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INTERSECTING FAMILIES IN CLASSICAL COXETER GROUPS

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Abstract Let Ω be a finite set and let G be a permutation group acting on it. A subset H of G is called *t-intersecting* if any two elements in H agree on at least t points. Let S_n^D and S_n^B be the classical Coxeter group of type D_n and type B_n, respectively. We show that the maximum-sized $(2t)$-intersecting families in S_n^D and S_n^B are precisely cosets of stabilizers of t points in $[n] := \{1, 2, \dots, n\}$, provided n is sufficiently large depending on t.

Keywords: intersecting families of permutations: Erdős–Ko–Rado theorem; representation theory; classical coxeter groups

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1. Introduction

The classical Erdős–Ko–Rado theorem [**6**] is a central result in extremal combinatorics. It has been generalized in many ways, including for permutations.

Let S_n be the symmetric group on $[n]$. Let t be a positive integer and let $t < n$. A subset $H \subset S_n$ is called *t-intersecting* if, for any two permutations $\sigma, \pi \in H$, there exist $i_1, \dots, i_t \in [n]$ such that $\sigma(i_l) = \pi(i_l)$ for $l \in [t]$.

In 1977, Deza and Frankl [**3**] showed that if $H \subset S_n$ is 1-intersecting, then $|H| \leqslant (n-1)!$. Later, Cameron and Ku [**1**] and Larose and Malvenuto [**8**] independently proved that equality holds if and only if H is a coset of the stabilizer of a point. Deza and Frankl [**3**] also conjectured that, for n sufficiently large depending on t, the cosets of stabilizers of t points are maximum-sized t-intersecting subsets of S_n. This has recently been proved by Ellis *et al.* [**5**].

Theorem 1.1 (see Ellis *et al.* [5]). *For n sufficiently large depending on t, if $H \subset S_n$ is t-intersecting, then $|H| \leqslant (n-t)!$. Equality holds if and only if H is a coset of the stabilizer of t points.*

The classical Coxeter group S_n^B is a group of signed permutations of $[n]$. More specifically, S_n^B consists of all permutations w of $[\pm n] := \{-n, \dots, -1, 1, \dots, n\}$ such that $w(-i) = -w(i)$ for each $i \in [n]$. The group S_n^D is a normal subgroup of S_n^B consisting of all those permutations $w \in S_n^B$ such that $|\{i \in [n] \colon w(i) \notin [n]\}|$ is even.

Note that, for any $\sigma, \pi \in S_n^B$, one has that

$$2|\{i \in [n] \colon \sigma(i) = \pi(i)\}| = |\{i \in [\pm n] \colon \sigma(i) = \pi(i)\}|.$$

We say a subset $H \subset S_n^B$ is $(2t)$-*intersecting* if, for any $\sigma, \pi \in S_n^B$, $|\{i \in [n] \colon \sigma(i) = \pi(i)\}| \geqslant t$. In this paper, we deal with the analogues of Theorem 1.1 for classical Coxeter groups S_n^B and S_n^D. We state our main theorem below.

Theorem 1.2. *For n sufficiently large depending on t, we have the following.*

(1) *If $H \subset S_n^B$ is a $(2t)$-intersecting subset of S_n^B, then $|H| \leqslant 2^{n-t}(n-t)!$; equality holds if and only if H is a coset of the stabilizer of t points in $[n]$.*

(2) *If $H \subset S_n^D$ is a $(2t)$-intersecting subset of S_n^D, then $|H| \leqslant 2^{n-t-1}(n-t)!$; equality holds if and only if H is a coset of the stabilizer of t points in $[n]$.*

The case $t = 1$ of Theorem 1.2 has been proved by Wang and Zhang [**14**], using completely combinatorial techniques. We will follow the ideas given by Ellis *et al.* in [**5**], i.e. using the representation theory of groups to prove Theorem 1.2. This method was also used by Godsil and Meagher to prove the case $t = 1$ of Theorem 1.1 in [**7**].

