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LINEAR ISOMETRIES OF SOME NORMED SPACES OF ANALYTIC FUNCTIONS

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1. Introduction. For $1 \leqq p < \infty$ let H^p denote the familiar Hardy space of analytic functions on the open unit disc D and let $\|\cdot\|_p$ denote the H^p norm. Let S^p denote the space of analytic functions f on D such that $f' \in H^p$. In this paper we will describe the linear isometries of S^p into itself when S^p is equipped with either of two norms. The first norm we consider is given by

(1) $\quad \|f\| = |f(0)| + \|f'\|_p$

and the second by

(2) $\quad \|f\| = \|f\|_\infty + \|f'\|_p.$

(It is well known [**1**, Theorem 3.11] that $f' \in H^p$ implies continuity for f on $\bar{D}$, the closure of D. Thus (2) actually defines a norm on S^p.) In the former case, with the norm defined by (1), we will show that an isometry of S^p induces, in a sense to be made precise in Section 2, an isometry of H^p and that Forelli's characterization [**2**] of the isometries of H^p can thus be used to describe the isometries of S^p. In the latter case, with the norm defined by (2), the approach begins similarly. But here the material is essentially self-contained and we obtain a very simple description of the isometries.

2. The isometries of S^p with $\|f\| = |f(0)| + \|f'\|_p$.

THEOREM 2.1. *Let T be a linear isometry of S^p into S^p. Then there is a linear isometry τ of H^p into H^p and a unimodular complex number λ such that*

$$Tf(z) = \lambda[f(0) + \int_0^z \tau f'(\zeta)d\zeta] \quad (f \in S^p, z \in D).$$

Proof. Let n be a positive integer, t be a real number and consider the function $f = 1 + tZ^n$. This polynomial is in S^p and hence

$$\begin{aligned} 1 + n|t| &= \|f\| = \|Tf\| \\ &= |T1(0) + tTZ^n(0)| + \|(T1)' + t(TZ^n)'\|_p \end{aligned}$$

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$$\leqq |T1(0)| + |t|\,|TZ^n(0)| + \|(T1)'\|_p + |t|\,\|(TZ^n)'\|_p$$
$$= |T1(0)| + \|(T1)'\|_p + |t|\,(|TZ^n(0)| + \|(TZ^n)'\|_p)$$
$$= \|T1\| + |t|\,\|TZ^n\| = \|1\| + |t|\,\|Z^n\| = 1 + n|t|.$$

Thus it follows that

(3) $\quad |T1(0) + tTZ^n(0)| = |T1(0)| + |t|\,|TZ^n(0)|$

and

(4) $\quad \|(T1)' + t(TZ^n)'\|_p = \|(T1)'\|_p + |t|\,\|(TZ^n)'\|_p.$

Now choose a value for n such that TZ^n is a non-constant function. (There is at most one value for n for which this is not the case.) Consider the function

$$\rho(t) = \|(T1)' + t(TZ^n)'\|_p.$$

Since $\|(TZ^n)'\|_p \neq 0$, it follows from (4) that ρ is not a differentiable function of t at $t = 0$. That is, in the terminology of [**3**], the L^p norm is not weakly differentiable at $(T1)'$. Consequently (cf. [**3**, p. 350])

$$(T1)'(e^{it}) = 0$$

on a set of positive measure if $p = 1$, and

$$(T1)'(e^{it}) = 0 \text{ a.e. if } p > 1.$$

But since $(T1)' \in H^p$, $(T1)' \equiv 0$ in either case [**1**, Theorem 2.2]. Thus $T1$ is a constant function λ, necessarily of modulus one, and (3) now implies that

$$TZ^n(0) = 0 \quad \text{for } n = 1, 2, \ldots .$$

Multiplying the operator T by the unimodular complex number $\bar{\lambda}$, we assume that $T1 = 1$. Now for any $f \in S^p$, the equations

$$1 + |t|\,\|f'\|_p = \|1 + t(f - f(0))\|$$
$$= \|T1 + tT(f - f(0))\| = \|1 + t(Tf - f(0))\|$$
$$= |1 + t(Tf(0) - f(0))| + |t|\,\|(Tf)'\|_p$$

imply that

$$1 + |t|\,(\|f'\|_p - \|(Tf)'\|_p) = |1 + t(Tf(0) - f(0))|.$$

Since the left hand side of the preceding equation depends only on $|t|$, it follows that

