10

Deep inelastic scattering

10.1 Kinematics for deep inelastic scattering

The processes that we are studying in the next few sections are shown schematically in Fig. 10.1. An initial neutrino with energy E hits a proton, producing a final state of a muon with energy E' and an undetected final hadronic state.

The lepton vertex is well known. All the interesting structure is included in the hadronic vertex. The kinematics are shown in the diagram and they involve

k_μ, the four-vector of the neutrino,
k'_μ, the four-vector of the muon,
$q = k - k'$, the four-momentum transferred from leptons to hadrons,
P_μ, the four-momentum of the target nucleon,
$\nu = pq/M$, energy transfer in the laboratory frame,
θ, the laboratory angle of the muon produced relative to the incident neutrino, and
$Q^2 = -q^2 = -m_\mu^2 + 2E\vec{k}\,'(1-\cos\theta) \approx 4EE'\sin^2\theta/2$, with $k' = \sqrt{E'^2 - m_\mu^2}$.

The above definitions hold also for electroproduction, when the initial neutrino is replaced by an electron and the exchange particle is the photon. The discussion of this and the following section is restricted to neutrino reactions. The cross section for such a process in the rest frame of the proton is given by

$$d\sigma = \frac{1}{(2E)(2M)} \sum\!\!\!\!\!\!\int_n |\mathcal{M}|^2 (2\pi)^4 \delta^4(k + p - k' - p_n) \frac{d^3k'}{2E'(2\pi)^3}, \qquad (10.1)$$

where

$$p_n^\mu = \sum_{i=1}^{n} p_i^\mu,$$

with the summation over all final particle configurations, each of which contains n particles with momenta p_i and $i = 1, \ldots, n$. The integration over the phase space

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

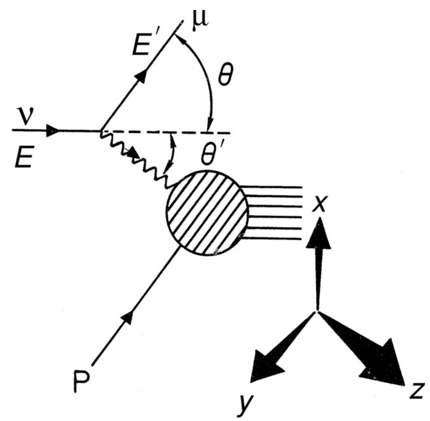

Figure 10.1. Inelastic neutrino–nucleon scattering, together with the coordinate system used in decomposing the leptonic current.

of final-state particles and the summation over the configurations is given by

$$\sum\!\!\!\!\!\!\int_n \cdots = \sum_n \int \prod_{i=1}^{n} \left[\frac{\mathrm{d}^3 p_i}{2E_i(2\pi)^3} \right] \cdots$$

Later on we shall specify the final state to be a single quark, with the product reduced to a single phase-space factor.

The matrix element is

$$\mathcal{M} = \frac{G}{\sqrt{2}} \bar{u}(k')\gamma_\mu(1-\gamma_5)u(k)\langle p_n|J^\mu|p\rangle. \tag{10.2}$$

We write the leptonic current as

$$j_\mu^{\text{lept}} = \bar{u}(k')\gamma_\mu(1-\gamma_5)u(k). \tag{10.3}$$

Neglecting the muon mass, the current is evaluated by multiplying it by a simple factor (Bjorken and Paschos, 1970):

$$j_\mu^{\text{lept}} = \sum_{s,s'} \bar{u}(k',s')\gamma_\mu(1-\gamma_5)u(k,s)\frac{\bar{u}(k,s)\gamma_0(1-\gamma_5)u(k',s')}{\bar{u}(k,s)\gamma_0(1-\gamma_5)u(k',s')}. \tag{10.4}$$

The factor of unity is introduced in order to change the numerator into a trace; the summation over spins does not change the lepton current because $(1-\gamma_5)$ is a chirality-projection operator, so the extra states introduced by $\sum_{s,s'}$ contribute zero:

$$\begin{aligned} j_\mu^{\text{lept}} &= \frac{2\,\mathrm{Tr}\big[\gamma_\mu(1-\gamma_5)\not{k}\,\gamma_0\,\not{k}'\big]}{\{2\,\mathrm{Tr}[\gamma_0(1-\gamma_5)\not{k}'\gamma_0\,\not{k}]\}^{1/2}} \\ &= \frac{8\big(k_\mu E' + k'_\mu E - g_{\mu o}\,k\cdot k' + \mathrm{i}\varepsilon_{\mu o\alpha\beta}k^\alpha k'^\beta\big)}{4\sqrt{EE'}\cos\theta/2}. \end{aligned} \tag{10.5}$$

Similarly, we calculate the square of the denominator, which produces a trace.

Using current conservation, we can eliminate one of the components in j_μ and expand the current in terms of three orthonormal polarization vectors whose spatial components lie along the axes shown in Fig. 10.1; the z-axis lies along q. This decomposition simplifies considerably in the high-energy limit $\nu \gg 2M \approx 2\,\mathrm{GeV}$; $Q^2 \ll \nu^2$, which is all we consider in this chapter. An alternative way would be to square the leptonic current and compute the leptonic tensor. The method is straightforward and the interested student can use it in order to reproduce some of the formulas in Sections 10.2 and 10.3. Here we find the method convenient for introducing helicity cross sections.

