses both the future part and the past part of the input, so we represent the input $\phi$ in the form of a marked point process in the continuous-parameter case and by $\phi = (\dots, \boldsymbol{\tau}_{-1}, \boldsymbol{\tau}_0, \boldsymbol{\tau}_1, \dots)$ in the discrete case.

**Definition 4.1.** We call the set $A(t)$ the Borovkov event if $A(t)$ has the following mapping $g$. In the continuous-parameter case, let $\phi_{[t,\infty)}$ be the restriction of $\phi$ to the half-open interval $[t,\infty)$. If $\phi \in A(t)$ then

$$\boldsymbol{x}_{t+s}^r(\phi, \boldsymbol{a}) = g(s, T_t\phi_{[t,\infty)})$$

for any $r \le t$, $s \ge L$, and $\boldsymbol{a} \in \bar{X}$. In the discrete case, if $\phi \in A(n)$ then

$$\boldsymbol{x}_{n+j}^r(\phi, \boldsymbol{a}) = g(j, \boldsymbol{\tau}_n, \boldsymbol{\tau}_{n+1}, \dots)$$

for any $r \le n$, $j \ge L$, and $\boldsymbol{a} \in \bar{X}$, where $\boldsymbol{\tau}_n$ is the mark at the position $n$.

Borovkov called this the renewing event in his model setting. However, neither the renewing event nor our event implies the probabilistic past–future independence, i.e. under the condition $\Phi \in A(t)$, the future $\{\boldsymbol{x}_{t+s}^r(\Phi, \boldsymbol{a}) \colon s > L(t)\}$ may be influenced by the stochastic behavior of the past $\{e(n), \boldsymbol{\tau}^K(n) \colon e(n) < t\}$. Thus the term 'renewing' or 'renovation' is liable to cause misunderstanding, so we call this the Borovkov event.

Clearly the Borovkov event is the untraceable event. The converse does not hold generally. Our definition of the untraceable event does not need even the time parameter of $\phi$. In the Borovkov event the future state functionally depends on neither the initial condition nor the past input. Our event can functionally depend on the past input. Consider an ordinary one-server queueing system in which the service time $\tau^{\mathrm{S}}(n)$ of the $n$th customer is represented by the moving average process $\tau^{\mathrm{S}}(n) = \sum_{i=0}^{\infty} 2^{-i}\varepsilon_{n-i}$. Define the input $\phi$ as the point process

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with mark $\{e(n+1)-e(n), \varepsilon_n\}$ instead of $\{e(n+1)-e(n), \tau^{\mathrm{S}}(n)\}$. We let

$$A(n) = \{\phi\colon \text{There is no customer in the system at } e(n+L) \text{ when the system starts with an arbitrary initial condition } (r, \boldsymbol{a}),\ r < n,\ \boldsymbol{a} \in \bar{X}\}.$$

Since the waiting time $W_n$ depends on $\varepsilon_k$ for any $k < n$, the future $\{W_i : i \geq n\}$ with $\phi \in A(n)$ is not determined only by the future input $\{\varepsilon_n, \varepsilon_{n+1}, \ldots\}$. Thus, if the state of the system is $W_n$, then $A(n)$ is not the Borovkov event, but it is the untraceable event. That is, our untraceable event does not always need the relation in Definition 4.1.

It seems difficult to extend Borovkov's proof to nonstationary or time-stationary models directly. The absorbing process is convenient in such cases and the untraceable event is naturally compatible with it, so here we use the untraceable events method.

## 5. Example of a queueing system

As a simple example, we will construct the untraceable event in the ordinary one-server queueing system with setup time. Customers arrive at our queueing system at times

$$\cdots \leq e(-1) \leq e(0) < 0 \leq e(1) \leq e(2) \leq \cdots.$$

Let $\tau^{\mathrm{S}}(n)$ be the service time of the $n$th customer. He has the associated setup time $\tau^{\mathrm{s}}_n$. If there are no customers in the system when the $n$th customer arrives, the server takes a setup time of length $\tau^{\mathrm{s}}_n$. Then the server begins service and continues to work according to a work-conserving service discipline until no customers remain in the system. If the $n$th customer finds other customers at his arrival, his setup time is not used. Brandt *et al.* [7, Theorem 5.8.4] showed the existence of a strong solution for the waiting time for customer-stationary arrival epochs. Here we consider the arrival epochs in as much generality as we can.