(5) $\quad Tf(0) = f(0),\ \|(Tf)'\|_p = \|f'\|_p.$

Now we complete the proof of the theorem. Let S_0^p denote the subspace of S^p of functions which are zero at 0. The differentiation operator D maps

S_0^p isometrically onto H^p, and its inverse I, given by

$$Ig(z) = \int_0^z g(\zeta)d\zeta, \quad (g \in H^p)$$

maps H^p isometrically onto S_0^p. Since T maps S_0^p into S_0^p (this follows from the first equation in (5)), we see that $\tau = D \circ T \circ I$ is a linear isometry from H^p into H^p. Finally, if $f \in S^p$ then

$$f - f(0) = I(f')$$

and thus

$$D(T(f - f(0))) = \tau f'.$$

This says that $(Tf)' = \tau(f')$, and so

$$Tf(z) = f(0) + \int_0^z \tau f'(\zeta)d\zeta, \quad (z \in D).$$

This completes the proof.

Remark. The converse of the preceding result is easily verified.

In [**2**] Forelli characterized the isometries of H^p. If we apply his theorems, then we obtain the following results.

COROLLARY 2.2. *Let T be as in* 2.1 *and assume that $p \neq 2$. Then there is a non-constant inner function ϕ and a function F in H^p such that*

$$Tf(z) = \lambda[f(0) + \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta] \quad (z \in D, f \in S^p).$$

Proof. This follows immediately from Theorem 2.1 and [**2**, Theorem 1] as applied to the isometry τ.

If we assume that the isometry T maps S^p onto S^p, then much more can be said about the functions F and ϕ that appear in the above corollary. Specifically, we get the following result.

THEOREM 2.3. *Let T be a linear isometry of S^p onto S^p, $p \neq 2$. Then there are unimodular complex numbers λ, μ and a conformal map ϕ of the unit disc D such that*

$$Tf(z) = \lambda[f(0) + \mu\int_0^z [\phi'(\zeta)]^{1/p}f'(\phi(\zeta))d\zeta] \quad (f \in S^p, z \in D).$$

Conversely, this equation defines an isometry T of S^p onto S^p.

Proof. This is a consequence of 2.1 and [**2**, Theorem 2]. For if T maps S^p onto itself, then the H^p isometry τ is also onto and Forelli's Theorem 2 says that

$$\tau g = \mu[\phi']^{1/p}g \circ \phi \quad (g \in H^p),$$

where $|\mu| = 1$ and ϕ is a conformal map of D. By substituting this for $\tau f'$ in 2.1 we obtain the formula for T. The converse is readily verified and we omit the details.

The isometries of H^1 onto itself were first found by de Leeuw, Rudin, and Wermer in [**4**] and it is interesting to look at Theorem 2.3 in the case $p = 1$. The conclusion is that an isometry T of S^1 has the form

$$Tf = \lambda[f(0) + \mu(f \circ \phi - f(\phi(0)))].$$

Now the work in the present paper was in large part motivated by a result due to R. Roan [**5**, Theorem 11] contained in a paper concerned with composition operators of various types on S^p. By a composition operator on S^p we mean an operator

$$T:S^p \to S^p$$

for which there is a function $\psi:D \to D$ such that

$$Tf = f \circ \psi \quad \text{for all } f \in S^p.$$

Roan shows [**5**, Theorem 11] that if T is a composition operator and if T is an isometry (not necessarily onto) of S^p with respect to the norm used in this section, then ψ is a conformal map of D onto itself such that $\psi(0) = 0$, i.e., a rotation. The point in our looking at the conclusion of Theorem 2.3 in the case $p = 1$ is that it suggests, at least for surjective isometries, that they are necessarily "close" to being composition operators. In fact, as we will now show, if we do assume that an arbitrary isometry T of S^p is a composition operator and is induced by some ψ, then Roan's result can be obtained as an immediate consequence of Corollary 2.2 as follows. For we have an H^p function F, an inner function ϕ and a map $\psi:D \to D$ such that $Tf = f \circ \psi$ and (by differentiation)