The three polarization vectors below correspond to the angular-momentum state $|J = 1, m\rangle$ with helicities $m = 0$, 1, and -1, respectively:

$$\begin{aligned}
\varepsilon_\mu^{\mathrm{S}} &= \frac{1}{\left[Q^2\right]^{1/2}}(q_z,\ 0,\ 0,\ q_0) \approx \frac{\nu}{\left[Q^2\right]^{1/2}}\left(1 + \frac{Q^2}{2\nu^2},\ 0,\ 0,\ 1\right),\\
\varepsilon_\mu^{\mathrm{R}} &= \frac{1}{\sqrt{2}}(0,\ 1,\ \mathrm{i},\ 0),\\
\varepsilon_\mu^{\mathrm{L}} &= \frac{1}{\sqrt{2}}(0,\ 1,\ -\mathrm{i},\ 0).
\end{aligned} \tag{10.6}$$

They satisfy the conditions $\varepsilon_{\mathrm{S}}^2 = +1$, $|\varepsilon_{\mathrm{L;R}}|^2 = -1$, and $\varepsilon_{\mathrm{S,L,R}} \cdot q = 0$. In the high-energy approximation the current, evaluated in the laboratory frame, becomes

$$j_\mu^\ell \approx 4\frac{(EE'Q^2)^{1/2}}{\nu}\left[\varepsilon_\mu^{\mathrm{S}} + \left(\frac{E'}{2E}\right)^{\frac{1}{2}}\varepsilon_\mu^{\mathrm{R}} + \left(\frac{E}{2E'}\right)^{\frac{1}{2}}\varepsilon_\mu^{\mathrm{L}}\right]. \tag{10.7}$$

The only change in j_μ^{lept} on going over to antineutrino-induced processes is the interchange $\mathrm{R} \leftrightarrow \mathrm{L}$.

The integration over the phase space of the muon can be carried out,

$$\frac{\mathrm{d}^3k'}{2E'(2\pi)^3} = \frac{E'\,\mathrm{d}E'\,\mathrm{d}\Omega}{2(2\pi)^3}.$$

In addition, we can transform to an invariant phase-space element,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}\nu} = \frac{\pi}{EE'}\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega\,\mathrm{d}E'},$$

arriving at

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}\nu} = \frac{G^2}{2\pi^2}\,\frac{E'}{E}\,\frac{Q^2}{\nu}\left(\frac{1}{(2\nu)(2M)}\sum\!\!\!\!\!\!\int_n |\langle n|\tilde{j}_\mu J^\mu|p\rangle|^2(2\pi)^4\delta^4(p_n - p - q)\right). \tag{10.8}$$

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

Here

$$\tilde{j}_\mu^{\text{lept}} = \varepsilon_\mu^{\text{S}} + \left(\frac{E'}{2E}\right)^{\frac{1}{2}} \varepsilon_\mu^{\text{R}} + \left(\frac{E}{2E'}\right)^{\frac{1}{2}} \varepsilon_\mu^{\text{L}}.$$

It is evident now that the amplitude $\langle n|\tilde{j}_\mu J^\mu|p\rangle$ is the sum of three helicity amplitudes: scalar (A_{S}), right-handed (A_{R}), and left-handed (A_{L}). The cross section is the sum of three helicity cross sections and three interference terms. When we average over the azimuthal angles of the hadrons produced, the three interference terms average to zero, as indicated by the following argument.

Let Γ be a fixed set of final-state hadron momenta that are measured. Let $\Gamma' = R\Gamma$ be the set of momenta obtained by rigid rotation of Γ about $\vec{q}$ (the z-axis) by the angle ϕ. We kept the neutrino and muon momenta fixed and rotated the hadronic system. This is equivalent to keeping the hadrons fixed and rotating the neutrino–muon plane in the opposite direction. Under this rotation the only change in the cross section is to replace $\tilde{j}_\mu^{\text{lept}}$ as follows:

$$\tilde{j}_\mu^{\text{lept}} = \varepsilon_\mu^{\text{S}} + \sqrt{\frac{E'}{2E}}\, \varepsilon_\mu^{\text{R}} e^{i\phi} + \sqrt{\frac{E}{2E'}}\, \varepsilon_\mu^{\text{L}} e^{-i\phi}. \tag{10.9}$$

The rotation is equivalent to a rotation of the two polarization vectors $\vec{\varepsilon}_{\text{R,L}}$ around the z-axis. $\vec{\varepsilon}^{\,\text{S}}$, which is parallel to $\vec{q}$, does not change. Accordingly, only the interference terms $2\Re(A_{\text{S}}A_{\text{R}}^* e^{i\phi})$, $2\Re(A_{\text{S}}A_{\text{L}}^* e^{-i\phi})$, and $2\Re(A_{\text{R}}A_{\text{L}}^* e^{2i\phi})$ change. They produce terms linear in $\cos(\phi)$ and $\sin(\phi)$. By averaging over the azimuthal orientations of the final hardons, i.e. integrating over ϕ from 0 to 2π, the interference terms are made to vanish. Should one wish to isolate the interference terms, then it is necessary to construct appropriate moments over the angle ϕ.

To sum up, Γ denotes a set of hadronic momenta in the final state, whose angles relative to each other are kept fixed; the rigid rotation around $\vec{q}$ has been averaged, i.e. integrated out. In this manner only helicity cross sections survive.