We consider the continuous-parameter case. The input $\phi$ is the counting measure whose position is $e(n)$. The mark of $e(n)$ is $\boldsymbol{\tau}(n) = \{\tau^{\mathrm{S}}(n), \tau^{\mathrm{s}}_n\}$. The elements of the state vector $\boldsymbol{x}_t$ consist of the remaining setup time, $x(0)$, the remaining service time, $x(1)$, of the customer receiving the service, and the service times, $x(2), x(3), \ldots$, of the waiting customers. If there is no corresponding customer, we let $x(i) = -1$. Then the state is an infinite-dimensional vector and the state space $X$ is a subset of $\mathbb{R}^{\infty}$. This model has the relation $\boldsymbol{x}_t = f(\boldsymbol{x}_s, T_s\phi, t-s)$, which is of the same type as (2.1). We let $|\boldsymbol{x}_t| = \sum_i \{x(i) : x(i) \geq 0,\ i \geq 0\}$ and $\bar{X} = \{x : |x| \leq \tilde{a}\}$, for a certain number $\tilde{a}$.

First we will show that the direct method like those of [11] and [13] is impossible. Consider the following special input: $e(n) = n-1$, $\tau^{\mathrm{S}}_n = 0.5$, and $\tau^{\mathrm{s}}_n = 0.6$, for all $n$. Assume that the initial state is idle and that the customer who arrives at $r$ receives service after his setup time. Then, if $r$ is an even number, there is one customer just before 0. If $r$ is an odd number, there is no customer at this epoch. That is, $\boldsymbol{x}^r_0(\phi, \boldsymbol{a})$ depends on $r$.

We will choose the untraceable event satisfying (3.4) and (3.5) under the following assumptions.

(a) With probability 1, for a certain constant $\lambda$,

$$\limsup_{n\to-\infty} \frac{-n}{-e(n)} \leq \lambda \quad \text{and} \quad \limsup_{n\to\infty} \frac{n}{e(n)} \leq \lambda.$$

(b) The sequence of the two variables $\{\tau^{\mathrm{S}}(n), \tau^{\mathrm{s}}_n\}$ is independent, identically distributed, and independent of arrival epochs.

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(c) $\lambda\,\mathrm{E}(\tau^{\mathrm{S}}(n)) < 1$, $\mathrm{E}(\tau^{\mathrm{s}}_n) < \infty$, and $\mathrm{P}(\tau^{\mathrm{S}}(n) + \tau^{\mathrm{s}}_n \le \mathrm{E}(\tau^{\mathrm{S}}(n))) > 0$.

(d) The input $\Phi$ has the AMS distribution of Corollary 7.1, or a related one.

Assumption (b) is not an essential condition, although it does affect the choice of untraceable event.

In order to construct $M_1(t)$, we must find $Y$ in (3.1). From (b) and (c) we have

$$\lim_{n\to-\infty} \frac{\tau^{\mathrm{s}}_n}{-n} = 0 \quad \text{and} \quad \lim_{n\to-\infty} \frac{1}{-n}\sum_{i=n}^{0} \tau^{\mathrm{S}}(i) = \mathrm{E}(\tau^{\mathrm{S}}(n))$$

with probability 1. To satisfy (c) we choose a positive number $\varepsilon$ such that $\mathrm{E}(\tau^{\mathrm{S}}(n)) + 2\varepsilon < \lambda^{-1}$. For sufficiently large $-n$ we have

$$\frac{1}{-n}\left\{\tilde{a} + \tau^{\mathrm{s}}_n + \sum_{i=n}^{0} \tau^{\mathrm{S}}(i)\right\} < \mathrm{E}(\tau^{\mathrm{S}}(n)) + \varepsilon < \frac{1}{\lambda} - \varepsilon < \frac{-e(n)}{-n}.$$

This implies that this $\phi$ is contained in the measurable set

$$C(\delta) = \left\{\phi\colon \sup_{n\le 0}\left\{\sum_{i=n}^{0} \tau^{\mathrm{S}}(i) + \tau^{\mathrm{s}}_n + \tilde{a} + e(n)\right\} < \delta\right\}$$

for a certain value of $\delta$. Hence, we have $\lim_{\delta\to\infty} \mathrm{P}(C(\delta)) = 1$.

