

BOUNDARY BEHAVIOR OF SUPERHARMONIC FUNCTIONS SATISFYING NONLINEAR INEQUALITIES IN A PLANAR SMOOTH DOMAIN

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(Received 9 November 2007; accepted 22 December 2008)

Communicated by A. M. Hassell

Abstract

This paper presents a sharp boundary growth estimate for all positive superharmonic functions u in a smooth domain Ω in \mathbb{R}^2 satisfying the nonlinear inequality

$$-\Delta u(x) \leq c\delta_\Omega(x)^{-\alpha}u(x)^p \quad \text{for all } x \in \Omega,$$

where $c > 0$, $\alpha \in \mathbb{R}$ and $p > 0$, and $\delta_\Omega(x)$ stands for the distance from a point x to the boundary of Ω . A result is applied to show the existence of nontangential limits of such superharmonic functions.

2000 Mathematics subject classification: primary 31A20; secondary 31A05, 35J60.

Keywords and phrases: superharmonic function, boundary growth, nontangential limit, nonlinear elliptic equation.

1. Introduction

Many elliptic equations involving nonlinear terms have been studied widely from the viewpoints of not only differential equations but also (probabilistic) potential theory. See [2, 4, 10–12] and references therein. In this paper, we present a sharp boundary growth estimate for positive superharmonic functions satisfying a certain nonlinear inequality in a planar smooth domain. Furthermore, we apply the result to show the existence of nontangential limits of such superharmonic functions. Let us start with the definition of superharmonic functions. By $D(x, r)$, we denote the open disk of center x and radius r in \mathbb{R}^2 . Let Ω be a domain in \mathbb{R}^2 . A lower semicontinuous function $u : \Omega \rightarrow (-\infty, +\infty]$, where $u \not\equiv +\infty$, is called *superharmonic* on Ω if it satisfies the mean value inequality

$$u(x) \geq \frac{1}{\pi r^2} \int_{D(x,r)} u(y) dy,$$

This work was partially supported by Grant-in-Aid for Young Scientists (B) (No. 19740062), Japan Society for the Promotion of Science.

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whenever the closure of $D(x, r)$ is contained in Ω . Let Δ be the Laplacian on \mathbb{R}^2 . It is well known that if u is superharmonic on Ω , then there exists a unique (Radon) measure μ_u on Ω such that

$$\int_{\Omega} \phi(x) d\mu_u(x) = - \int_{\Omega} u(x) \Delta \phi(x) dx \quad \text{for all } \phi \in C_0^{\infty}(\Omega),$$

where $C_0^{\infty}(\Omega)$ is the collection of all infinitely differentiable functions vanishing outside a compact set in Ω . The measure μ_u is called the *Riesz measure* associated with u . If μ_u is absolutely continuous with respect to Lebesgue measure and $d\mu_u(x) = f_u(x) dx$ where f_u is a nonnegative locally integrable function on Ω , then we call f_u the *Riesz function* associated with u . If $u \in C^2(\Omega)$, then $f_u = -\Delta u$.

Throughout the paper, we suppose that Ω is a bounded $C^{1,1}$ -domain in \mathbb{R}^2 . By $\delta_{\Omega}(x)$ we denote the distance from a point x to the boundary $\partial\Omega$ of Ω . The symbol A stands for an absolute positive constant whose value is unimportant and may change from line to line. We discuss positive superharmonic functions u on Ω having an associated Riesz function f_u and satisfying the nonlinear inequality

$$f_u(x) \leq c\delta_{\Omega}(x)^{-\alpha}u(x)^p \quad \text{for almost every } x \in \Omega, \quad (1.1)$$

where $c > 0$, $\alpha \in \mathbb{R}$ and $p > 0$. Our results are as follows.

THEOREM 1.1. *Let $c > 0$ and $p > 0$. Suppose that*

$$\alpha \leq 3 - p.$$

Let u be a positive superharmonic function on Ω having an associated Riesz function f_u which satisfies (1.1). Then there exists a constant A depending only on u , c , α , p and Ω such that

$$\frac{\delta_{\Omega}(x)}{A} \leq u(x) \leq \frac{A}{\delta_{\Omega}(x)} \quad \text{for all } x \in \Omega. \quad (1.2)$$