$$(f' \circ \psi)\psi' = Ff' \circ \phi \quad (f \in S^p).$$

Taking $f = Z$ we get $\psi' = F$ so $\psi \in S^p$ and

$$(f' \circ \psi)\psi' = (f' \circ \phi)\psi' \quad (f \in S^p).$$

Next taking $f = Z^2$ yields $\psi\psi' = \phi\psi'$ and since $\psi' \neq 0$ a.e. on $|z| = 1$, it follows that $\psi = \phi$. Consequently, ψ is an inner function in S^p and

$$0 = Z(0) = TZ(0) = \psi(0).$$

Also, since ψ is continuous on $\bar{D}$, ψ must be a finite Blaschke product and consequently, maps the unit circle onto itself. It follows that

$$1 \leqq \|\psi'\|_1 \leqq \|\psi'\|_p = \|(TZ)'\|_p = 1,$$

hence equality holds throughout and so

$$\begin{aligned}
2\pi &= \int_0^{2\pi} |\psi'(e^{it})|\,dt \\
&\geqq \left| \int_0^{2\pi} \frac{\psi'(e^{it})}{\psi(e^{it})}\, ie^{it}dt \right| \\
&= \left| \int_{\partial D} \frac{\psi'(\zeta)}{\psi(\zeta)}\, d\zeta \right| \\
&= 2\pi \text{ (number of zeros of } \psi \text{ in } D).
\end{aligned}$$

Thus ψ has exactly one zero in D and since $\psi(0) = 0$, ψ is necessarily a rotation of D.

In the next section we will consider the isometries of S^p when the norm is given by (2) and will show that these are all composition operators induced by conformal maps of D. In fact when $p > 1$ these conformal maps must actually be rotations of D.

3. The isometries of S^p with $||f|| = ||f||_\infty + ||f'||_p$.

THEOREM 3.1. *Let T be an isometry of S^p into S^p. Then T has the form*

$$Tf = \lambda f \circ \phi$$

where λ is a unimodular complex number and ϕ is a conformal map of D. If $p > 1$, ϕ is necessarily a rotation of D.

Proof. The argument is rather lengthy and will be given by establishing several claims and lemmas.

CLAIM (i) $T1$ *is a constant function of modulus one.*

Proof. As in 2.1, let n be a positive integer, t be a complex number, and $f = 1 + tZ^n$. Then, as before, we obtain

$$\begin{aligned}
1 + |t| + n|t| &= ||f|| = ||Tf|| \\
&= ||T1 + tTZ^n||_\infty + ||(T1)' + t(TZ^n)'||_p \\
&\leqq ||T1||_\infty + |t|\,||TZ^n||_\infty + ||(T1)'||_p + |t|\,||(TZ^n)'||_p \\
&= ||T1|| + |t|\,||TZ^n|| = ||1|| + |t|\,||Z^n|| \\
&= 1 + |t| + n|t|.
\end{aligned}$$

Thus we have

$$(6) \qquad ||T1 + tTZ^n||_\infty = ||T1||_\infty + |t|\,||TZ^n||_\infty$$

and

$$(7) \qquad ||(T1)' + t(TZ^n)'||_p = ||(T1)'||_p + |t|\,||(TZ^n)'||_p.$$

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Again (7) implies that $(T1)' \equiv 0$, hence $T1$ is a constant function of modulus one. Thus we can assume that $T1 = 1$.

CLAIM (ii) $\|TZ^n\|_\infty \geqq 1$.

Proof. Let m be a positive integer with $m \neq n$. Then for any real number t,

$$\begin{aligned} &\|TZ^m + tTZ^n\|_\infty + \|(TZ^m)' + t(TZ^n)'\|_p \\ &= \|Z^m + tZ^n\|_\infty + \|mZ^{m-1} + ntZ^{n-1}\|_p \\ &= \|1 + tZ^{n-m}\|_\infty + \|mZ^{m-1} + ntZ^{n-1}\|_p \\ &= 1 + |t| + \|mZ^{m-1} + ntZ^{n-1}\|_p. \end{aligned}$$

Thus

$$\begin{aligned} (8) \qquad &\|TZ^m + tTZ^n\|_\infty - |t| \\ &= 1 + \|mZ^{m-1} + ntZ^{n-1}\|_p - \|(TZ^m)' + t(TZ^n)'\|_p. \end{aligned}$$

