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The Time Change Method and SDEs with Nonnegative Drift

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Abstract. Using the time change method we show how to construct a solution to the stochastic equation $dX_t = b(X_{t-})dZ_t + a(X_t)dt$ with a nonnegative drift a provided there exists a solution to the auxilary equation $dL_t = [a^{-1/\alpha}b](L_{t-})d\bar{Z}_t + dt$ where $Z, \bar{Z}$ are two symmetric stable processes of the same index $\alpha \in (0, 2]$. This approach allows us to prove the existence of solutions for both stochastic equations for the values $0 < \alpha < 1$ and only measurable coefficients a and b satisfying some conditions of boundedness. The existence proof for the auxilary equation uses the method of integral estimates in the sense of Krylov.

1 Introduction

The use of the time change method in constructing of solutions of one-dimensional Itô equations is well known. Usually, if the equation involves the drift term, one has then to apply the time change method in conjunction with a particular space transformation (drift transformation).

Here we shall use the time change method to construct a solution to the equation

$$dX_t = b(X_{t-})dZ_t + a(X_t)dt, \quad t \geq 0, X_0 = x_0 \in \mathbb{R}, \tag{1.1}$$

where Z is a symmetric stable process of index $\alpha \in (0, 2]$. The coefficients a and b are assumed to be only Borel measurable satisfying some boundedness conditions.

Equation (1.1) without drift ($a = 0$) is well studied. The case of $\alpha = 2$ was treated in detail in series of papers by H.J. Engelbert and W. Schmidt in the 1980's. They were able to find sufficient and necessary conditions for the existence of solutions. We refer here, for example, to [6] and [8]. The general case with arbitrary $\alpha \in (0, 2]$ but still with $a = 0$ was studied by P. Zanzotto in [19] and [20] who, in particular, generalized the results of Engelbert and Schmidt for $\alpha \in (1, 2]$. The main method used for SDEs (1.1) without drift was the time change method.

Equation (1.1) with drift and $\alpha = 2$ was studied by H. J. Engelbert and W. Schmidt in [7] where they proved the existence of solutions under very general assumptions on the coefficients a and b combining the time change method and the method of drift transformation due to A. Zvonkin.

The case of equation (1.1) with $\alpha \in (1, 2)$ was considered in [11]. In particular, it was shown in [11] how one can obtain a solution X to (1.1) for any $\alpha \in (0, 2]$ by the

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time change method if one has a process Y satisfying the equation

$$dY_t = d\tilde{Z}_t + c(Y_t)dt, \quad t \geq 0, \quad Y_0 = 0, \tag{1.2}$$

where $c = a|b|^{-\alpha}$ with $|b|^{-\alpha} := 1/|b|^{\alpha}$ and $\tilde{Z}$ is a symmetric stable process of the same index α. In order to solve equation (1.2), we used the method of so-called Krylov's estimates for processes Y. The tools developed there worked only for $\alpha \in (1,2)$ but not for the case $\alpha \leq 1$. Some results for equation (1.1) with $\alpha = 1$ and measurable coefficients were obtained in [12]. To our knowledge, there are no existence results known for equation (1.1) with $\alpha < 1$ in the case of only measurable coefficients.

The purpose of this paper is twofold. First, we suggest a method to construct a process Y satisfying equation (1.2) applicable for all values of $\alpha \in (0,2]$. More precisely, we consider the auxililary equation

$$dL_t = [a^{-1/\alpha}b](L_{t-})d\bar{Z}_t + dt, \quad t \geq 0, \quad L_0 = 0, \tag{1.3}$$

where $\bar{Z}$ is a symmetric stable process of the same index α and $a^{-1/\alpha} := 1/a^{1/\alpha}$. Provided there is a solution L to equation (1.3), a suitable time change in the process L will lead to a solution Y for equation (1.2). However, in order for the time change to work, one has to require the drift coefficient a to be nonnegative. To prove the existence of solutions to equation (1.3), we shall use the method of Krylov's estimates for processes L. Conceptually, to obtain the corresponding integral estimates, we follow an idea similar to that used in [11] for processes Y and X. Second, the use of the method based on equation (1.3) allows us to prove the existence of solutions to equation (1.1) with $\alpha < 1$ and only measurable coefficients a and b satisfying some assumptions of boundedness.

It should be noted that the existence of solutions to equation (1.1) in the case of $0 < \alpha < 1$ and $a = 0$ with measurable coefficient b was proven in [19] where it was assumed that the coefficient b satisfied some additional assumption of local integrability. The main novelty of the results obtained here compare with those in [19] is the presence of the drift term a. Moreover, the handling of equation (1.1) with drift seems to be more complicated and requires different approaches, as for equation (1.1) in the Brownian motion case ($\alpha = 2$).