We define the helicity cross sections for absorption of the "virtual" W nucleon into final hadronic states by

$$\sigma^{(\lambda)}(\nu, Q^2) = \frac{1}{(2\nu)(2M)} \int |\langle n|\varepsilon_\mu^\lambda \cdot J^\mu(0)|p\rangle|^2 (2\pi)^4 \delta^{(4)}(p' - p - q) \frac{d^3 p'}{2E_{p'}(2\pi)^3}. \tag{10.10}$$

Here we have assumed that there is only one particle in the final hadronic state. When there are many particles produced, the phase space is replaced by a product of phase-space factors. These cross sections depend only on ν and Q^2. The final formula reads

$$\frac{d\sigma}{dQ^2\, d\nu} = \frac{G^2}{4\pi^2} \frac{E'}{E} \frac{Q^2}{\nu} \left(2\sigma_{\text{S}} + \frac{E'}{E}\sigma_{\text{R}} + \frac{E}{E'}\sigma_{\text{L}}\right). \tag{10.11}$$

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

The absorption cross sections are not uniquely defined; for $q^2 = 0$ the flux factor for the exchanged particle is 2ν. This is a convention and sometimes the factor has been replaced by $2\nu[1 - Q^2/(2M\nu)]$. To avoid the zeros which appear for elastic scattering, we chose the overall factor in the helicity cross sections $F = 4M\nu$ (Eq. (10.10))

On introducing the structure function

$$W_2(\nu, Q^2) = \frac{1}{2\pi}\frac{Q^2}{\nu}(2\sigma_S + \sigma_R + \sigma_L) \tag{10.12}$$

and the ratios

$$(L) = \frac{\sigma_L}{2\sigma_S + \sigma_R + \sigma_L} \leq 1,$$

$$(R) = \frac{\sigma_R}{2\sigma_S + \sigma_R + \sigma_L} \leq 1, \tag{10.13}$$

it follows that

$$\frac{d\sigma}{dQ^2\, d\nu} = \frac{G^2}{2\pi}\frac{E'}{E} W_2(\nu, Q^2)\left[1 + \frac{\nu}{E'}(L) - \frac{\nu}{E}(R)\right]. \tag{10.14}$$

The antineutrino–nucleon cross section is obtained from (10.14) by the interchange $L \leftrightarrow R$. We shall find these cross sections useful in several applications later on.

An alternative notation introduces the structure functions $W_1(\nu, Q^2)$, $W_2(\nu, Q^2)$, and $W_3(\nu, Q^2)$ defined in the next section. They are related to the absorption cross sections through (10.12) and the following:

$$W_1(\nu, Q^2) = W_2(\nu, Q^2)\left(1 + \frac{\nu^2}{Q^2}\right)[(L) + (R)], \tag{10.15}$$

$$W_3(\nu, Q^2) = W_2(\nu, Q^2)\frac{2M}{Q}\left(1 + \frac{\nu^2}{Q^2}\right)^{\frac{1}{2}} [(L) - (R)]. \tag{10.16}$$

In the limit $\nu^2/Q^2 \gg 1$ they reduce to

$$W_1(\nu, Q^2) = \nu W_2(\nu, Q^2)\frac{\nu}{Q^2}\,[(L) + (R)], \tag{10.17}$$

$$\nu W_3(\nu, Q^2) = \nu W_2(\nu, Q^2)\frac{2M\nu}{Q^2}\,[(L) - (R)]. \tag{10.18}$$

10.2 Hadronic structure functions

In the previous section we introduced structure functions that describe the hadronic vertex. Here we describe their connection with products of currents and their commutators. The formalism of this section is convenient in discussing sum rules or the light-cone behavior of the product of weak currents. We define the hadronic tensor as

$$W_{\mu\nu} = (2\pi)^3 \overline{\sum_{S_n}} \sum\!\!\!\!\!\!\int_n \langle P|J_\mu^+(0)|P_n\rangle\langle P_n|J_\nu(0)|P\rangle\delta^{(4)}(p_n - p - q). \quad (10.19)$$

Here $\sum\!\!\!\!\!\int_n$ sums over final states, and $\overline{\sum}_{S_n}$ averages over the spins of the target nucleon. By exponentiating the delta function and using translation invariance,

$$J_\mu(x) = \mathrm{e}^{\mathrm{i}Px} J_\mu(0)\mathrm{e}^{-\mathrm{i}Px}, \quad (10.20)$$

one obtains

$$W_{\mu\nu} = \frac{1}{2\pi}\overline{\sum_{S_n}} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle P|J_\mu^+(x)J_\nu(0)|P\rangle, \quad (10.21)$$

where the unitary relation $\sum\!\!\!\!\!\!\int_n |p_n\rangle\langle p_n| \equiv 1$ was used. We may change $W_{\mu\nu}$ into a commutator,

$$W_{\mu\nu} = \frac{1}{2\pi}\overline{\sum_{S_n}} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle P|\left[J_\mu^+(x), J_\nu(0)\right]|P\rangle, \quad (10.22)$$

since the second term of the commutator,

$$\frac{1}{2\pi} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\langle P|J_\nu(0)J_\mu^+(x)|P\rangle, \quad (10.23)$$

vanishes. This is proven by reversing the steps and showing that (10.22) reduces to

$$\sum\!\!\!\!\!\!\int_n \langle P|J_\nu(0)|p_n\rangle\langle p_n|J_\mu^+(0)|P\rangle\delta^{(4)}(p_n - P + q) = 0, \quad (10.24)$$

since in the physical process $q_0 = E_n - M_{\text{proton}} \geq 0$, but in Eq. (10.24) the δ-function argument implies

$$q_0 = P_0 - p_n^0 = M - E_n < 0.$$

By virtue of Lorentz and gauge invariance, $W_{\mu\nu}$ can be written in terms of six scalar functions, which are better known as structure functions. In neutrino scattering, however, only three contribute to the inelastic cross section because the lepton current is conserved (for $m_\mu = 0$). The tensor relevant to deep inelastic scattering is

$$W_{\mu\nu} = -g_{\mu\nu}W_1 + \frac{P_\mu P_\nu}{M^2}W_2 - \mathrm{i}\frac{\varepsilon_{\mu\nu\alpha\beta}P^\alpha q^\beta}{2M^2}W_3, \quad (10.25)$$

where the structure functions $W_1(Q^2, \nu)$ and $W_2(Q^2, \nu)$ arise from the product of vector $\otimes$ vector currents and axial $\otimes$ axial currents, whereas $W_3(Q^2, \nu)$ is the interference of an axial current ⊗ a vector current. The additional three terms are

$$\frac{q_\mu q_\nu}{M^2} W_4 + \frac{P_\mu q_\nu + P_\nu q_\mu}{M^2} W_5 + \mathrm{i}\frac{P_\mu q_\nu - P_\nu q_\mu}{M^2} W_6.$$

Their contributions to the matrix elements and the cross section are proportional to lepton masses and will be neglected.