Since $(TZ^m)' \neq 0$ a.e., the right hand side of (8) is a differentiable function of t at $t = 0$, with derivative equal to, say, c. Thus we can write the left hand side as

$$(9) \qquad \|TZ^m\|_\infty + ct + o(t).$$

Also,

$$\|TZ^m + tTZ^n\|_\infty \leqq \|TZ^m\|_\infty + |t|\,\|TZ^n\|_\infty$$

so it follows from (8) and (9) that

$$|t|\,\|TZ^n\|_\infty \geqq |t| + ct + o(t).$$

Hence

$$\|TZ^n\|_\infty \geqq 1 + c\frac{t}{|t|} + \frac{o(t)}{|t|}.$$

From this inequality we get that

$$\|TZ^n\|_\infty \geqq 1 + |c|,$$

so that $\|TZ^n\|_\infty \geqq 1$ as claimed.

CLAIM (iii). *TZ is a conformal map of D onto D which is a rotation if $p > 1$.*

In order to establish (iii) we will need the following elementary lemma and remark.

LEMMA 3.2. *Suppose that the range of a function $f:[a, b] \to \mathbf{C}$ contains a circular arc $\{Me^{it}:0 \leqq t \leqq \theta\}$. Then the variation of f on $[a, b]$ is $\geqq M\theta$, the length of the arc.*

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Proof. Let $\epsilon > 0$ and choose consecutive and equally spaced points $z_0, z_1, \ldots, z_n$ along with the arc with $z_0 = M$, $z_n = Me^{i\theta}$, and such that

$$\sum_{j=1}^{n} |z_j - z_{j-1}| > M\theta - \epsilon.$$

Since f maps $[a, b]$ onto the arc, there are points $t_0, t_1, \ldots, t_n$ in $[a, b]$ such that

$$f(t_j) = z_j, \quad j = 0, 1, \ldots, n.$$

Let $s_0, s_1, \ldots, s_n$ be the increasing rearrangement of $t_0, t_1, \ldots, t_n$. Then

$$|f(s_j) - f(s_{j-1})| \geqq |z_j - z_{j-1}|,$$

so

$$\sum_{j=1}^{n} |f(s_j) - f(s_{j-1})| \geqq M\theta - \epsilon.$$

Remark 3.3. For $n = 1, 2, \ldots$ the function TZ^n maps the unit circle $|z| = 1$ onto a set that contains the circle with center at the origin and radius equal to $\|TZ^n\|_\infty$. (This follows from the equation

$$\|1 + tTZ^n\|_\infty = 1 + |t| \, \|TZ^n\|_\infty, \quad (t \in \mathbf{C}),$$

which is just (6) with $T1 = 1$.)

Proof of (iii). We are going to apply the preceding lemma to

$$f(t) = TZ(e^{it}), \quad 0 \leqq t \leqq 2\pi.$$

Now the function TZ maps the unit circle onto a set that contains the circle

$$|z| = \|TZ\|_\infty \geqq 1.$$

Thus by Lemma 3.2 (since TZ is absolutely continuous)

$$2\pi\|TZ\|_\infty \leqq \int_0^{2\pi} |(TZ)'(e^{it})| dt.$$

Hence,

$$1 \leqq \|TZ\|_\infty \leqq \|(TZ)'\|_1 \leqq \|(TZ)'\|_p = 2 - \|TZ\|_\infty \leqq 1.$$

In particular,

$$1 = \|TZ\|_\infty = \|(TZ)'\|_1 = \|(TZ)'\|_p.$$

But from $\|(TZ)'\|_1 = 1$ and the fact that TZ maps $|z| = 1$ onto a set containing the unit circle, we can conclude that $|TZ| \equiv 1$. Thus TZ must be a finite Blaschke product. Then the equation

$$\| (TZ)'\|_1 = 1$$

implies as before that TZ is a conformal map of D onto itself. And if $p > 1$, then the equality

$$\| (TZ)'\|_p = \| (TZ)'\|_1$$

implies that $|(TZ)'| \equiv 1$, so that the conformal map TZ must actually be a rotation of D in this case. This completes the proof of (iii).

Set $\phi = TZ$. We know now that ϕ is a conformal map of D and we are first going to prove the theorem in the special case $\phi = Z$. We will then derive the general result from this special case. Thus our next step is to establish the following.

CLAIM (iv). *If* $T1 = 1$ *and* $TZ = Z$, *then* $TZ^n = Z^n$ *for all* n.