The paper has the following structure. In § 2, we give a sketch of the proof of Theorem 1.2. In § 3, we prove some necessary results and lemmas pertaining to representations and characters of S_n^B and S_n^D. In § 4, we prove Theorem 1.2 for S_n^D. In § 5, we prove Theorem 1.2 for S_n^B. In § 6, we give some comments on the generalization of Theorem 1.2 to imprimitive reflection groups.

In the following sections of this paper, we always assume that t is a fixed positive integer and that $n > 3t + 1$ is a large enough positive integer depending on t.

2. Overview of the proof

Readers can refer to [**4**, § 2] and [**5**, § 2] for any definition or result on the general representation theory of finite groups and Cayley graphs that is not explained herein.

The proof of Theorem 1.2 will rely crucially on the following two lemmas.

Lemma 2.1 (see [4, Theorem 4]). *Let G be a finite group, let X be an inverse-closed, conjugation-invariant subset of G and let Γ be the normal Cayley graph on G generated by X. Let A be the adjacency matrix of Γ. Let $V_1, \dots, V_k$ be a complete set of non-isomorphic irreducible modules of G. For each $i \in [k]$, let U_i denote the sum of all submodules of the group algebra $\mathbb{C}G$ that are isomorphic to V_i. Then, U_i is the eigenspace of A, and the corresponding eigenvalue is*

$$\lambda_{V_i} = \frac{\sum_{x \in X} \chi_{V_i}(x)}{\dim(V_i)}.$$

Note that the eigenvalues of the adjacency matrix A of the graph Γ are usually referred to as eigenvalues of Γ.

Lemma 2.2 (see [5, Theorem 12]). *Let $\Gamma = (V, E)$ be an N-vertex graph. Let $\Gamma_1, \dots, \Gamma_k$ be regular, spanning subgraphs of Γ, all having $\{v_1, v_2, \dots, v_N\}$ as an orthogonal system of eigenvectors (where v_1 is the all-$1'$ vector). For $i \in [N]$ and $j \in [k]$, let λ_i^j be the eigenvalue of v_i in Γ_j. Let $\beta_1, \beta_2, \dots, \beta_k \in \mathbb{R}$ and take*

$$\lambda_i = \sum_{j=1}^{k} \beta_j \lambda_i^j.$$

If $\lambda_{\min} = \min\{\lambda_i\}_{i \in [N]}$ and I is an independent set in Γ, then

$$\frac{|I|}{|V|} \leqslant \frac{-\lambda_{\min}}{\lambda_1 - \lambda_{\min}}.$$

And equality implies that

$$1_I \in \operatorname{span}(\{v_1\} \cup \{v_i \colon \lambda_i = \lambda_{\min}\}).$$

We think of the λ_i above as eigenvalues of the linear combination of graphs

$$\Upsilon = \sum_{j=1}^{k} \beta_j \Gamma_j.$$

The corresponding linear combination of adjacent matrices

$$A = \sum_{j=1}^{k} \beta_j A_j$$

is a so-called *pseudo-adjacent matrix* for Γ (see [**5**, Theorem 12] for the definition).

We give a sketch of the proof of Theorem 1.2. Take S_n^D as an example. We will proceed in the following manner.

First, we bound the size of a $(2t)$-intersecting subset of S_n^D. Let Γ_{FPF_t} be the Cayley graph on S_n^D generated by the set

$$\mathrm{FPF}_t = \{\sigma \in S_n^D \colon \sigma \text{ has fewer than } t \text{ fixed points in } [n]\}.$$

Following [**5**], we will choose various subgraphs of Γ_{FPF_t} and construct a pseudo-adjacency matrix A for Γ_{FPF_t} that is a suitable real linear combination of the adjacency matrices of these subgraphs, such that the minimal eigenvalue $\lambda_{\min}$ of A is

$$\varpi := -\frac{1}{2^t n(n-1)\cdots(n-t+1) - 1},$$

and the eigenvalue of the all-$1'$ eigenvector is $\lambda_1 = 1$. Most of the work of the proof is in showing that such a suitable linear combination exists, which turns out to be more difficult to handle than in [**5**]. We then apply Lemma 2.2 to this linear combination and obtain the following.