Next, if the customer arriving at $e(n) < 0$ finds no customer and remains until the time epoch 0, we have $|\boldsymbol{x}^r_0(\phi, \boldsymbol{a})| = \sum_{i=n}^{0} \tau^{\mathrm{S}}(i) + \tau^{\mathrm{s}}_n + e(n)$. Therefore, $|\boldsymbol{x}^r_0(\phi, \boldsymbol{a})| < \delta$ for $\phi$ in $C(\delta)$. Hence, if $\phi \in T_{-t}C(\delta)$ then $|\boldsymbol{x}^{r+t}_t(\phi, \boldsymbol{a})| = |\boldsymbol{x}^r_0(T_t\phi, \boldsymbol{a})| < \delta$. We will choose a subset of $T_{-t}C(\delta)$ to be $M_1(t)$, because $T_{-t}C(\delta)$ satisfies (3.1) for $Y = \{x\colon |x| < \delta\}$.

We use Corollary 7.1. We define the set of the input

$$H = \left\{\phi\colon \limsup_{n\to\infty} \frac{n}{e(n)} \le \lambda\right\}.$$

According to (c), there are positive numbers $\tilde{S}$, $\eta$, and $\upsilon$ satisfying $(\lambda + \eta)\tilde{S} < 1 - \upsilon$ and $\mathrm{E}(\tau^{\mathrm{S}}(n)) < \tilde{S} < \lambda^{-1}$. We choose a number $L_\delta$ satisfying $\upsilon L_\delta > \delta$. Let $n(\phi, \delta)$ be the number of customers arriving during the half-open interval $[t, t + L_\delta)$. We let

$$M_1(t) = T_{-t}C(\delta) \cap \{\phi\colon n(\phi, \delta) < (\lambda + \eta)L_\delta\} \cap H. \tag{5.1}$$

From (a), the probability of $H$ is 1 and $\lim_{\delta\to\infty} \mathrm{P}(n(\Phi, \delta) \le (\lambda + \eta)L_\delta) = 1$. Hence, (3.2) holds.

The set $C(t, \delta) = M_1(t)$ satisfies parts (ii), (iii), and (v) of Corollary 7.1. With respect to part (iv), the set $C(\delta)$ of (5.1) is the nondecreasing set of $\delta$. The elements of $\{\phi\colon n(\phi, \delta) < (\lambda + \eta)L_\delta\} \cap H$ satisfy $n(\phi, \xi) < (\lambda + \eta)L_\xi$ for all $\xi$ larger than a sufficiently large $\xi_\phi$. Therefore, $C(0, \delta)$ satisfies (iv) and $M_1(t)$ satisfies (3.5). The set $H$ is unnecessary in (5.1) if we use Theorem 3.1.

Next, we choose $M_2(t)$ such that

$$M_2(t) = \{\phi\colon \tau^{\mathrm{S}}(n) + \tau^{\mathrm{s}}_n < \tilde{S} \text{ if } t \le e(n) < t + L_\delta\}. \tag{5.2}$$

Equation (3.4) holds from condition (b). That is,

$$\inf_{-\infty<j<\infty} \inf_{D\in\pi_j} \mathrm{P}(M_2(t_j) \mid D) \geq \mathrm{P}(\tau^{\mathrm{S}}(1)+\tau_1^{\mathrm{s}} < \tilde{S})^{n(\delta)} > 0,$$

where $n(\delta)$ is the maximum integer smaller than $(\lambda+\eta)L_\delta$. We let $A(t) = M_1(t) \cap M_2(t)$.

Finally, we must prove that $A(t)$ is the untraceable event. Note that the condition (5.2) of this set removes the counterexample ($e(n) = n-1$, $\tau_n^{\mathrm{S}} = 0.5$, $\tau_n^{\mathrm{s}} = 0.6$) which we showed first.