To prove this we will need two additional lemmas.

LEMMA 3.4. *Given* $0 < \epsilon < 1$, *there exists* $\delta > 0$ *such that if* $|w| \leqq 1$, $|w - 1| \geqq \epsilon$ *and* $|z| \leqq 1$, $|1 - z| \leqq \delta$, *then*

$$|w + rz| \leqq 1 + r(1 - \delta), \quad 0 < r < 1.$$

Proof. The inequality to be established is equivalent to

$$(10) \quad |w|^2 + r^2|z|^2 + 2r\mathrm{Re}(w\bar{z})$$
$$\leqq 1 + r^2(1 - \delta)^2 + 2r(1 - \delta), \quad 0 < r < 1.$$

The inequalities $|w| \leqq 1$, $|w - 1| \geqq \epsilon$ imply the existence of some small $\eta = \eta(\epsilon) > 0$ such that

$$(11) \quad \mathrm{Re}(w\bar{z}) \leqq 1 - \eta$$

whenever $\delta > 0$ is suitably small and $|z - 1| < \delta$. (In fact, if $\delta \leqq \epsilon/2$, then we can take $\eta = \epsilon^2/8$.) With (11), (10) will hold whenever

$$1 + r^2 + 2r(1 - \eta) \leqq 1 + r^2(1 - \delta)^2 + 2r(1 - \delta), 0 < r < 1.$$

But this latter inequality holds if $\delta < \eta/2$.

LEMMA 3.5.

$$\|\alpha + (TZ^n)'\|_p = 1 - \|TZ^n\|_\infty + \| \,|\alpha| + nZ^{n-1}\|_p + o(\,|\alpha|\,)$$

provided the argument of α *assumes only finitely many values as* $\alpha \to 0$.

Proof. We are assuming now that $T1 = 1$ and $TZ = Z$. Then for complex α and $n > 1$,

$$|\alpha| + 1 + \|\alpha + nZ^{n-1}\|_p$$
$$= \|\alpha Z + Z^n\|_\infty + \|\alpha + nZ^{n-1}\|_p$$
$$= \|\alpha Z + TZ^n\|_\infty + \|\alpha + (TZ^n)'\|_p.$$

Thus

(12) $\|\alpha + (TZ^n)'\|_p = |\alpha| + 1 + \|\alpha + nZ^{n-1}\|_p - \|\alpha Z + TZ^n\|_\infty.$

Consider

$$|\alpha| - \|\alpha Z + TZ^n\|_\infty + \|TZ^n\|_\infty.$$

This expression is 0 if $\alpha = 0$ and by the triangle inequality is non-negative for all α. Furthermore, by (12),

(13) $|\alpha| - \|\alpha Z + TZ^n\|_\infty = \|\alpha + (TZ^n)'\|_p - \|\alpha + nZ^{n-1}\|_p - 1.$

So if α is restricted to a line through the origin, say $\alpha = te^{i\theta}$, then the right-hand side of (13) is a differentiable function of t at $t = 0$. Thus the same is true of

$$|\alpha| - \|\alpha Z + TZ^n\|_\infty + \|TZ^n\|_\infty.$$

Consequently,

$$|\alpha| - \|\alpha Z + TZ^n\|_\infty + \|TZ^n\|_\infty = o(|\alpha|)$$

provided that the argument of α assumes only finitely many values as $\alpha \to 0$. If we substitute

$$o(|\alpha|) - \|TZ^n\|_\infty$$

for

$$|\alpha| - \|\alpha Z + TZ^n\|_\infty$$

in (12) and note that

$$\|\alpha + nZ^{n-1}\|_p = \|\, |\alpha| + nZ^{n-1}\|_p,$$

we obtain the lemma.