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Theorem 2.3. *For t fixed and n sufficiently large depending on t, if $I \subset S_n^D$ is $(2t)$-intersecting, then $|I| \leqslant 2^{n-t-1}(n-t)!$.*

Next, we will characterize the maximum-sized $(2t)$-intersecting subset of S_n^D. Let U_t be the direct sum of all submodules of group algebra $\mathbb{C}S_n^D$ that are isomorphic to the irreducible modules V, where the eigenvalue λ_V of the pseudo-adjacency matrix A constructed above is λ_1 or $\lambda_{\min}$. We will then show the following.

Theorem 2.4. *U_t is spanned by the characteristic functions of the cosets of the stabilizer of t points in $[n]$.*

Now, apply Lemma 2.2 again to that linear combination (constructed above); we see that the characteristic function of a maximum-sized $(2t)$-intersecting subset of S_n^D is spanned by those of the cosets of stabilizers of t points in $[n]$. We will complete our proof by showing the following.

Theorem 2.5. *If 1_I is spanned by the characteristic functions of the cosets of stabilizers of t points in $[n]$, where I is a maximum-sized $(2t)$-intersecting subset of S_n^D, then I is a coset of the stabilizer of t points in $[n]$.*

3. Representation theory of S_n^B and S_n^D

3.1. Basic results on the representations of S_n^B and S_n^D

Readers can refer to [**5**, § 3] for any definition and basic result regarding the representation theory of S_n that is not explained herein.

Each element $w \in S_n^B$ is a product of disjoint cycles $w = \theta_1 \cdots \theta_h$, where

$$\theta_i = \begin{pmatrix} b_{i1} & b_{i2} & \cdots & b_{il_i} \\ (-1)^{k_{i1}} b_{i2} & (-1)^{k_{i2}} b_{i3} & \cdots & (-1)^{k_{il_i}} b_{i1} \end{pmatrix} \quad (b_{ij} \in [n],\ k_{ij} \in \{0,1\}). \tag{3.1}$$

Define $l(\theta_i) := l_i$. Define

$$\tau(\theta_i) := \sum_{j=1}^{l_i} k_{ij} \quad \text{and} \quad \tau(w) := \sum_{i=1}^{h} \tau(\theta_i).$$

A cycle θ_i of w is called *positive* if $\tau(\theta_i) \equiv 0 \pmod 2$, and it is called *negative* if $\tau(\theta_i) \equiv 1 \pmod 2$. We associate with each $w \in S_n^B$ a double partition (α, β) of n, which is denoted by $\mathrm{Ty}(w)$, such that the length of each positive cycle of w is assigned to be a part of α, and the length of each negative cycle of w is assigned to be a part of β (where a *double partition* (α, β) of n is an ordered pair of partitions α and β such that $|\alpha| + |\beta| = n$). We have the following.

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Proposition 3.1 (see [11]).

(1) *Two elements $w, w' \in S_n^B$ are conjugate if and only if* $\mathrm{Ty}(w) = \mathrm{Ty}(w')$.

(2) *Let $w = \theta_1 \cdots \theta_h \in S_n^D$ be a product of disjoint cycles. Set*

$$d(w) = \gcd(\tau(\theta_1), \dots, \tau(\theta_h), l(\theta_1), \dots, l(\theta_h), 2).$$

The conjugacy class of S_n^B containing w then splits into the union of $d(w)$ conjugacy classes in S_n^D.