The introduction would be incomplete without mentioning the results known for equation (1.1) with $b = 1$ (similarly, for (1.2) with $c = a$). Thus, in [18] the authors studied the solutions under some conditions different for cases $0 < \alpha < 1$, $\alpha = 1$, and $1 < \alpha < 2$. Without going into detail, we only mention that, in the case of $\alpha < 1$, the coefficient a is required to satisfy some smoothness properties. Moreover, the method used in [18] was a purely analytical one based on properties of corresponding Markov processes. More recently, N. Portenko [14] proved the existence of solutions to equation (1.1) with $b = 1$ and $\alpha \in (1,2)$ under the assumption $|a|^p \in L(\mathbb{R})$ for $p > 1/(\alpha - 1)$ where he used his own estimates for transition probability density function of the solution process.

2 Preliminaries

We shall denote by $\mathbf{D}_{[0,\infty)}(\mathbb{R})$ the Skorokhod space, *i.e.*, the set of all real valued functions $z\colon [0,\infty) \to \mathbb{R}$ with right-continuous trajectories and with finite left limits (also called *cádlag* functions). For simplicity, we shall write $\mathbf{D}$ instead of $\mathbf{D}_{[0,\infty)}(\mathbb{R})$. We will equip $\mathbf{D}$ with the σ-algebra $\mathcal{D}$ generated by the Skorokhod topology. Under $\mathbf{D}^n$ we will understand the n-dimensional Skorokhod space defined as $\mathbf{D}^n = \mathbf{D} \times \cdots \times \mathbf{D}$ with the corresponding σ-algebra $\mathcal{D}^n$ being the direct product of n one-dimensional σ-algebras $\mathcal{D}$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space carrying a process Z with $Z_0 = 0$ and let $\mathbb{F} = (\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. The notation $(Z, \mathbb{F})$ means that Z is adapted to the filtration $\mathbb{F}$. We call $(Z, \mathbb{F})$ a symmetric stable process of index $\alpha \in (0, 2]$ if trajectories of Z belong to $\mathbf{D}$ and

$$\mathbf{E}\Big(e^{i\xi(Z_t - Z_s)} | \mathcal{F}_s \Big) = e^{-(t-s)|\xi|^\alpha}$$

for all $t > s \geq 0$ and $\xi \in \mathbb{R}$. If $\alpha = 2$, then $Z = W$ is a process of Brownian motion with variance $2t$. For $\alpha = 1$ we have a Cauchy process with unbounded second moment. In general, $\mathbf{E}|Z_t|^\beta < \infty$ for $\beta < \alpha$. It is well known that the process of Brownian motion W is the only symmetric stable process with continuous paths.

If $\alpha \in (0, 2)$, then one has the following quasi-isometrical property proven by J. Rosinski and W. Woyczynski [17]: there exist constants c_α and C_α depending on α only such that for all $t > 0$

$$c_\alpha \mathbf{E} \int_0^t |f_s|^\alpha ds \leq \sup_{\lambda > 0} \lambda^\alpha \mathbf{P}\Big(\sup_{s \leq t} \Big| \int_0^s f_u dZ_u \Big| > \lambda \Big) \leq C_\alpha \mathbf{E} \int_0^t |f_s|^\alpha ds. \tag{2.1}$$

For all $0 < \alpha \leq 2$, Z is a Markov process and can be characterized in terms of analytic characteristics of Markov processes. First, for any function $f \in L^\infty(\mathbb{R})$ and $t \geq 0$, we can define the operator

$$(P_t f)(x) := \int_\Omega f(x + Z_t) d\mathbf{P}(\omega)$$

where $L^\infty(\mathbb{R})$ is the Banach space of functions $f\colon \mathbb{R} \to \mathbb{R}$ with the norm $\|f\|_\infty = \mathit{ess}\sup |f(x)|$. The family $(P_t)_{t \geq 0}$ is called the family of convolution operators associated with Z. Formally, for a suitable class of functions $g(x)$, let

$$(\mathcal{L} g)(x) = \lim_{t \downarrow 0} \frac{(P_t g)(x) - g(x)}{t},$$

which is called the infinitesimal generator of the process Z.

It is known that for $\alpha < 2$

$$(\mathcal{L} g)(x) = \int_{\mathbb{R} \setminus \{0\}} [g(x + z) - g(x) - \mathbf{1}_{\{|z| < 1\}} g'(x) z] \frac{k_\alpha}{|z|^{1+\alpha}} dz$$

for any $g \in C^2$, where C^2 is the set of all bounded and twice continuously differentiable functions $g\colon \mathbb{R} \to \mathbb{R}$ and k_α is a suitable constant. Contrary to the case of $\alpha \in (0,2)$, the infinitesimal generator of a Brownian motion process ($\alpha = 2$) is the Laplacian, that is, the second derivative operator.