Now we can prove claim (iv) which is that

$$TZ^n(e^{i\theta}) = e^{in\theta} \quad \text{for all } \theta.$$

First choose θ_1 such that

$$TZ^n(e^{i\theta_1}) = \|TZ^n\|_\infty.$$

(Such a θ_1 exists because TZ^n maps $|z| = 1$ onto the circle $|z| = \|TZ^n\|_\infty$.) We will show that

(14) $TZ^n(e^{i\theta_1}e^{i\theta}) = e^{in\theta}\|TZ^n\|_\infty$

for all θ from which it will follow that

(15) $TZ^n = e^{-in\theta_1}\|TZ^n\|_\infty Z^n.$

But once (15) is obtained, the equation

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$$\|Z^n\|_\infty + \|nZ^{n-1}\|_p = \|TZ^n\|_\infty + \|(TZ^n)'\|_p$$

implies that

$$1 + n = \|TZ^n\|_\infty (1 + n).$$

Thus $\|TZ^n\|_\infty = 1$ and so

(16) $TZ^n = e^{-in\theta_1}Z^n.$

Finally, from (16) and the equation

$$\|1 + Z + Z^n\|_\infty + \|1 + nZ^{n-1}\|_p$$
$$= \|1 + Z + e^{-in\theta_1}Z^n\|_\infty + \|1 + ne^{-in\theta_1}Z^{n-1}\|_p,$$

we find, since

$$\|1 + nZ^{n-1}\|_p = \|1 + ne^{-in\theta_1}Z^{n-1}\|_p,$$

that

$$3 = \|1 + Z + Z^n\|_\infty = \|1 + Z + e^{-in\theta_1}Z^n\|_\infty.$$

Hence $e^{-in\theta_1} = 1$. So the proof of claim (iv) will be complete when we establish (14). Now suppose (14) did fail for some θ, say θ_0. Then by continuity there exist $\epsilon_1 > 0$ and $\delta_1 > 0$ such that if

$$|e^{i\theta} - e^{i\theta_0}| < \delta_1,$$

then

(17) $|TZ^n(e^{i\theta_1}e^{i\theta}) - e^{in\theta_0}\|TZ^n\|_\infty| \geqq \epsilon_1.$

Now apply Lemma 3.4 with

$$\epsilon = \frac{\epsilon_1}{\|TZ^n\|_\infty}$$

to obtain a positive number δ such that the conclusion of that lemma holds. By choosing a smaller δ_1 we can have

(18) $\left|\dfrac{1 + e^{i\theta}e^{-i\theta_0}}{2} - 1\right| < \delta$

along with (17) if

$$|e^{i\theta} - e^{i\theta_0}| < \delta_1.$$

Now suppose that

$$|e^{i\theta} - e^{i\theta_0}| < \delta_1.$$

Then (17) implies that

$$\left|\frac{e^{-in\theta_0}TZ^n(e^{i\theta_1}e^{i\theta})}{\|TZ^n\|_\infty} - 1\right| \geqq \epsilon$$

and so Lemma 3.4, with

$$w = \frac{e^{-in\theta_0}TZ^n(e^{i\theta_1}e^{i\theta})}{\|TZ^n\|_\infty},\ z = \frac{1 + e^{i\theta}e^{-i\theta_0}}{2},$$

yields by (18),

$$\left|\frac{e^{-in\theta_0}TZ^n(e^{i\theta_1}e^{i\theta})}{\|TZ^n\|_\infty} + r\frac{1 + e^{i\theta}e^{-i\theta_0}}{2}\right|$$

$$\leqq 1 + r(1 - \delta) \quad \text{for } 0 < r < 1.$$

Equivalently,

$$\left|TZ^n(e^{i\theta_1}e^{i\theta}) + re^{in\theta_0}\frac{1 + e^{i\theta}e^{-i\theta_0}}{2}\right|$$

$$\leqq \|TZ^n\|_\infty + r(1 - \delta) \quad \text{for } 0 < r < \|TZ^n\|_\infty.$$

But there is a small $\delta_2 > 0$ such that if

$$|e^{i\theta} - e^{i\theta_0}| \geqq \delta_1,$$

then

$$\left|\frac{1 + e^{i\theta}e^{-i\theta_0}}{2}\right| \leqq 1 - \delta_2.$$

Thus if we replace $1 - \delta$ with the larger of $1 - \delta$ and $1 - \delta_2$, we have that

$$\left|TZ^n(e^{i\theta_1}e^{i\theta}) + re^{in\theta_0}\frac{1 + e^{i\theta}e^{-i\theta_0}}{2}\right|$$

$$\leqq \|TZ^n\|_\infty + r(1 - \delta)$$

$$\text{for } 0 < r < \|TZ^n\|_\infty \text{ and all } \theta.$$

After the change of variable $z \to ze^{-i\theta_1}$, this is

$$\|TZ^n + re^{in\theta_0}\frac{1 + Ze^{-i(\theta_1+\theta_0)}}{2}\|_\infty \tag{19}$$