Actually, the lower bound estimate in (1.2) is valid for all positive superharmonic functions (see Lemma 3.2). Thus, the interesting object is the relation between the boundary growth estimate and the ranges of p and α . The following theorem shows that the bound $\alpha \leq 3 - p$ is sharp for the upper bound estimate in (1.2). For $\xi \in \partial\Omega$ and $\theta > 0$, let

$$\Gamma_{\theta}(\xi) = \{x \in \Omega : |x - \xi| < (1 + \theta)\delta_{\Omega}(x)\}.$$

THEOREM 1.2. *Let $c > 0$ and $p > 0$. Suppose that*

$$\alpha > 3 - p.$$

Let $\xi \in \partial\Omega$. Then there exists a positive C^2 -function u on Ω satisfying

$$0 \leq -\Delta u(x) \leq c\delta_{\Omega}(x)^{-\alpha}u(x)^p \quad \text{for all } x \in \Omega \quad (1.3)$$

such that

$$\limsup_{\Gamma_{\theta}(\xi) \ni x \rightarrow \xi} \delta_{\Omega}(x)u(x) = +\infty$$

for any $\theta > 0$. In particular, u does not satisfy the upper bound estimate in (1.2).

Higher dimensional results analogous with Theorems 1.1 and 1.2 are presented in another paper [5]. The above results are proved using similar ideas given there. However, the proof of Theorem 1.1 is much simpler than the higher dimensional case, because of the higher-order integrability of the logarithmic kernel. In contrast, the proof of Theorem 1.2 requires a careful choice of sequences of balls and numbers. Furthermore, combining Theorem 1.1 and the earlier result due to Arsove and Huber, we will obtain a result concerning the existence of nontangential limits in the next section.

2. Application to nontangential limits

It is well known that every positive superharmonic function in the unit disk has radial limits almost everywhere on the boundary (see Littlewood [7]). However, nontangential limits do not necessarily exist. The existence of nontangential limits was proved by Arsove and Huber [1] (see also references therein).

THEOREM A. *Let u be a positive superharmonic function on the unit disk D having an associated Riesz function f_u . Suppose that there exists a constant A such that*

$$f_u(x) \leq \frac{A}{(1 - |x|)^2} \quad \text{for almost every } x \in D.$$

Then u has nontangential limits almost everywhere on ∂D .

Now, we suppose that $p > 0$ and $\alpha \leq 2 - p$. Let u be a positive superharmonic function on the unit disk D having an associated Riesz function f_u which satisfies

$$f_u(x) \leq c(1 - |x|)^{-\alpha} u(x)^p \quad \text{for almost every } x \in D. \quad (2.1)$$

Then it follows from Theorem 1.1 that

$$f_u(x) \leq A \frac{u(x)^p}{(1 - |x|)^\alpha} \leq \frac{A}{(1 - |x|)^{\alpha+p}} \leq \frac{A}{(1 - |x|)^2} \quad \text{for almost every } x \in D.$$

Therefore Theorem A implies that u has nontangential limits almost everywhere on ∂D . Thus we obtain the following result.

COROLLARY 2.1. *Let $c > 0$ and $p > 0$. Suppose that $\alpha \leq 2 - p$. Let u be a positive superharmonic function on the unit disk D having an associated Riesz function f_u which satisfies (2.1). Then u has nontangential limits almost everywhere on ∂D .*

3. Proof of Theorem 1.1

For two positive functions f and g , we write $f \approx g$ if there exists a constant $A > 1$ such that $A^{-1}f \leq g \leq Af$. The constant A will be called the constant of comparison.

Let G and K denote the Green’s function for Ω and the Martin kernel of Ω with pole at $\xi \in \partial\Omega$, respectively. It is known that

$$G(x, y) \approx \log \left(1 + \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x - y|^2} \right) \quad \text{for all } x, y \in \Omega, \tag{3.1}$$

where the constant of comparison depends only on Ω (see [3]). Let $x_0 \in \Omega$ be fixed and let $\{y_j\}$ be a sequence in Ω converging to $\xi \in \partial\Omega$. Since the Martin boundary of Ω coincides with the Euclidean boundary (see [6]), the ratio $G(x, y_j)/G(x_0, y_j)$ converges to $K(x, \xi)$. Therefore we obtain

$$K(x, \xi) \approx \frac{\delta_\Omega(x)}{|x - \xi|^2} \quad \text{for all } x \in \Omega, \tag{3.2}$$

where the constant of comparison depends only on Ω .