We note also that the use of Fourier transforms can simplify calculations when working with the infinitesimal generator $\mathcal{L}$. Let $g \in L_1(\mathbb{R})$ and

$$\hat{g}(\xi) := \int_{\mathbb{R}} e^{iz\xi} g(z)dz$$

be the Fourier transform of g. The following fact will be used later (cf. [11, Proposition 2.1]).

Proposition 2.1 *Let $\mathcal{L}$ be the infinitesimal generator of a symmetric stable process Z. Assume that $g \in C^2(\mathbb{R})$ and $\mathcal{L}g \in L_1(\mathbb{R})$. Then $\widehat{(\mathcal{L}g)}(\xi) = -|\xi|^\alpha \hat{g}(\xi)$.*

The existence of solutions to stochastic equations (1.1)–(1.3) is understood here in the weak sense. For instance, we say that equation (1.1) has a solution if there exist a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $\mathbb{F}$ and processes X and Z on it such that

$$X_t = x_0 + \int_0^t b(X_{s-})dZ_s + \int_0^t a(X_s)ds, \quad t \geq 0 \quad \mathbf{P} - \text{a.s.}, \tag{2.2}$$

where $(Z, \mathbb{F})$ is a symmetric stable process of given index α. The definitions for equations (1.2) and (1.3) are similar.

3 The Time Change Method

Here we are going to show how to construct a solution to equation (1.1) for any $\alpha \in (0,2]$ using the time change method and equations (1.2) and (1.3). The method of time change is well known in the theory of stochastic processes but also plays an important role in many applications, including the area of mathematical finance. There is an extensive application literature; we only mention [2,3] and the references therein.

Recall first that a process A is called an $\mathbb{F}$-time change if it is an increasing, right-continuous process with $A_0 = 0$ such that A_t is an $\mathbb{F}$-stopping time for any $t \geq 0$ (see [9, Chapter 4]). Define $T_t =: \inf\{s \geq 0 : A_s > t\}$ called the right-continuous inverse process to A. By definition, T is an increasing process starting at zero. It is easy to see that T is an $\mathbb{F}$-adapted process if and only if A is an $\mathbb{F}$-time change.

Proposition 3.1 *Let $\alpha \in (0,2]$ and assume that there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 \leq |b| \leq \delta_2$. Then for any initial value $x_0 \in \mathbb{R}$, equation* (1.1) *has a solution if and only if equation* (1.2) *has a solution.*

Proof Suppose first that X is a solution to equation (1.1) which means that equation (2.2) is satisfied. The integrals on the right-hand side of (2.2) are well defined and are $\mathbf{P}$-a.s. finite for all $t \geq 0$. Let

$$A_t = \int_0^t |b|^\alpha (X_s)ds \quad \text{and} \quad T_t = \inf\{s \geq 0 : A_s > t\}.$$

In can be easily verified that the process T satisfies the relation

$$T_t = \int_0^t |b|^{-\alpha}(X_{T_s})ds.$$

By definition, the process A is $\mathbb{F}$-adapted so that its right-inverse process T is an $\mathbb{F}$-time change process defined for all $t \geq 0$. We note that (T_t) is a global time change[1] because $A_\infty = \lim_{t\uparrow\infty} A_t = \infty$. Now define $Y_t = X_{T_t}$, $\mathcal{G}_t = \mathcal{F}_{T_t}$. Applying the time change $t \to T_t$ to the semimartingale X in (2.2) (see [10, Chapter 10]) and using the change of variables rule in Lebesgue–Stieltjes integral (see [16, Ch. 0, (4.9)]) yields

$$Y_t = x_0 + \int_0^{T_t} b(X_{s-})dZ_s + \int_0^t a(Y_s)dT_s.$$

It remains to note that the process

$$\tilde{Z}_t := \int_0^{T_t} b(X_{s-})dZ_s$$

is nothing but a symmetric stable process of the index α (see [17, Theorem 3.1]). Hence Y is a solution to equation (1.2).

The proof of the opposite direction is a very similar one. Suppose the process Y is a solution to equation (1.2) defined on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ with a filtration $\mathbb{G}$ where $\tilde{Z}$ is a symmetric stable process adapted to $\mathbb{G}$. Define

$$T_t = \int_0^t |b|^{-\alpha}(Y_s)ds \quad \text{and let} \quad X_t = Y_{A_t}, \mathcal{F}_t = \mathcal{G}_{A_t}$$

for all $t \geq 0$ where A is the right inverse to T and $T_\infty = \lim_{t\uparrow\infty} T_t = \infty$. By applying the global time change $t \to A_t$ to the semimartingale Y in (1.2) we obtain

$$\tilde{Z}_{A_t} = X_t - x_0 - \int_0^t a(X_s)ds.$$

Using simple time change arguments (cf. [5]), we can conclude that there exists a symmetric stable process Z defined on the same probability space such that

$$\tilde{Z}_{A_t} = \int_0^t b(X_{s-})dZ_s.$$

This proves that X is a solution to equation (1.1). ■

Proposition 3.2 *In addition to the assumptions of Proposition* 3.1*, suppose that there exist strictly positive constants K_1 and K_2 such that $K_1 \leq a(x) \leq K_2$ for all $x \in \mathbb{R}$. Then for any $x_0 \in \mathbb{R}$, equation* (1.2) *has a solution if and only if equation* (1.3) *has a solution.*