**Theorem 5.1.** *$A(t) = M_1(t) \cap M_2(t)$ is an untraceable event with interval length $L_\delta$.*

*Proof.* Consider an input $\phi \in M_1(t) \cap M_2(t)$. Let $e(n)$, $n \geq p, t \leq e(p) \leq e(p+1) \leq \cdots \leq e(p+n(\phi,\delta)-1) < t+L_\delta$, be the arrival epochs of $\phi$ in $[t, t+L_\delta)$. We have $|\boldsymbol{x}_t^r(\phi,\boldsymbol{a})| < \delta$ and $\tau^{\mathrm{S}}(n)+\tau_n^{\mathrm{s}} < \tilde{S}$ on $[t, t+L_\delta)$. Therefore, if there is an integer $i \geq p$ such that

$$\begin{aligned} &\delta + (j-p)\tilde{S} \geq e(j) - t, \qquad j = p, \dots, i-1, \\ &\delta + (i-p)\tilde{S} < e(i) - t, \end{aligned} \tag{5.3}$$

then there is no customer just before $e(i)$, for every $r < t$ and $\boldsymbol{a} \in \bar{X}$. When (5.3) does not hold for $i = p, \dots, p+n(\phi,\delta)-1$, we have

$$\delta + n(\phi,\delta)\tilde{S} < \delta + (\lambda+\eta)\tilde{S}L_\delta < \delta + L_\delta - \upsilon L_\delta < L_\delta.$$

Hence, the system becomes idle at $t+L_\delta$. Consequently, after $t+L_\delta$ the state does not depend on $(r, \boldsymbol{a})$.

By Corollary 7.1 and Theorem 3.2, the absorbing process $\boldsymbol{x}_t^*(\phi)$ exists with probability 1 under the assumptions (a), (b), (c), and (d).

## 6. Proof of Theorem 3.2

We will consider the probability $\mathrm{P}(\Theta(A))$ or $\mathrm{P}(\Theta(A_\delta))$ under a general input distribution. If the decomposition representation $A(t) = M_1(t) \cap M_2(t)$ holds for $t$ on a doubly infinite sequence $\{t_j\} \subset \tilde{Z}$, we have the relation

$$M^* \supset \Theta(A) \supset \Theta(A(t_j : j \in \mathbb{N})) \subset \Theta(M_1(t_j : j \in \mathbb{N})).$$

In this section we show that $\mathrm{P}(\Theta(A(t_j))) = \mathrm{P}(\Theta(M_1(t_j)))$ under a certain condition.

We fix a doubly infinite sequence $\dots, t_{-1}, t_0, t_1, \dots, \cdots < t_{-1} < t_0 < t_1 < \cdots$. Let $\pi_j$ be the class defined in Section 3. Assumption (6.1), below, means that if $\Phi \in M_1(t_j)$ then the event $M_2(t_j)$ occurs with probability larger than a positive constant irrespective of the occurrence of the past untraceable events $A(t_i)$.

**Theorem 6.1.** *Assume that*

$$\inf_{-\infty<j<\infty} \inf_{D\in\pi_j} \mathrm{P}(M_2(t_j) \mid D) > 0. \tag{6.1}$$

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*Then*

$$\mathrm{P}\Big(\bigcap_{i=-\infty}^{\infty}\bigcup_{j\geq i}M_1(t_j)-\bigcap_{i=-\infty}^{\infty}\bigcup_{j\geq i}(M_1(t_j)\cap M_2(t_j))\Big)=0, \tag{6.2}$$

$$\mathrm{P}\Big(\bigcap_{i=-\infty}^{\infty}\bigcup_{j\leq i}M_1(t_j)-\bigcap_{i=-\infty}^{\infty}\bigcup_{j\leq i}(M_1(t_j)\cap M_2(t_j))\Big)=0. \tag{6.3}$$

*Proof.* As a preparation, we define, for a pair $(p,\phi)\in\mathbb{N}\times M$, the integers $s_1,\dots,s_v$, $p\leq s_1<\cdots<s_v$, by $\phi\in M_1(t_{s_i})$. The $v\equiv v(\phi)\leq\infty$ in this definition is the maximum integer $i$ such that $\phi\in M_1(t_{s_i})$. We define

$$D_{p,q}=\{\phi\colon v(\phi)\geq q \text{ and } \phi\notin M_2(t_{s_i}) \text{ for all } i\leq q\}.$$