LEMMA 3.1. *If h is a positive harmonic function on Ω , then there exists a constant A depending only on h and Ω such that*

$$\frac{\delta_\Omega(x)}{A} \leq h(x) \leq \frac{A}{\delta_\Omega(x)} \quad \text{for all } x \in \Omega.$$

PROOF. By the Martin representation theorem,

$$h(x) = \int_{\partial\Omega} K(x, \xi) \, d\nu(\xi),$$

where ν is a measure on $\partial\Omega$ such that $\nu(\partial\Omega) = h(x_0) > 0$. Hence (3.2) yields

$$\frac{\delta_\Omega(x)}{A} h(x_0) \leq h(x) \leq \frac{A}{\delta_\Omega(x)} h(x_0),$$

and so the lemma follows. □

LEMMA 3.2. *If u is a positive superharmonic function on Ω , then there exists a constant A depending only on u and Ω such that*

$$u(x) \geq \frac{1}{A} \delta_\Omega(x) \quad \text{for all } x \in \Omega. \tag{3.3}$$

PROOF. Let μ_u be the Riesz measure associated with u . By the Riesz decomposition theorem,

$$u(x) = h(x) + \int_{\Omega} G(x, y) \, d\mu_u(y) \quad \text{for all } x \in \Omega,$$

where h is a nonnegative harmonic function on Ω . If $\mu_u(\Omega) = 0$, then $u = h$. Therefore we obtain (3.3) from Lemma 3.1 in this case. If $\mu_u(\Omega) > 0$, then we find $r > 0$ such that $\mu_u(E) > 0$, where $E = \{x \in \Omega \mid \delta_\Omega(x) \geq r\}$. It follows from (3.1) that

$$u(x) \geq \int_E G(x, y) \, d\mu_u(y) \geq \frac{\delta_\Omega(x)}{A} \mu_u(E) \quad \text{whenever } \delta_\Omega(x) < \frac{r}{2}.$$

Also, u has a positive minimum on $\{x \in \Omega \mid \delta_\Omega(x) \geq 2^{-1}r\}$ by the lower semicontinuity of u . Hence (3.3) follows. □

In the rest of this section, we suppose that u is a positive superharmonic function on Ω having an associated Riesz function f_u which satisfies (1.1). Then, by the Riesz decomposition theorem,

$$u(x) = h(x) + \int_{\Omega} G(x, y) f_u(y) dy \quad \text{for all } x \in \Omega, \tag{3.4}$$

where h is the greatest harmonic minorant of u on Ω . Note that h is nonnegative.

LEMMA 3.3. *There exists a constant A depending only on u and Ω such that*

$$\int_{\Omega} \delta_{\Omega}(y) f_u(y) dy \leq A.$$

PROOF. Since u is finite almost everywhere on Ω , we can find a point $x_1 \in \Omega$ with $u(x_1) < \infty$. Observe from (3.1) that $G(x_1, y) \geq A^{-1} \delta_{\Omega}(y)$ for all $y \in \Omega$. Since $f_u \geq 0$, the conclusion follows from (3.4). □

LEMMA 3.4. *Let $x \in \Omega$ and let $4^{-1} \delta_{\Omega}(x) \leq \rho \leq 2^{-1} \delta_{\Omega}(x)$. Then, for all $z \in D(x, 2^{-1} \rho)$,*

$$u(z) \leq \frac{A_1}{\delta_{\Omega}(x)} + A_1 \int_{D(x, \rho)} f_u(y) \log \frac{3\delta_{\Omega}(x)}{|z - y|} dy,$$

where A_1 is a positive constant depending only on u and Ω . □

PROOF. Let $x \in \Omega$ and $z \in D(x, 2^{-1} \rho)$. By (3.1),

$$G(z, y) \leq A \frac{\delta_{\Omega}(y)}{\delta_{\Omega}(x)} \quad \text{for all } y \in \Omega \setminus D(x, \rho).$$

Therefore, the superharmonicity $f_u \geq 0$ and Lemma 3.3 give

$$\int_{\Omega \setminus D(x, \rho)} G(z, y) f_u(y) dy \leq \frac{A}{\delta_{\Omega}(x)}.$$

Also, by (3.1),

$$G(z, y) \leq A \log \frac{3\delta_{\Omega}(x)}{|z - y|} \quad \text{for all } y \in D(x, \rho).$$