10.3 Scaling and the total cross section

The structure functions are functions of ν and Q^2, but at high energies both variables are very large. It was suggested by Bjorken (1969) that, in the limit $\nu \to \infty$, $Q^2 \to \infty$, with the ratio

$$x = \frac{Q^2}{2M\nu} = \text{finite}, \tag{10.26}$$

the structure functions become functions of x only, i.e.

$$\nu W_{2,3}(\nu, Q^2) \to F_{2,3}(x), \tag{10.27}$$

$$M W_1(\nu, Q^2) \to F_1(x). \tag{10.28}$$

This was established in experiments on deep inelastic electron–proton scattering, for which the limit is reached at relatively low values of $Q^2, 2M\nu \approx (1\ \text{GeV})^2$. Inelastic electron–proton scattering is closely related to neutrino reactions and we mention it later on in this section.

In the scaling limit the relations (10.18) and (10.19) reduce to

$$\begin{aligned} 2xF_1(x) &= F_2(x)[(L) + (R)], \\ xF_3(x) &= F_2(x)[(L) - (R)]. \end{aligned}$$

With this notation we can rewrite the cross section in a convenient form. For variables we use $x = Q^2/(2M\nu)$ and the inelasticity $y = \nu/E$, then we substitute the scaling functions into the cross section, Eq. (10.14), and change the phase-space variables to arrive at

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}y} = \frac{G^2ME}{\pi}\left[xy^2F_1(x) + (1-y)F_2(x) + xy\left(1 - \frac{1}{2}y\right)F_3(x)\right],$$

where the structure functions depend on the process under consideration. In order to obtain the corresponding cross section for an antineutrino-induced reaction, one should change the sign of the F_3 term and replace the structure functions with the charge conjugate.

The main difference between electroproduction and neutrino-induced reactions is the nature of the particle exchanged. In electroproduction the particle exchanged

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

is the photon, which has only a vector coupling

$$j_\mu^{\mathrm{lept}} = \bar{u}(k')\gamma_\mu u(k). \tag{10.29}$$

The vector–axial interference term is now absent and the cross sections σ_{R} and σ_{L} are equal. Following steps similar to those of the previous section, one finds

$$\frac{\mathrm{d}\sigma(\mathrm{ep})}{\mathrm{d}Q^2\,\mathrm{d}\nu} = \frac{E'}{E}\frac{4\pi\alpha^2}{Q^4}\left[W_2^{\mathrm{e}}\cos^2\left(\frac{\theta}{2}\right) + 2W_1^{\mathrm{e}}\sin^2\left(\frac{\theta}{2}\right)\right], \tag{10.30}$$

where W_1^{e} and W_2^{e} are electroproduction structure functions analogous to those introduced in Eq. (10.25). In the cross section we kept the scattering angle θ. However, we can substitute it in terms of Q^2 and the energies E and E' and arrive at a formula analogous to (10.14). The superscript e indicates their electromagnetic origin. Numerous experiments have shown that the limits

$$\nu W_2^{\mathrm{e}}(\nu, Q^2) \to F_2^{\mathrm{e}}(x), \tag{10.31}$$

$$M W_1^{\mathrm{e}}(\nu, Q^2) \to F_1^{\mathrm{e}}(x) \tag{10.32}$$

are reached for relatively low values of ν and Q^2. This is shown in Fig. 10.2, where the structure function $F_2(x)$ is plotted for a range of Q^2. Deviations from the scaling law have also been established, and we return to this topic in Chapter 11. We show next that scaling predicts $\sigma_{\mathrm{tot}} \sim E_\nu$: namely a linear rise with neutrino energy.

From (10.14) and scale invariance (10.27) we find the averaged cross section over protons and neutrons

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\nu} &= \frac{G^2}{2\pi}\frac{E'}{E}\int_{\sim 0}^{2M\nu}\frac{\mathrm{d}Q^2}{\nu}\nu W_2(\nu, Q^2)\Big(1 + \frac{\nu}{E'}(L) - \frac{\nu}{E}(R)\Big)\\ &= \frac{G^2 M}{\pi}\frac{E'}{E}\Big(1 + \frac{\nu}{E'}\langle L\rangle - \frac{\nu}{E}\langle R\rangle\Big)\int_0^1 \mathrm{d}x\,\frac{1}{2}\big[F_2(x)_{\mathrm{p}} + F_2(x)_{\mathrm{n}}\big],\end{aligned} \tag{10.33}$$

where $\langle R\rangle$ and $\langle L\rangle$ imply that the appropriate averages of x have been taken. Then the total cross section is

$$\sigma_{\mathrm{tot}} = \frac{G^2 M E}{\pi}\int_0^1 \mathrm{d}x\,\frac{1}{2}\big[F_2(x)_{\mathrm{p}} + F_2(x)_{\mathrm{n}}\big]\left\{\frac{1}{2} + \frac{\langle L\rangle}{2} - \frac{\langle R\rangle}{6}\right\}. \tag{10.34}$$