[1] That is, $T_t \in [0, \infty)$ for all $t \geq 0$.

Proof Let $(L, \mathbb{F})$ be a solution to equation (1.3) defined on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$ with a filtration $\bar{\mathbb{F}}$. Set

$$\tau_t := \int_0^t a^{-1}|b|^\alpha(L_s)ds$$

and define by τ^{-1} the right-inverse process to τ. It is easy to see that τ^{-1} is a strictly increasing, continuous $\bar{\mathbb{F}}$-time change. Moreover, it is a global time change because of the assumptions of the Proposition. It can be directly verified that

$$\tau_t^{-1} = \int_0^t a|b|^{-\alpha}(L_{\tau_s^{-1}})ds.$$

Now, letting $Y_t = L_{\tau_t^{-1}}$ and applying the time change $t \to \tau_t^{-1}$ to the relation (1.3) yields

$$Y_t = \int_0^{\tau_t^{-1}} [a^{-1/\alpha}b](L_{s-})d\bar{Z}_s + \tau_t^{-1} = \int_0^{\tau_t^{-1}} [a^{-1/\alpha}b](L_{s-})d\bar{Z}_s + \int_0^t [a|b|^{-\alpha}](Y_s)ds.$$

It remains to note that the first integral on the right-hand side of last relation is nothing but a symmetric stable process $\tilde{Z}$ of the same index α (cf. [17]) proving that Y is a solution to equation (1.2).

On the other hand, assuming that Y is a solution to equation (1.2) defined of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ with a filtration $\tilde{\mathbb{F}}$, we let

$$T_t = \int_0^t [a|b|^{-\alpha}](Y_s)ds$$

so that the right-inverse process T^{-1} has the form

$$T_t^{-1} = \int_0^t [a^{-1}|b|^\alpha](L_s)ds$$

where $L_t := Y_{T_t^{-1}}$.

After the time change $t \to T_t^{-1}$ in (1.2), we obtain $L_t = \tilde{Z}_{T_t^{-1}} + t$. Once again, by standard time change arguments (see [5]), there is a symmetric stable process $\bar{Z}$ of the same index α such that

$$\tilde{Z}_{T_t^{-1}} = \int_0^t [a^{-1/\alpha}b](L_{t-})d\bar{Z}_t.$$

Hence L satisfies equation (1.3). ∎

Remark 3.3 *In Propositions* 3.1 *and* 3.2 *we required the coefficients a and b to be bounded from above and "away from zero". This allowed for simple time change proofs and lead to so-called nonexploding solutions for equations* (1.1)–(1.3)*. Those assumptions on a and b could be relaxed to allow solutions to have explosions. However, in this note we do not consider the case of exploding solutions.*

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4 Some Integral Estimates

Let K be a strictly positive constant and Z be a symmetric stable process of index $0 < \alpha < 1$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F}$. By $\mathcal{I}$ we denote the class of all $\mathbb{F}$-predictable one-dimensional processes γ_t such that $|\gamma_t|^\alpha \le K$.

Let $C_0^\infty(\mathbb{R})$ denote the class of all infinitely many times differentiable real valued functions with compact support defined on $\mathbb{R}$. For any $x \in \mathbb{R}, \lambda > 0$, and any nonnegative, measurable function $f \in C_0^\infty(\mathbb{R})$ define the value function $v(x)$ as

$$v(x) = \sup_{\gamma \in \mathcal{I}} \mathbf{E} \int_0^\infty e^{-\lambda s} f(x + X_s^\gamma) ds,$$

where the process X^γ is given by $dX_t^\gamma = \gamma_t dZ_t + dt$. Then for the value function v and the process X^γ, the Bellman principle of optimality can be formulated as follows [13]. For any $[0, \infty)$-valued $\mathbb{F}$-stopping time τ it holds that

$$v(x) = \sup_{\gamma \in \mathcal{I}} \mathbf{E} \Big\{ \int_0^\tau e^{-\lambda s} f(x + X_s^\gamma) ds + e^{-\lambda \tau} v(x + X_\tau^\gamma) \Big\}.$$

Using standard arguments, one can derive from the principle above the corresponding Bellman equation (γ is deterministic)

$$\sup_{|\gamma|^\alpha \le K} \Big\{ |\gamma|^\alpha \mathcal{L} v(x) - \lambda v(x) + v_x(x) + f(x) \Big\} = 0, \tag{4.1}$$

which holds a.e. in $\mathbb{R}$.