Hence the conclusion follows from (3.4) and Lemma 3.1. □

PROOF OF THEOREM 1.1. The lower bound estimate in (1.2) has been already proved in Lemma 3.2. We show the upper bound estimate. Let $x \in \Omega$. By Lemma 3.3,

$$\delta_{\Omega}(x) \int_{D(x, 2^{-1} \delta_{\Omega}(x))} f_u(y) dy \leq A.$$

Let $r = \delta_\Omega(x)$ and let $\psi_x(\zeta) = r^3 f_u(x + r\zeta)$. For simplicity, we write $D(r) = D(0, r)$. By the change of variables $z = x + r\eta$ and $y = x + r\zeta$,

$$\int_{D(2^{-1})} \psi_x(\zeta) \, d\zeta \leq A, \tag{3.5}$$

and from Lemma 3.4

$$ru(x) \leq A_1 + A_1 \int_{D(4^{-1})} \psi_x(\zeta) \log \frac{3}{|\zeta|} \, d\zeta, \tag{3.6}$$

$$ru(x + r\eta) \leq \Psi_x(\eta) \quad \text{for all } \eta \in D(4^{-1}), \tag{3.7}$$

where

$$\Psi_x(\eta) = A_1 + A_1 \int_{D(2^{-1})} \psi_x(\zeta) \log \frac{3}{|\eta - \zeta|} \, d\zeta.$$

Suppose that $p > 0$ and $\alpha \leq 3 - p$. Let $q > \max\{1, p\}$ and $s = q/p > 1$. By (3.6) and the Hölder inequality,

$$ru(x) \leq A_1 + A \left(\int_{D(4^{-1})} \psi_x(\zeta)^s \, d\zeta \right)^{1/s}.$$

Therefore, in order to show the upper bound estimate in (1.2), it suffices to prove that there exists a constant A independent of x such that

$$\int_{D(4^{-1})} \psi_x(\zeta)^s \, d\zeta \leq A. \tag{3.8}$$

Using the inequality $(a + b)^t \leq 2^t(a^t + b^t)$ for $a, b, t \geq 0$,

$$\begin{aligned} & \left(\int_{D(2^{-1})} \Psi_x(\eta)^q \, d\eta \right)^{1/q} \\ & \leq A + A \left(\int_{D(2^{-1})} \left(\int_{D(2^{-1})} \psi_x(\zeta) \log \frac{3}{|\eta - \zeta|} \, d\zeta \right)^q \, d\eta \right)^{1/q}. \end{aligned}$$

By the Minkowski inequality, the right hand side is bounded from above by

$$A + A \int_{D(2^{-1})} \left(\int_{D(2^{-1})} \left(\log \frac{3}{|\eta - \zeta|} \right)^q \, d\eta \right)^{1/q} \psi_x(\zeta) \, d\zeta.$$

Therefore

$$\left(\int_{D(2^{-1})} \Psi_x(\eta)^q \, d\eta \right)^{1/q} \leq A + A \int_{D(2^{-1})} \psi_x(\zeta) \, d\zeta. \tag{3.9}$$

Since $2^{-1}r \leq \delta_\Omega(x + r\eta) \leq 2r$ for all $\eta \in D(2^{-1})$ and $r \leq \text{diam } \Omega$, it follows from (1.1), (3.7) and $\alpha \leq 3 - p$ that

$$0 \leq \psi_x(\eta) = r^3 f_u(x + r\eta) \leq cr^3 \delta_\Omega(x + r\eta)^{-\alpha} u(x + r\eta)^p \leq A \Psi_x(\eta)^p,$$

for almost every $\eta \in D(4^{-1})$. This and (3.9) give

$$\left(\int_{D(4^{-1})} \psi_x(\zeta)^s d\zeta \right)^{1/q} \leq A + A \int_{D(2^{-1})} \psi_x(\zeta) d\zeta,$$

because $s = q/p$. Hence (3.5) yields (3.8). This completes the proof. □

4. Proof of Theorem 1.2

PROOF OF THEOREM 1.2. Suppose that $p > 0$ and $\alpha > 3 - p$. Let

$$\lambda = p + \alpha \quad \text{and} \quad \gamma = \frac{\lambda - 1}{2}.$$

Then $\gamma > 1$ and

$$\lambda - 2\gamma = 1. \tag{4.1}$$

Let $\xi \in \partial\Omega$. Since Ω is a $C^{1,1}$ -domain, there is a disk $D(z, r)$ contained in Ω such that $\xi \in \partial D(z, r)$. Without loss of generality, we may assume that $\xi = (0, 0)$, $z = (10, 0)$ and $r = 10$. For each $j \in \mathbb{N}$, let $x_j = (e^{-j^3+3}, 0)$ and $r_j = e^{-\gamma j^3}$. Then $D(x_j, 8r_j) \subset \Omega$ and $D(x_j, 2r_j) \cap D(x_k, 2r_k) = \emptyset$ if $j \neq k$. Let A_2 be a constant to be determined below and let f_j be a nonnegative smooth function on Ω such that

$$f_j \leq A_2 \frac{e^{\lambda j^3}}{j^2} \quad \text{on } \Omega \quad \text{and} \quad f_j = \begin{cases} A_2 \frac{e^{\lambda j^3}}{j^2} & \text{on } D(x_j, r_j), \\ 0 & \text{on } \Omega \setminus D(x_j, 2r_j). \end{cases}$$