Since $\{\phi\colon s_1=i\}=\{M\setminus\bigcup_{j=p}^{i-1}M_1(t_j)\}\cap M_1(t_i)$, as a function of $\phi$ $s_1\equiv s_1(\phi)$ is measurable. Similarly, $s_i$, $v$ and $D_{p,q}$ are all measurable. Thus,

$$\begin{aligned}\mathrm{P}(D_{p,q})&=\sum_{p\leq j_1<\cdots<j_q<\infty}\mathrm{P}(\phi\colon s_i(\phi)=j_i \text{ and } \phi\notin M_2(t_{j_i}) \text{ for all } i\leq q)\\&=\sum_{p\leq j_1<\cdots<j_q<\infty}\mathrm{P}((M\setminus M_2(t_{j_q}))\cap\tilde{D}(j_1,\dots,j_q)),\end{aligned}$$

where

$$\begin{aligned}&\tilde{D}(j_1,\dots,j_q)\\&\quad:=\{\phi\colon s_i(\phi)=j_i \text{ for all } i,\ 1\leq i\leq q,\ \text{ and } \phi\notin M_2(t_{j_i}) \text{ for all } i,\ 1\leq i\leq q-1\}.\end{aligned}$$

If the probability of this set is positive, it is contained in $\pi_{j_q}$. Denoting the value of the left-hand side of (6.1) by $\xi$, we obtain

$$\mathrm{P}(D_{p,q})\leq(1-\xi)\sum_{p\leq j_1<\cdots<j_q<\infty}\cdots\sum_{p\leq j_1<\cdots<j_q<\infty}\mathrm{P}(\tilde{D}(j_1,\cdots,j_q))\leq(1-\xi)\,\mathrm{P}(D_{p,q-1}).$$

Repeating this process yields $\mathrm{P}(D_{p,q})\leq(1-\xi)^q$, which decreases to 0 as $q\to\infty$.

Now consider (6.2). Let $A=\bigcap_i\bigcup_{j\geq i}(M_1(t_j)\cap M_2(t_j))$ and $B=\bigcap_i\bigcup_{j\geq i}M_1(t_j)$. Clearly $A\subset B$. Assume that $\mathrm{P}(B\setminus A)>\varepsilon>0$. Then we find that there is an integer $p$ such that

$$\mathrm{P}(\phi\colon\phi\in B \text{ and if } \phi\in M_1(t_j) \text{ for } j\geq p,\ \text{ then } \phi\notin M_2(t_j))>\varepsilon. \tag{6.4}$$

For this integer $p$, the left-hand side of (6.4) is contained in $D_{p,q}$ for any $q$, meaning that (6.4) is impossible, since $\lim_{q\to\infty}\mathrm{P}(D_{p,q})=0$. Hence, (6.2) holds.

Next we consider (6.3). Let $A=\bigcap_i\bigcup_{j\leq i}(M_1(t_j)\cap M_2(t_j))$ and $B=\bigcap_i\bigcup_{j\leq i}M_1(t_j)$. Let $C_{k,v}$ be the set of inputs $\phi$ for which

(i) there are an infinite number of epochs $t_{j_i}$, $i\leq 0$, $\cdots<t_{j_{-1}}<t_{j_0}$, such that $\phi\in M_1(t_{j_i})$, and

(ii) if $\phi\in M_1(t_j)$ for $j$, $k\leq j\leq v$, then $\phi\notin M_2(t_j)$.

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Assume that $\mathrm{P}(B \setminus A) > \varepsilon > 0$. If $\phi \in B \setminus A$ then $\phi \in C_{-\infty,v}$ for small $v$, meaning that $\mathrm{P}(C_{-\infty,v}) > \varepsilon$ for a certain $v$. Let $n_{k,v}$ be the number of time epochs for which $k \le j_i \le v$. If $\phi \in C_{k,v}$ then $\lim_{k\to-\infty} n_{k,v} = \infty$. Hence, we obtain

$$D_{k,q} \supset C_{k,v} \cap \{n_{k,v} \ge q\} \xrightarrow{k\to-\infty} C_{-\infty,v}$$

for any $q$. This is inconsistent with $\mathrm{P}(D_{k,q}) < (1-\xi)^q$.

*Proof of Theorem 3.2.* From Theorem 6.1, we have

$$\mathrm{P}(\Theta(M_1(t_j,\delta))) = \mathrm{P}(\Theta(M_1(t_j,\delta) \cap M_2(t_j,\delta))) = \mathrm{P}(\Theta(A(t_j,\delta))).$$

Since $\Delta$ is a countable set, we have $\mathrm{P}(\bigcup_\delta \Theta(A_\delta)) = \mathrm{P}(\bigcup_\delta \Theta(M_{1\delta}))$. Hence, the theorem follows from (3.5).