$$\leqq \|TZ^n\|_\infty + r(1 - \delta), \quad 0 < r < \|TZ^n\|_\infty.$$

Now we will deduce a contradiction from (19) and Lemma 3.5. Since θ_1 was chosen so that $TZ^n(e^{i\theta_1}) = \|TZ^n\|_\infty$,

$$\|TZ^n\|_\infty + r = \|TZ^n + r\frac{1 + Ze^{-i\theta_1}}{2}\|_\infty$$

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$$= \|Z^n + r\frac{1 + Ze^{-i\theta_1}}{2}\|_\infty + \|nZ^{n-1} + \frac{r}{2}e^{-i\theta_1}\|_p$$

$$- \|(TZ^n)' + \frac{r}{2}e^{-i\theta_1}\|_p =$$

(by change of variable $z \to e^{-i\theta_0}z$)

$$\|Z^n + re^{in\theta_0}\frac{1 + Ze^{-i(\theta_1+\theta_0)}}{2}\|_\infty$$

$$+ \|nZ^{n-1} + \frac{r}{2}e^{i(n\theta_0-\theta_1-\theta_0)}\|_p$$

$$- \|(TZ^n)' + \frac{r}{2}e^{-i\theta_1}\|_p$$

$$= \|TZ^n + re^{in\theta_0}\frac{1 + Ze^{-i(\theta_1+\theta_0)}}{2}\|_\infty$$

$$+ \| (TZ^n)' + \frac{r}{2}e^{i(n\theta_0-\theta_1-\theta_0)}\|_p$$

$$- \| (TZ^n)' + \frac{r}{2}e^{-i\theta_1}\|_p.$$

From (19) and Lemma 3.5 we conclude that the preceding expression is dominated by

$$\|TZ^n\|_\infty + r(1 - \delta) + o\left(\frac{r}{2}\right) \quad \text{as } r \to 0.$$

That is, we have shown that for $0 < r < \|TZ^n\|_\infty$

$$\|TZ^n\|_\infty + r \leqq \|TZ^n\|_\infty + r(1 - \delta) + o\left(\frac{r}{2}\right).$$

This contradiction establishes (14) and consequently claim (iv).

It is easy now to complete the proof of Theorem 3.1: Let T be an arbitrary isometry of S^p into S^p. By claim (i) $T1$ is a constant λ of modulus one so that $\bar{\lambda}T = T_1$ is an isometry of S^p such that $T_1 1 = 1$. Next by claim (iii), $T_1 Z$ is a conformal map ϕ of D onto D which is a rotation if $p > 1$. Let

$$T_2:S^p \to S^p$$

be defined by

$$T_2 f = f \circ \phi^{-1}.$$

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Then T_2 is an isometry of S^p. (If $p > 1$ this follows from the fact that ϕ is then a rotation, while if $p = 1$, one can verify it directly.) Now consider $T_3 = T_2 \circ T_1$. T_3 is an isometry of S^p such that $T_3 1 = 1$ and such that

$$T_3 Z = T_2 \phi = Z.$$

Thus by claim (iv),

$$T_3 Z^n = Z^n \quad \text{for all natural numbers } n.$$

Since the polynomials are dense in S^p, we have

$$T_3 f = f \quad \text{for all } f \in S^p.$$

But from this it follows that

$$f = T_2(T_1 f) = T_1 f \circ \phi^{-1}$$

so that $T_1 f = f \circ \phi$, hence

$$Tf = \lambda T_1 f = \lambda f \circ \phi \quad \text{for all } f \in S^p.$$

REFERENCES

1. P. L. Duren, *The theory of H^p spaces* (Academic Press, New York, 1970).
2. F. Forelli, *The isometries of H^p*, Can. J. Math. *16* (1964), 721-728.
3. G. Köthe, *Topological vector spaces I* (Springer-Verlag, New York, 1969).
4. K. de Leeuw, W. Rudin and J. Wermer, *The isometries of some function spaces*, Proc. Amer. Math. Soc. *11* (1960), 694-698.
5. R. C. Roan, *Composition operators on the space of functions with H^p-derivative*, Houston J. of Math. *4* (1978), 423-438.

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