The factor in the curly brackets lies between 1 and $\frac{1}{3}$. In particular,

$$\frac{1}{2} + \frac{1}{2}\langle L\rangle - \frac{1}{6}\langle R\rangle = \begin{cases} 1 & \text{if} \quad \sigma_{\mathrm{R}} = \sigma_{\mathrm{S}} = 0, \\ \dfrac{2}{3} & \text{if} \quad \sigma_{\mathrm{R}} = \sigma_{\mathrm{L}}, \sigma_{\mathrm{S}} = 0, \\ \dfrac{1}{2} & \text{if} \quad \sigma_{\mathrm{R}} = \sigma_{\mathrm{L}} = 0, \\ \dfrac{1}{3} & \text{if} \quad \sigma_{\mathrm{L}} = \sigma_{\mathrm{S}} = 0. \end{cases} \tag{10.35}$$

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

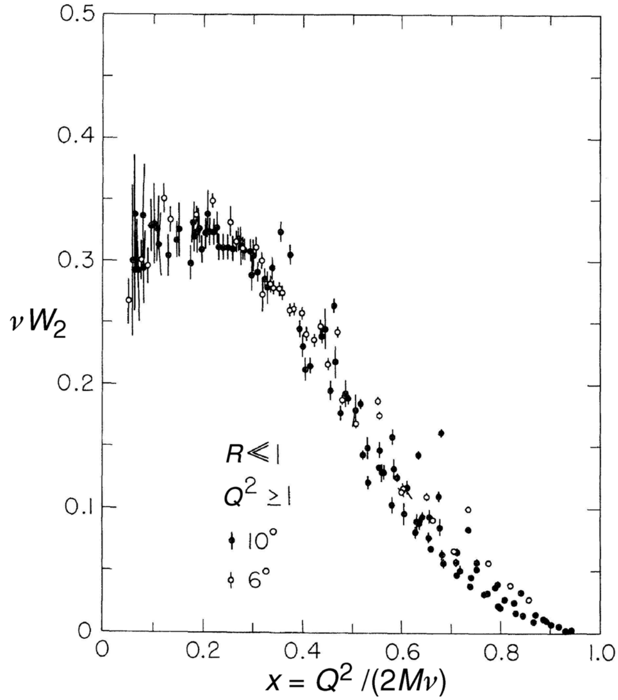

Figure 10.2. Scaling of the structure function $\nu W_2^{\mathrm{e}} = F_2(x)$.

From (10.33) we see that a linear rise in σ_{tot} depends on the property that νW_2 is scale-invariant and the absence of a W propagator. The neutrino measurements give

$$\sigma_{\text{tot}}^{\nu} = (0.677 \pm 0.014) \times 10^{-38}\,\text{cm}^2\,\frac{E_\nu}{\text{GeV}}.$$

We can also compare neutrino and antineutrino cross sections on isoscalar targets:

$$\frac{\sigma^{\bar{\nu}\text{N}}}{\sigma^{\nu\text{N}}} = \frac{\frac{1}{2} + \frac{1}{2}\langle R\rangle - \frac{1}{6}\langle L\rangle}{\frac{1}{2} + \frac{1}{2}\langle L\rangle - \frac{1}{6}\langle R\rangle}.$$

The ratio is bounded between $\frac{1}{3}$ and 3. The experimental data give

$$\sigma_{\text{tot}}^{\bar{\nu}} = (0.334 \pm 0.008) \times 10^{-38}\,\text{cm}^2\,\frac{E_{\bar{\nu}}}{\text{GeV}},$$

with the ratio of the two slopes being 0.501 ± 0.015, which is consistent with the above prediction and close to the lower bound.

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

10.4 The parton model

"Friends rush in where angels fear to tread."

(R. P. Feynman, at Fermilab, 1973)

A physical interpretation of the scaling phenomenon is given by the parton model, which considers the scattering as the incoherent sum of scattering from point-like constituents within the proton, called partons. The point-like nature of the constituents reproduces scaling. By studying several reactions it was possible to deduce properties of the constituents, such as electric charge, and identify the partons with quarks. The parton model has been applied to a wide range of high-energy reactions, many of which will be covered in this chapter. Deep inelastic reactions together with hadron spectroscopy supply the major evidence for the quark substructure of matter.

Neutrino–nucleon scattering

The basic idea in the parton model is to regard the deep inelastic scattering as quasi-free scattering from point-like constituents within the proton. This happens when the scattering is viewed from a frame in which the proton has infinite momentum. The neutrino–proton center-of-mass system is, at high energies, a good approximation of such a frame. In the infinite-momentum frame, the proton is Lorentz-contracted into a thin pancake, and the lepton scatters instantaneously. Furthermore, the proper motion of the constituents within the proton is slowed down by time dilatation. We estimate the interaction time and lifetime of the virtual states within the proton. In the notation of the previous section and Fig. 10.3, the initial electron and proton are collinear and in opposite directions:

$$\begin{aligned} \vec{k} &= -\vec{P}, \\ k_0 &\approx P_0 = P. \end{aligned} \tag{10.36}$$