Let A_3 be the constant of comparison in (3.1) and let j_0 be a natural number such that

$$\frac{c}{e^{3\alpha}} \left(\frac{2\pi A_2(\gamma - 1)}{A_3} \right)^p \frac{j_0^{p+2}}{A_2} \geq 1. \tag{4.2}$$

Define $f = \sum_{j=j_0}^{\infty} f_j$. Then, by (4.1),

$$\begin{aligned} \int_{\Omega} \delta_{\Omega}(y) f(y) dy &= \sum_{j=j_0}^{\infty} \int_{D(x_j, 2r_j)} \delta_{\Omega}(y) f_j(y) dy \\ &\leq \sum_{j=j_0}^{\infty} 2\delta_{\Omega}(x_j) A_2 \frac{e^{\lambda j^3}}{j^2} \pi (2r_j)^2 \\ &= 8e^3 \pi A_2 \sum_{j=j_0}^{\infty} \frac{e^{(-1+\lambda-2\gamma)j^3}}{j^2} < \infty. \end{aligned}$$

Hence $u := \int_{\Omega} G(\cdot, y) f(y) dy$ is well defined on Ω . Since f is locally Hölder continuous on Ω , it follows from [9, Theorem 6.6] that $u \in C^2(\Omega)$ and u satisfies $-\Delta u = f$ in Ω . For the moment, let $x \in \partial D(x_j, 2r_j)$. By (3.1),

$$\begin{aligned} G(x, x_j) &\geq \frac{1}{A_3} \log \frac{\delta_{\Omega}(x)\delta_{\Omega}(x_j)}{|x - x_j|^2} \geq \frac{1}{A_3} \log \frac{2^{-1}\delta_{\Omega}(x_j)^2}{(2r_j)^2} \\ &= \frac{1}{A_3} \log \frac{e^6}{8} e^{2(\gamma-1)j^3} \geq \frac{2(\gamma-1)}{A_3} j^3. \end{aligned}$$

Therefore the mean value equality and (4.1) give

$$\begin{aligned} u(x) &\geq \int_{D(x_j, r_j)} G(x, y) f_j(y) dy = A_2 \frac{e^{\lambda j^3}}{j^2} \int_{D(x_j, r_j)} G(x, y) dy \\ &= A_2 \frac{e^{\lambda j^3}}{j^2} \pi r_j^2 G(x, x_j) \geq \frac{2\pi A_2(\gamma-1)}{A_3} j e^{j^3}. \end{aligned}$$

By the minimum principle,

$$u(x) \geq \frac{2\pi A_2(\gamma-1)}{A_3} j e^{j^3} \quad \text{for all } x \in D(x_j, 2r_j). \quad (4.3)$$

Hence

$$u(x_j) \geq \frac{2e^3 \pi A_2(\gamma-1)}{A_3} j \delta_{\Omega}(x_j)^{-1},$$

and so

$$\lim_{j \rightarrow +\infty} \delta_{\Omega}(x_j) u(x_j) = +\infty.$$

Finally, we show that $-\Delta u \leq c \delta_{\Omega}(x)^{-\alpha} u^p$ in Ω . If $x \notin \bigcup_{j \geq j_0} D(x_j, 2r_j)$, then

$$c \delta_{\Omega}(x)^{-\alpha} u(x)^p \geq 0 = f(x) = -\Delta u(x).$$

Let $x \in D(x_j, 2r_j)$, where $j \geq j_0$. Then, by (4.2) and (4.3),

$$\begin{aligned} c \delta_{\Omega}(x)^{-\alpha} u(x)^p &\geq \frac{c}{e^{3\alpha}} \left(\frac{2\pi A_2(\gamma-1)}{A_3} \right)^p j^p e^{(p+\alpha)j^3} \\ &\geq f_j(x) = f(x) = -\Delta u(x). \end{aligned}$$

This completes the proof. \square

Acknowledgement

The author is grateful to the editor for several valuable comments.

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