Define $A = \{x : \mathcal{L}v(x) > 0\}$. Then the Bellman equation is equivalent to two equations

$$\begin{cases} K\mathcal{L}v - \lambda v + v_x + f = 0 & \text{on } A \\ -\lambda v + v_x + f = 0 & \text{on } A^c. \end{cases} \tag{4.2}$$

Lemma 4.1 *For all $x \in \mathbb{R}$, it holds that*

$$v(x) \le N \|f\|_2 := N \Big(\int_{\mathbb{R}} f^2(y) dy \Big)^{1/2}, \tag{4.3}$$

where the constant N depends on K and α only.

Proof For any function $h \colon \mathbb{R} \to \mathbb{R}$ such that $h \in L_1(\mathbb{R})$ and any $\varepsilon > 0$, we define

$$h^{(\varepsilon)}(x) = \frac{1}{\varepsilon} \int_{\mathbb{R}} h(x) q\Big(\frac{x - y}{\varepsilon}\Big) dy$$

to be the ε-convolution of h with a smooth function q such that $q \in C_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} q(x) dx = 1$.

For $\varepsilon > 0$, let

$$f^{(\varepsilon)} := \begin{cases} \lambda v^{(\varepsilon)} - K\mathcal{L}v^{(\varepsilon)} - v_x^{(\varepsilon)} & \text{on } A \\ \lambda v^{(\varepsilon)} - v_x^{(\varepsilon)} & \text{on } A^c. \end{cases}$$

It follows that $K\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)} = -f + K\mathcal{L}v^{(\varepsilon)}\mathbf{1}_{A^c}$, so that

$$\left(K\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)}\right)^2 \le 2(f^{(\varepsilon)})^2 + 2K^2\left(\mathcal{L}v^{(\varepsilon)}\right)^2.$$

Obviously, $f^{(\varepsilon)}$ is square integrable and (4.2) implies that $f^{(\varepsilon)} \to f$ as $\varepsilon\downarrow 0$ a.s. in $\mathbb{R}$.

Now, applying Proposition 2.1, the Parseval identity, and integration by parts to the inequality

$$\int_{\mathbb{R}} \left(K\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)}\right)^2 dx \le 2\int_{\mathbb{R}} (f^{(\varepsilon)})^2(x)dx + 2K^2\int_{\mathbb{R}} (\mathcal{L}v^{(\varepsilon)})^2(x)dx$$

yields

$$\int_{\mathbb{R}} |\hat{v}^{(\varepsilon)}(\xi)|^2\left([K|\xi|^\alpha + \lambda]^2 + |\xi|^2\right)d\xi \le \\ 2\int_{\mathbb{R}} |\hat{f}^{(\varepsilon)}(\xi)|^2 d\xi + 2K^2\int_{\mathbb{R}} |\xi|^{2\alpha}|\hat{v}^{(\varepsilon)}(\xi)|^2 d\xi. \tag{4.4}$$

One sees easily that there exists a constant $\lambda_0 > 0$ such that

$$[K|\xi|^\alpha + \lambda_0]^2 + |\xi|^2 \ge 4K^2|\xi|^{2\alpha} \tag{4.5}$$

for all $\xi \in \mathbb{R}$.

Combining the inequalities (4.4) and (4.5), we obtain for all $\lambda \ge \lambda_0$

$$\frac{1}{2}\int_{\mathbb{R}} |\hat{v}^{(\varepsilon)}(\xi)|^2\left([K|\xi|^\alpha + \lambda]^2 + |\xi|^2\right)d\xi \le 2\int_{\mathbb{R}} |\hat{f}^{(\varepsilon)}(\xi)|^2 d\xi. \tag{4.6}$$

Let

$$N_1 := \int_{\mathbb{R}} \frac{d\xi}{[K|\xi|^\alpha + \lambda]^2 + |\xi|^2}.$$

Clearly, the constant N_1 is finite and depends on K and α only.