In this frame

$$p \cdot q = M\nu = (q_0 + q_z)P, \tag{10.37}$$

$$k \cdot q = -Q^2/2 = (q_0 - q_z)P, \tag{10.38}$$

from which it follows that

$$q_0 = \frac{2M\nu - Q^2}{4P}. \tag{10.39}$$

The time of interaction is $\tau \approx 1/q_0$, which for moderate values of x decreases as

$$\tau = \frac{4P}{2M\nu(1-x)}. \tag{10.40}$$

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

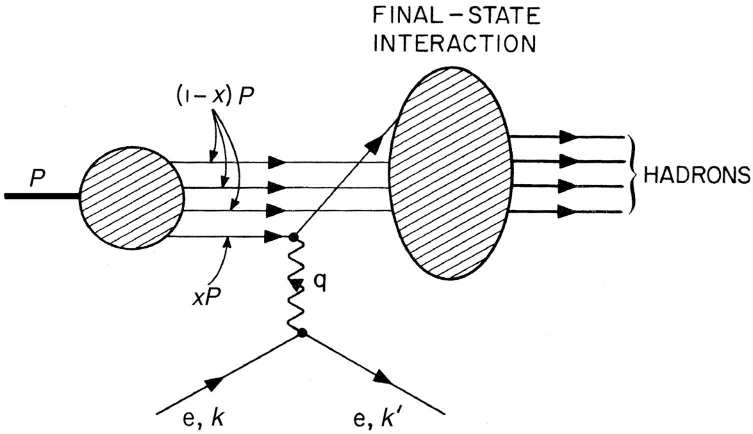

Figure 10.3. Kinematics for neutrino–nucleon scattering in the parton model.

We visualize the proton as composed of virtual states called partons. We denote by x the fraction of the proton's momentum carried by a constituent. The lifetime of the virtual states (Feynman, 1969; Bjorken and Paschos, 1969) is

$$T = \frac{1}{E_x + E_{1-x} - E_{\mathrm{p}}} = \frac{1}{\sqrt{(xP)^2 + \mu_1^2} + \sqrt{(1-x)^2P^2 + \mu_2^2} - \sqrt{P^2 + M^2}}$$

$$\approx \frac{2P}{\left(\mu_1^2 + P_{1\perp}^2\right)/x + \left(\mu_2^2 + P_{2\perp}^2\right)/(1-x) - M^2}. \tag{10.41}$$

If we now require that $\tau \ll T$, then we must consider the partons, contained in the proton, as free during the interaction. In this limit the current interacts with just one of the constituents, leaving the rest undisturbed, thus making the impulse approximation valid. The above conditions appear to be satisfied in high-energy and large-momentum-transfer electron–nucleon scattering and in high-energy neutrino–nucleon scattering. The model could fail for $x \to 0$ or 1, for which the expansion in (10.39) is no longer justified. The reader may have noticed that we use x with two meanings: the first one is Bjorken's variable x defined in Eq. (10.26) and the second is the fraction of the proton's momentum. This was done on purpose because the two variables are the same.

The cross section of a proton is the incoherent sum of cross sections of the individual constituents. We denote by $\mathrm{d}\sigma_i(x)/(\mathrm{d}Q^2\,\mathrm{d}\nu)$ the cross section of a neutrino on a parton of type i, which carries a fraction x of the proton's momentum,

$$p_i^\mu \approx xP^\mu. \tag{10.42}$$

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

We denote by $f_i(x)$ the probability of finding the ith constituent carrying a fraction x of the proton's momentum. Then the cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}\nu} = \sum_i \int_0^1 \frac{\mathrm{d}\sigma_i(x)}{\mathrm{d}Q^2\,\mathrm{d}\nu} f_i(x)\mathrm{d}x. \tag{10.43}$$

The summation here is over all types of constituents within the proton and the integral is over the momentum fraction x. Thus the complicated hadronic structure is reduced to the incoherent scattering from point-like constituents times the structure functions $f_i(x)$.

The point cross sections have already been derived in Section 8.3. For neutrino–parton scattering

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}(p_i q/m_i)} = \frac{G^2}{\pi}\delta\left(\frac{p_i q}{m_i} - \frac{Q^2}{2m_i}\right) \tag{10.44}$$

or

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}(p_i q)} = \frac{G^2}{\pi}\delta\left(p_i q - \frac{Q^2}{2}\right). \tag{10.45}$$

For neutrino–antiparton scattering

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}(p_i q)} = \frac{G^2}{\pi}\left(1 - \frac{p_i q}{p_i k^\mu}\right)^2 \delta\left(p_i q - \frac{Q^2}{2}\right), \tag{10.46}$$

with k^μ the four-momentum of the neutrino.

The proton is built from two up quarks and one down quark, which constitute the valence quarks and give the proton its quantum numbers. In addition, there is in the proton a cloud of quark–antiquark pairs produced by the radiation of gluons and their subsequent conversion into pairs. The number of the pairs is infinite, but their momentum distributions have not been calculated explicitly. We denote the probability of finding an up quark carrying a fraction x of the proton's momentum by $u(x)$. Similarly, we denote by $d(x)$ the probability of finding a down quark carrying a fraction x of the proton's momentum. The cloud of quark–antiquark pairs of any flavor necessitates the introduction of additional quark distribution functions. For instance, $\bar{u}(x)$ and $\bar{d}(x)$ correspond to up and down antiquarks. Similarly, there are distributions $s(x)$, $\bar{s}(x)$, $c(x)$, $\bar{c}(x)$, ... for strange, charm, and other flavors.