## 7. $\lim_{\delta\to\infty} \mathrm{P}(\Theta(M_{1\delta})) = 1$ for AMS input

The purpose of this section is to obtain Corollary 7.1, which is used to mitigate the difficulty of proving condition (3.5) of Theorem 3.2.

**Lemma 7.1.** *Let* $\{M_j(\delta) : j \in \mathbb{N},\ \delta > 0\}$ *be the sets in* $\sigma(M)$. *If*

$$\lim_{\delta\to\infty} \limsup_{j\to-\infty} \mathrm{P}(M_j(\delta)) = \lim_{\delta\to\infty} \limsup_{j\to\infty} \mathrm{P}(M_j(\delta)) = 1,$$

*then* $\lim_{\delta\to\infty} \mathrm{P}(\Theta(M_j(\delta))) = 1$.

*Proof.* We have

$$\mathrm{P}\left\{\phi : \phi \notin \bigcup_{j>k} M_j(\delta) \text{ for an integer } k\right\} \le 1 - \limsup_{j\to\infty} \mathrm{P}(M_j(\delta)) \xrightarrow{\delta\to\infty} 0.$$

Since a similar argument holds for $j < k$, the lemma follows.

Let $\mu_Z$ be the Lebesgue measure on $Z$ when $Z = \mathbb{R}$. When $Z = \mathbb{N}$, it is the measure such that $\mu_Z(\{n\}) = 1$ for $n \in \mathbb{N}$ and $\mu_Z(\mathbb{R} \setminus \mathbb{N}) = 0$. Let $\{C(t,\delta) : t \in Z,\ \delta > 0\}$ be a set in $\sigma(M)$. In applications, we let $C(t,\delta) = M_1(t,\delta)$. Equation (7.2), below, means that, for properly chosen $\delta$ and most $u$ in $Z$, the random input $\Phi$ takes an element in $C(u,\delta)$ with high probability. We define $[D]_t = [-t,t] \cap D$ for a set $D \subset Z$ and use the formula

$$\mathrm{P}\left(\bigcap_{i=1}^{k} E_i\right) \ge 1 - \sum_{i=1}^{k}(1 - \mathrm{P}(E_i)) \tag{7.1}$$

for arbitrary sets $E_i$.

**Theorem 7.1.** *Assume that, for any numbers $R$ and $q$ such that $0 \le R < 1$ and $0 \le q < 1$, we can choose a $\delta$ such that*

$$\liminf_{t\to\infty} \frac{\mu_Z([u : \mathrm{P}(C(u,\delta)) \ge q]_t)}{2t} > R. \tag{7.2}$$

*Then there is a sequence $\{\delta_h : h = 1, 2, \ldots\}$ of parameter values such that*

$$\lim_{h\to\infty} \mathrm{P}(\Theta(C(s_j,\delta_h))) = 1$$

*for some doubly infinite sequence $\{s_j\}$.*

*Proof.* Let $q_i$ be numbers such that $0 < q_i < 1$ and $\lim_{i\to\infty} q_i = 1$. Let $n$ be a positive integer. Let $\delta_i$ be the value of $\delta$ in (7.2) corresponding to $q = q_i$ and $R = 1 - n^{-i}$. We will show the existence of a doubly infinite sequence $s_j$, $\cdots \le s_{-1} \le s_0 = 0 \le s_1 \le \cdots$, such that

$$\begin{gathered}\lim_{j\to-\infty} s_j = -\infty, \qquad \lim_{j\to\infty} s_j = \infty, \\ \mathrm{P}(\Phi \in C(s_j, \delta_i)) \ge q_i \quad \text{for any } j \le -i \text{ and any } j \ge i.\end{gathered} \tag{7.3}$$

The theorem then follows from Lemma 7.1.

Let $D_i = \{u\colon\ \mathrm{P}(C(u,\delta_i)) \ge q_i\}$. Considering $\mu_Z([D_i]_t)/(2t)$ as a probability, from (7.1) we obtain

$$\frac{\mu_Z([\bigcap_{i=1}^j D_i]_t)}{2t} \ge 1 - \sum_{i=1}^j \left\{1 - \frac{\mu_Z([D_i]_t)}{2t}\right\}. \tag{7.4}$$

Since $\liminf_{t\to\infty} \mu_Z([D_i]_t)/(2t) > 1 - n^{-i}$ from (7.2), (7.4) yields

$$\liminf_{t\to\infty} \frac{\mu_Z([\bigcap_{i=1}^j D_i]_t)}{2t} > 1 - \sum_{i=1}^j n^{-i}.$$