Using estimate (4.6) and the inverse Fourier transform yields for all $x \in \mathbb{R}$ and $\lambda \ge \lambda_0$

$$\left(v^{(\varepsilon)}(x)\right)^2 \le \frac{N_1}{4\pi^2}\int_{\mathbb{R}} |\hat{v}^{(\varepsilon)}(\xi)|^2\left([K|\xi|^\alpha + \lambda]^2 + |\xi|^2\right)^2 d\xi \le \frac{N_1}{\pi^2}\int_{\mathbb{R}}\left(f^{(\varepsilon)}(z)\right)^2 dz.$$

The result follows then by taking the limit $\varepsilon \to 0$ in the above inequality and using the Lebesgue dominated convergence theorem. ∎

Now we assume that there exists a constant $K > 0$ such that

$$[a^{-1}|b|^{\alpha}](x) \le K \quad \text{for all} \quad x \in \mathbb{R}, \tag{4.7}$$

and we are interested in L_2-estimates of the form

$$\mathbf{E}\int_0^{\infty} e^{-\lambda s} f(x_0 + L_s)ds \le N\|f\|_2. \tag{4.8}$$

Theorem 4.2 *Let L be a solution to equation* (1.3) *and let the assumption* (4.7) *hold. Then for any* $x_0 \in \mathbb{R}, \lambda \ge \lambda_0$, *and any measurable function* $f\colon \mathbb{R} \to [0,\infty)$, *the estimate* (4.8) *holds, where the constant N depends on K and* α *only.*

Proof Assume first that $f \in C_0^{\infty}(\mathbb{R})$ so that there is a solution v to equation (4.1) satisfying the inequality (4.3). By taking the ε-convolution on both sides of (4.1), we obtain for all $0 \le r \le K$

$$r\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)} + f^{(\varepsilon)} \le 0.$$

Therefore, for $s \ge 0$, applying Itô's formula to the expression $v^{(\varepsilon)}(x_0 + L_s)e^{-\lambda s}$, yields

$$\begin{aligned}
&\mathbf{E}v^{(\varepsilon)}(x_0 + L_s)e^{-\lambda s} - v^{(\varepsilon)}(x_0)\\
&\quad = \mathbf{E}\int_0^s e^{-\lambda_u}\left([a^{-1}|b|^{\alpha}](L_u)\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + v_x^{(\varepsilon)}\right)(x_0 + L_u)du\\
&\quad \le -\mathbf{E}\int_0^s e^{-\lambda u} f^{(\varepsilon)}(x_0 + L_u)du.
\end{aligned}$$

By Lemma 4.1

$$\mathbf{E}\int_0^s e^{-\lambda u} f^{(\varepsilon)}(x_0 + L_u)du \le \sup_{x_0} v^{(\varepsilon)}(x_0) \le N\|f^{(\varepsilon)}\|_2.$$

It remains to pass to the limit in the above inequality letting $\varepsilon \to 0, s \to \infty$ and using Fatou's lemma.

The inequality (4.8) can be extended in a standard way first to any function $f \in L_2(\mathbb{R})$ and then to any nonnegative, measurable function using the monotone class theorem arguments (see, for example, [4, Theorem 20]). ∎

Corollary 4.3 *Let L be a solution to equation* (1.3) *and let assumption* (4.7) *be true. Then for any* $x_0 \in \mathbb{R}, \lambda \ge \lambda_0, m \in \mathbb{N}, t \ge 0$, *and any measurable function* $f\colon \mathbb{R} \to [0,\infty)$, *it holds that*

$$\mathbf{E}\int_0^{t\wedge\tau_m(L)} f(x_0 + L_s)ds \le N\|f\|_{2,m} := N\Big(\int_{[-m,m]} f^2(y)dy\Big)^{1/2},$$

where $\tau_m(L) = \inf\{t \ge 0 : |x_0 + L_t| > m\}$ *and the constant N depends on* K, m, t, *and* α *only.*

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5 Existence of Solutions

Here we shall first prove the existence of solutions to equation (1.3) with $0 < \alpha < 1$ using the estimates derived in the previous section. Combined with the results of Section 3, it will allow us to formulate the corresponding results for the existence of solutions to equation (1.1).

Theorem 5.1 *Assume that* $0 < \alpha < 1$ *and that assumption* (4.7) *is satisfied. Then for any* $x_0 \in \mathbb{R}$*, there exists a solution to equation* (1.3).

Proof First, by assumption (4.7), there exists a sequence of functions $h_n, n \geq 1$, being Lipshitz-continuous and uniformly bounded by K such that $h_n \to [a^{-1/\alpha}b]$ as $n \to \infty$ pointwise and in $\|\cdot\|_{2,m}$-norm for all $m \in \mathbb{N}$. For any $n = 1, 2, \dots$, the equation

$$dL^n_t = h_n(L^n_{t-})dZ_t + dt \tag{5.1}$$

has a unique strong solution (see, for example, [9, Theorem 9.1]) where the process Z is defined on *a priori* fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Our goal is to show that the sequence of processes $\{L^n\}, n \geq 1$ converges to a process L that satisfies equation (1.3).

Let $Y^n_t := \int_0^t h_n(L^n_{s-})dZ_s$.