When we substitute $p_i^\mu = xP^\mu$ into (10.44) we obtain the point cross section

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}Q^2\,\mathrm{d}\nu} = \frac{G^2}{\pi} Mx\delta\left(xM\nu - \frac{Q^2}{2}\right). \tag{10.47}$$

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

Finally, on substituting the point cross sections in (10.42) and integrating over x, we arrive at the neutrino–proton scattering

$$\frac{d\sigma^{\nu p}}{dQ^2\,d\nu} = \frac{G^2}{\pi}\frac{x}{\nu}\left[d(x) + \bar{u}(x)\left(1 - \frac{\nu}{E}\right)^2\right]. \tag{10.48}$$

Here we omit the contribution from the strange and heavier quarks and their antiparticles. In the case of antineutrino–proton scattering we obtain

$$\frac{d\sigma^{\bar{\nu} p}}{dQ^2\,d\nu} = \frac{G^2}{\pi}\frac{x}{\nu}\left[\bar{d}(x) + u(x)\left(1 - \frac{\nu}{E}\right)^2\right]. \tag{10.49}$$

It is now evident that the momentum fraction $x = Q^2/(2M\nu)$ is indeed the Bjorken scaling variable.

The general case with many families of quarks can be easily written down. The contribution from a quark $q(x)$ and an antiquark $\bar{q}(x)$ is

$$\frac{d\sigma^{\nu N}}{dx\,dy} = \frac{G^2}{\pi} 2MEx\left[q(x) + (1-y)^2\bar{q}(x)\right], \tag{10.50}$$

provided that the quark under consideration is allowed by charge conservation. Similarly, the antineutrino–nucleon cross section is

$$\frac{d\sigma^{\bar{\nu} N}}{dx\,dy} = \frac{G^2}{\pi} 2MEx\left[(1-y)^2 q(x) + \bar{q}(x)\right]. \tag{10.51}$$

These relations are used to determine the antiquark content of the proton. For instance, the antineutrino–nucleon cross section at $y = 1$ measures $\bar{q}$. The total cross sections are also easily derived. They grow linearly with neutrino or antineutrino energy.

Finally the ratio of the total cross sections for an isoscalar target, such as deuterium or oxygen, is

$$\frac{\sigma^{\bar{\nu} d}}{\sigma^{\nu d}} = \frac{\int_0^1 dx\, x\left[\frac{1}{3}(u+d) + (\bar{u}+\bar{d})\right]}{\int_0^1 dx\, x\left[(u+d) + \frac{1}{3}(\bar{u}+\bar{d})\right]}. \tag{10.52}$$

Taking the experimental ratio of the cross sections to be approximately 0.50, we arrive at the conclusion that the integrated antiquark contribution is approximately 20% of the quark contribution.

We close this section with a few remarks. We derived the general formulas for neutrino- and antineutrino-induced reactions using helicity cross sections. We have also shown explicitly that they are related to the structure functions. The formalism can be carried over to electroproduction, for which similar formulas hold. We also emphasized that high-energy neutrino reactions are closely related to electroproduction in the deep inelastic region.

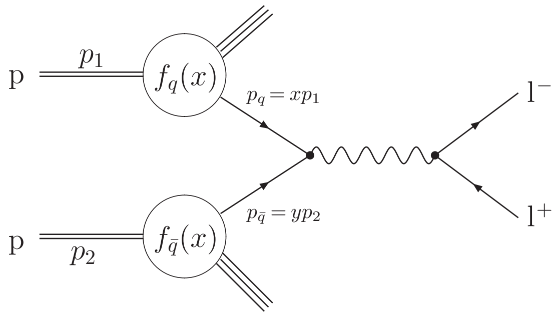

Figure 10.4. The Drell–Yan process.

Both reactions were analyzed in terms of the parton model, assuming that the constituents of protons are the quarks (Bjorken and Paschos, 1969). This will be further developed in the next two chapters, where the quark-parton content of hadrons becomes more evident.

10.5 The Drell–Yan process

The production of a massive photon or of a $W^{\pm}$ in hadron–hadron collisions and its subsequent decay has been successfully analyzed in terms of the parton model. The reactions (Drell and Yan, 1970)

$$\mathrm{p} + \mathrm{p} \to \gamma + \cdots \to \mu^+\mu^- + \mathrm{X}, \tag{10.53}$$

$$\bar{\mathrm{p}} + \mathrm{p} \to \mathrm{W} + \cdots \to \mathrm{e}^-\bar{\nu} + \mathrm{X} \tag{10.54}$$

are known as Drell–Yan processes. Together with deep inelastic scattering and electron–positron annihilation, these processes play an important role in determining the structure functions and in testing the parton model, including QCD corrections. The Drell–Yan process was especially important in formulating a strategy for seeking and discovering the W bosons.

To calculate the cross section corresponding to Fig. 10.4, we begin with the parton subprocess,

$$\sigma(\bar{\mathrm{q}}\mathrm{q} \to \ell^+\ell^-) = \frac{4\pi\alpha^2}{3Q^2} e_q^2. \tag{10.55}$$

In order to embed it in the hadronic process, we rewrite it as a differential cross section, $\mathrm{d}\sigma/\mathrm{d}Q^2$, for the production of a lepton pair with invariantmass $\sqrt{Q^2}$,

https://doi.org/10.1017/9781009402378.011 Published online by Cambridge University Press

where

$$Q^2 = \hat{s} = (p_q + p_{\bar{q}})^2, \tag{10.56}$$

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{4\pi\alpha^2}{3Q^2} e_q^2 \delta(Q^2 - \hat{s}). \tag{10.57}$$

We envisage each hadron of momentum P being made up of partons carrying a longitudinal momentum xP. We make the idealization that the partons carry negligible transverse momentum. When the mass of the produced pair is very large, the cross section is the incoherent sum of the elementary subprocesses. In this case a quark of type q from one hadron annihilates with an antiquark of the same type from the other hadron. The probability of finding the quark with fractional momentum x is given by $f_q(x)$ and that for the antiquark by $f_{\bar{q}}(y)$. The hadronic cross section can now be obtained (Drell and Yan, 1970):

$$\frac{d\sigma}{dQ^2}(\mathrm{pp} \to \ell^+\ell^-\mathrm{X}) = \overline{\sum_q} e_q^2 \int dx \int dy\, f_q(x) f_{\bar{q}}(y) \frac{d\hat{\sigma}}{dQ^2}, \tag{10.58}$$

where the sum is over all possible $q\bar{q}$ pairs that can be formed from the constituents of the colliding protons and the average is over the number of initial $q\bar{q}$ states. This gives in the end an overall factor of $\frac{1}{3}$.