We select an $n$ larger than 3. Then $1 - \sum_{i=1}^j n^{-i} > (n-2)/(n-1) \ge \frac{2}{3}$. Hence, the region $\bigcap_{i=1}^j D_i$ expands in both directions in $Z$. We can thus select a doubly infinite sequence $\{s_j\}$ such that

$$s_{-j} \in \bigcap_{i=1}^j D_i \quad \text{and} \quad s_j \in \bigcap_{i=1}^j D_i,$$

for any positive $j$. This sequence satisfies (7.3).

We define the interval $J(t) = (-t, t)$.

**Theorem 7.2.** *Property (7.2) holds if, for any positive $\varepsilon$, there is a number $\delta$ such that*

$$\liminf_{t\to\infty} \mu_Z(J(t))^{-1} \int_{J(t)} \mathrm{P}(\Phi \in C(u,\delta))\,\mathrm{d}\mu_Z(u) > 1 - \varepsilon. \tag{7.5}$$

*Proof.* We select nonnegative numbers $R < 1$ and $q < 1$ arbitrarily. Choose an $\varepsilon$ such that $(1-\varepsilon-q)/(1-q) > R$. Assume that $\delta$ satisfies (7.5) for this $\varepsilon$. Define the set $D = \{u\colon\ \mathrm{P}(\Phi \in C(u,\delta)) \ge q\}$. We replace $\mathrm{P}(\Phi \in C(u,\delta))$ in (7.5) by 1 on $D$ and $q$ on $J(t) \setminus D$. We then have

$$\liminf_{t\to\infty} \frac{q(2t - \mu_Z([D]_t)) + \mu_Z([D]_t)}{2t} \ge 1 - \varepsilon.$$

Hence,

$$\liminf_{t\to\infty} \frac{\mu_Z([D]_t)}{2t} \ge \frac{1-\varepsilon-q}{1-q} > R.$$

**Corollary 7.1.** *Property (7.2) holds under the following assumptions:*

(i) *$\Phi$ is doubly ended AMS.*

(ii) *$T_s C(t,\delta) = C(t-s,\delta)$.*

(iii) *$\bigcup_{0<\delta<\infty} C(t,\delta) = \bigcup_{0<\delta<\infty} C(0,\delta)$ for any $t$.*

(iv) $C(0,\delta) \in \bigcup_{0<\eta<\infty} \bigcap_{\eta \le \xi < \infty} C(0,\xi)$ *for any* $\delta$.

(v) $\lim_{\delta\to\infty} \mathrm{P}(C(0,\delta)) = 1$.

*Proof.* Let $D = \bigcup_{0<\delta<\infty} C(0,\delta)$. Then $D$ is the invariant set in (ii) and (iii), meaning that $\mathrm{P}(T_u\Phi \in D) = \mathrm{P}(\Phi \in D) \ge \mathrm{P}(C(0,\delta)) \to 1$ as $\delta \to \infty$. Hence, $\tilde{P}(D) = 1$, for the stationary distribution $\tilde{P}$ in the definition of doubly ended AMS in [16].

Moreover, if $\phi \in D$ then $\phi \in C(0,\delta)$ for some $\delta$, meaning that $\phi \in \bigcap_{\eta<\xi<\infty} C(0,\xi)$ for some $\eta$, from (iv). Thus, $\lim_{\xi\to\infty} \tilde{P}(C(0,\xi)) \ge \tilde{P}(D) = 1$.

From (ii), the left-hand side of (7.5) is equal to $\tilde{P}(C(0,\delta))$, so $\lim_{\xi\to\infty} \tilde{P}(C(0,\xi)) = 1$ implies the existence of a $\delta$ satisfying (7.5). The corollary follows.

We can weaken condition (i). For example, it can be replaced by '$\Phi$ is both right-side AMS and left-side AMS' or 'some other input $\tilde{\Phi}$ satisfies the conditions of Corollary 7.1 and $\mathrm{P}(\Phi \in C(u,\delta)) \ge \mathrm{P}(\tilde{\Phi} \in C(u,\delta))$ holds for all $u$'.

If $C(0,\delta)$ is nondecreasing with respect to $\delta$, then condition (iv) holds.

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