We shall show that the sequence of processes $Q^n := (L^n, Y^n, Z)$, $n \geq 1$, is tight in the sense of weak convergence in $(\mathbf{D}^3, \mathcal{D}^3)$. Due to the Aldous' criterion [1], it is enough to verify that

$$\lim_{C\to\infty} \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{0\leq s\leq t} \|Q^n_s\| > C\Big) = 0 \tag{5.2}$$

for all $t \geq 0$ and

$$\limsup_{n\to\infty} \mathbf{P}\Big(\|Q^n_{t\wedge(\tau^n + r_n)} - Q^n_{t\wedge\tau^n}\| > \varepsilon\Big) = 0$$

for all $t \geq 0, \varepsilon > 0$, every sequence of $\mathbb{F}$-stopping times τ^n, and every sequence of real numbers r_n such that $r_n \downarrow 0$. We use $\|\cdot\|$ to denote the Euclidean norm of a vector.

On the other hand, for the tightness of the sequence Q^n it suffices to prove the tightness of the sequence of processes R^n where

$$R^n_t = \int_0^t |h_n|^\alpha (L^n_s) ds.$$

However, the sequence of processes R^n trivially satisfies the Aldous' conditions because of the uniform boundness of the coefficients h_n for all $n \geq 1$.

Since the sequence $\{Q^n\}$ is tight, there exists a subsequence $\{n_k\}, k = 1, 2, \dots$, a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$, and the process $\bar{Q}$ on it with values in $(\mathbf{D}^3, \mathcal{D}^3)$ such that Q^{n_k} converges weakly (in distribution) to the process $\bar{Q}$ as $k \to \infty$. For simplicity, let $\{n_k\} = \{n\}$.

According to the embedding principle of Skorokhod (see, *e.g.*, [9, Theorem 2.7]), there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and the processes $\tilde{Q} = (\tilde{L}, \tilde{Y}, \tilde{Z})$, $\tilde{Q}^n = (\tilde{L}^n, \tilde{Y}^n, \tilde{Z}^n)$, $n = 1, 2, \dots$, on it such that

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(i) $\tilde{Q}^n \to \tilde{Q}$ as $n \to \infty$ $\tilde{\mathbf{P}}$-a.s.
(ii) $\tilde{Q}^n = Q^n$ in distribution for all $n = 1, 2, \dots$.

Using standard measurability arguments [13, Chapter 2], one can prove that the processes $\tilde{Z}^n, \tilde{Z}$ are symmetric stable processes of the given index $0 < \alpha < 1$ with respect to the augmented filtrations $\tilde{\mathbb{F}}^n$ and $\tilde{\mathbb{F}}$ generated by processes $\tilde{Q}^n$ and $\tilde{Q}$, respectively.

Using the properties (i), (ii), and equation (5.1), one can show (cf. [13, Chapter 2]) that

$$\tilde{L}^n_t = x_0 + \int_0^t h_n(\tilde{L}^n_{s-}) d\tilde{Z}^n_s + dt, \quad t \geq 0, \quad \tilde{\mathbf{P}}\text{-a.s.}$$

On the other hand, the same properties and the quasi-left continuity of the processes $\tilde{Q}^n$ yield $\lim_{n\to\infty} \tilde{L}^n_t = \tilde{L}_t$ $\tilde{\mathbf{P}}$-a.s. Therefore, in order to show that the process $\tilde{L}$ is a solution to equation (1.3), it suffices to verify that, for all $t \geq 0$,

$$\lim_{n\to\infty} \int_0^t h_n(\tilde{L}^n_{s-}) d\tilde{Z}^n_s = \int_0^t [a^{-1/\alpha} b](\tilde{L}_{s-}) d\tilde{Z}_s \quad \tilde{\mathbf{P}}\text{- a.s.} \tag{5.3}$$

The following fact can be proven in a similar way as Lemma 4.2 in [11].

Lemma 5.2 *For any Borel measurable function $f\colon \mathbb{R} \to [0, \infty)$ and any $t \geq 0$, there exists a sequence $m_k \in (0, \infty)$, $k = 1, 2, \dots$ such that $m_k \uparrow \infty$ as $k \to \infty$ and it holds that*

$$\tilde{\mathbf{E}} \int_0^{t\wedge\tau_{m_k}(\tilde{L})} f(\tilde{L}_s) ds \leq N \|f\|_{2,m_k},$$

where the constant N depends on K, α, t, and m_k only.