The q and $\bar{q}$ carry the fractions x and y of the proton momenta and the invariant mass becomes

$$\hat{s} = (xp_1 + yp_2)^2 \approx xys, \tag{10.59}$$

with $s \approx 2p_1 \cdot p_2$. The cross section now takes the form

$$\frac{d\sigma}{dQ^2} = \frac{4}{9}\frac{\pi\alpha^2}{Q^2} \int dx \int dy\, f_q(x) f_{\bar{q}}(y) \delta(Q^2 - xys). \tag{10.60}$$

After integration over y, we obtain the final result

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{9Q^2 s} \int_{Q^2/s}^{1} \frac{dx}{x} f_q(x) f_{\bar{q}}\left(\frac{Q^2}{xs}\right). \tag{10.61}$$

To lowest order (without gluon emission) we expect a scaling result: the last integral depends on the ratio $\tau = Q^2/s$. The scaling is satisfied but the overall rate is modified by QCD corrections, which involve gluons. In this case the corrections are substantial and the reader should consult specialized articles for more details.

In addition to the quarks, the proton contains also gluons, i.e. vector mesons that mediate the strong interactions. This requires that for each hadron we must introduce a gluon distribution function: $g(x)$. In several processes gluons play an important role. For instance, the production of Higgses in high-energy colliders proceeds through the fusion of two gluons,

$$\mathrm{g} + \mathrm{g} \rightarrow \mathrm{H} \rightarrow \mathrm{ZZ}, \tag{10.62}$$

with g denoting gluons and the Higgs decaying to two lighter particles (in this case Z bosons). The hadronic reaction can be analyzed as a Drell–Yan process with the quarks of the intermediate states replaced by gluons.

Let us consider the process

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{H} + \text{hadrons} \rightarrow \mathrm{Z\,Z} + \text{hadrons} \tag{10.63}$$

and denote by $\sigma_0(\mathrm{gg} \rightarrow \mathrm{H} \rightarrow \mathrm{ZZ})$ the point cross section for the production of two Z bosons. The gluon distribution function for protons has been measured in DESY experiments to be large at small values of x. The cross section for the production of Z pairs through two gluons with moments xp_1 and yp_2, respectively, is given by

$$\frac{\mathrm{d}\sigma_0}{\mathrm{d}Q^2} = \sigma_0(Q^2)\delta(Q^2 - xys). \tag{10.64}$$

The cross section for the proton–proton collision is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \int \frac{\sigma_0(Q^2)}{s}\frac{\mathrm{d}x}{x}\, g(x) g\left(\frac{Q^2}{xs}\right). \tag{10.65}$$

One usually takes the gluon structure functions from electron–proton-scattering experiments and extrapolates them to regions of small x and large Q^2 by means of the renormalization-group equations. In addition to the corrected structure functions, the calculations must include corrections to the gluon–Higgs-boson coupling induced again by virtual gluons.

Problems for Chapter 10

1. Show that Eq. (10.5) is determined up to an overall phase.
2. Determine the behaviour of $\varepsilon_\mu^{\mathrm{R,L}}$ under rotations around the z-axis.
   This can be done easily if you split $\varepsilon_\mu^{\mathrm{R,L}}$ into $\varepsilon_\mu^{(x)} = (0,\ 1,\ 0,\ 0)$ and $\varepsilon_\mu^{(y)} = (0,\ 0,\ 1,\ 0)$.
3. Prove Eq. (10.7) with the following *Ansatz*: $j_\mu^{\mathrm{lept}} = a\varepsilon_\mu^{\mathrm{S}} + b\varepsilon_\mu^{\mathrm{R}} + c\varepsilon_\mu^{\mathrm{L}}$.
   Determine a, b, and c using $k^\mu = (E,\ k_x,\ 0,\ k_z)$, $q^\mu = (\nu,\ 0,\ 0,\ q_z)$, and momentum conservation. Apply the high-energy limit $\nu \gg 2M$ and $Q^2 \ll \nu^2$ in order to obtain Eq. (10.7).
4. Derive Eq. (10.22) starting from Eq. (10.19).
5. In order to prove Eq. (10.30) rewrite j_μ^{lept} as

   $$j_\mu^{\mathrm{lept}} = \frac{1}{2}\bar{u}'\gamma_\mu(1 + \gamma_5 + 1 - \gamma_5)u$$

   and follow steps similar to those in the case of neutrino–hadron scattering.

6. Check the individual steps leading to Eq. (10.34).
7. Carry out the various steps leading to Eqs. (10.47) and (10.48).
8. Calculate the helicity cross section for a left-handed W scattered on quarks and show that it reproduces the result in Eq. (8.45).

References

Bjorken, J. D. (1969), *Phys. Rev*. **179**, 1547
Bjorken, J. D., and Paschos, E. A. (1969), *Phys. Rev*. **185**, 1975
(1970), *Phys. Rev*. **D1**, 3151
Drell, S. D., and Yan, T. M. (1970), *Phys. Rev. Lett*. **25**, 25
Feynman, R. P. (1969), *Phys. Rev. Lett*. **23**, 1415