Without loss of generality, we can assume in Lemma 5.2 that $\{m_k\} = \{m\}$.
For (5.3) to be true, it is enough to verify that for all $t \geq 0$ and $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \tilde{\mathbf{P}}\Big(\Big| \int_0^t h_n(\tilde{L}^n_{s-}) s\tilde{Z}^n_s - \int_0^t [a^{-1/\alpha} b](\tilde{L}_{s-}) d\tilde{Z}_s) ds \Big| > \varepsilon \Big) = 0. \tag{5.4}$$

In order to show (5.4) we estimate for a fixed $n_0 \in \mathbb{N}$

$$\begin{aligned}
&\tilde{\mathbf{P}}\Big(\Big| \int_0^t h_n(\tilde{L}^n_{s-}) s\tilde{Z}^n_s - \int_0^t [a^{-1/\alpha} b](\tilde{L}_{s-}) d\tilde{Z}_s) ds \Big| > \varepsilon \Big) \\
&\quad \leq \tilde{\mathbf{P}}\Big(\Big| \int_0^t h_{n_0}(\tilde{L}^n_{s-}) d\tilde{Z}^n_s - \int_0^t h_{n_0}(\tilde{L}_{s-}) d\tilde{Z}_s \Big| > \frac{\varepsilon}{3} \Big) \\
&\qquad + \tilde{\mathbf{P}}\Big(\Big| \int_0^{t\wedge\tau_m(\tilde{L}^n)} [h_n(\tilde{L}^n_{s-}) - h_{n_0}(\tilde{L}^n_{s-})] d\tilde{Z}^n_s \Big| > \frac{\varepsilon}{3} \Big) \\
&\qquad + \tilde{\mathbf{P}}\Big(\Big| \int_0^{t\wedge\tau_m(\tilde{L})} [h_{n_0}(\tilde{L}_{s-}) - [a^{-1/\alpha} b](\tilde{L}_{s-})] d\tilde{Z}_s \Big| > \frac{\varepsilon}{3} \Big) \\
&\qquad + \tilde{\mathbf{P}}\big(\tau_m(\tilde{L}^n) < t \big) + \tilde{\mathbf{P}}\big(\tau_m(\tilde{L}) < t \big) \\
&\quad = J^1_{n,n_0} + J^2_{n,n_0,m} + J^3_{n_0,m} + \tilde{\mathbf{P}}\big(\tau_m(\tilde{L}^n) < t \big) + \tilde{\mathbf{P}}\big(\tau_m(\tilde{L}) < t \big).
\end{aligned}$$

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By fixed n_0, $J^1_{n,n_0} \to 0$ as $n \to \infty$ by Skorokhod's Lemma about the convergence of stochastic integrals with respect to symmetric stable processes (cf. Lemma 2.3 in [15]). To show that $J^2_{n,n_0,m} \to 0$ as $n \to \infty$ and $J^3_{n_0,m} \to 0$ as $n_0 \to \infty$, we use first the Chebyshev's inequality and (2.1) and then Theorem 4.2 and Lemma 5.2, respectively, to estimate

$$J^2_{n,n_0,m} \le \frac{3C_\alpha}{\varepsilon} N \| |h_n - h_{n_0}|^\alpha \|_{2,m} \tag{5.5}$$

and

$$J^3_{n_0,m} \le \frac{3C_\alpha}{\varepsilon} N \| |h_{n_0} - [a^{-1/\alpha} b]|^\alpha \|_{2,m} \tag{5.6}$$

where the constant N depends on K, α, m, and t only. Obviously,

$$\| |h_n - [a^{-1/\alpha} b]|^\alpha \|_{2,m} \to 0 \text{ as } n \to \infty,$$

implying that the right-hand sides in (5.5) and (5.6) converge to 0 by letting first $n \to \infty$ and then $n_0 \to \infty$.

Because of the property $\tau_m(\tilde{L}^n) \to \tau_m(\tilde{L})$ as $n \to \infty$ $\tilde{\mathbf{P}}$-a.s.,

$$\tilde{\mathbf{P}}\big(\tau_m(\tilde{L}^n) < t\big) \to \tilde{\mathbf{P}}\big(\tau_m(\tilde{L}) < t\big) \text{ as } n \to \infty$$

for all $m \in \mathbb{N}, t > 0$. Therefore, the last two terms can be made arbitrarily small by choosing large enough m for all n due to the fact that the sequence of processes $\tilde{L}^n$ satisfies the property (5.2). This proves (5.4). ∎

From Theorem 5.1 and Propositions 3.1 and 3.2 we obtain

Theorem 5.3 *Let $0 < \alpha < 1$ and there exist strictly positive constants δ_1, δ_2, K_1, and K_2 such that*

(i) *$\delta_1 \le |b|(x) \le \delta_2$ for all $x \in \mathbb{R}$;*

(ii) *$K_1 \le a(x) \le K_2$ for all $x \in \mathbb{R}$.*

Then for any initial value $x_0 \in \mathbb{R}$, equation (1.1) *has a solution.*

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