Let T be a subgroup of S_n^B generated by $\{\binom{i}{-i} : i \in [n]\}$. Then, $S_n^B = S_n \ltimes T$. Let (α, β) be a double partition of n. Set $|\alpha| = a$ and $|\beta| = b$. Let S_a^B be a subgroup of S_n^B of signed permutations of $\{1, 2, \dots, a\}$ and let S_b^B be a subgroup of S_n^B of signed permutations of $\{a+1, a+2, \dots, n\}$. Let S_a and S_b be the symmetric groups of permutations of $\{1, 2, \dots, a\}$ and $\{a+1, a+2, \dots, n\}$, respectively. Let S^α be the Specht module for the symmetric group S_a labelled by the partition α. Extend this module to the group S_a^B by letting the signs, i.e. the elements $\{\binom{i}{-i} : i \in [a]\}$, act trivially. Let S^β be the Specht module for the symmetric group S_b labelled by the partition β. Extend this module to the group S_b^B by letting the elements $\{\binom{i}{-i} : a+1 \leqslant i \leqslant n\}$ act by -1 on S^β. Then, S^α and S^β are irreducible modules for the groups S_a^B and S_b^B, and they are the modules ordinarily denoted by $S^{(\alpha,\emptyset)}$ and $S^{(\emptyset,\beta)}$, respectively. Let $\chi_{(\alpha,\emptyset)}$ and $\chi_{(\emptyset,\beta)}$ be the characters of the modules $S^{(\alpha,\emptyset)}$ and $S^{(\emptyset,\beta)}$, respectively. Let

$$S^{(\alpha,\beta)} := \mathrm{Ind}_{S_a^B \times S_b^B}^{S_n^B}(S^{(\alpha,\emptyset)} \otimes S^{(\emptyset,\beta)}).$$

By using the standard formula for induced characters, the character $\chi_{(\alpha,\beta)}$ for $S^{(\alpha,\beta)}$ is

$$\chi_{(\alpha,\beta)}(w) = \sum_{g^{-1}wg \in S_a^B \times S_b^B} \chi_{(\alpha,\emptyset)}(g^{-1}wg)\chi_{(\emptyset,\beta)}(g^{-1}wg), \tag{3.2}$$

where the sum is over coset representatives g of $S_n^B/(S_a^B \times S_b^B)$ such that $g^{-1}wg \in S_a^B \times S_b^B$. Denote $g^{-1}wg$ by w^g for simplicity. Write that $w^g = w_1^g \cdot w_2^g$, where $w_1^g \in S_a^B$ and $w_2^g \in S_b^B$. Then, (3.2) becomes

$$\chi_{(\alpha,\beta)}(w) = \sum_{w^g \in S_a^B \times S_b^B} \chi_{(\alpha,\emptyset)}(w_1^g)\chi_{(\emptyset,\beta)}(w_2^g) = \sum_{w^g \in S_a^B \times S_b^B} \chi_\alpha(\overline{w_1^g})\chi_\beta(\overline{w_2^g})(-1)^{\tau(w_2^g)}, \tag{3.3}$$

where χ_α, χ_β are the characters of the Specht modules S^α, S^β for the symmetric groups S_a, S_b, respectively, and, for $w \in S_n^B$, there is a unique $\bar{w} \in S_n$ such that $\bar{w}(i) = |w(i)|$ for each $i \in [n]$. In particular, the dimension of the module $S^{(\alpha,\beta)}$ is

$$\dim(S^{\alpha,\beta}) = \chi_{(\alpha,\beta)}(\mathrm{id}) = \frac{|S_n^B|}{|S_a^B \times S_b^B|}\chi_{(\alpha,\emptyset)}(\mathrm{id})\chi_{(\emptyset,\beta)}(\mathrm{id}) = \binom{n}{a}\dim(S^\alpha)\dim(S^\beta), \tag{3.4}$$

where id denotes the identity of the group.

The following proposition describes the irreducible representations of S_n^B and S_n^D.

Proposition 3.2 (see [10, Theorem 4.18, Proposition 5.8, Theorem 5.9]).

(1) *The modules $S^{(\alpha,\beta)}$, where (α,β) runs over all double partitions of n, form a complete set of non-isomorphic irreducible representations of S_n^B.*

(2) *For each double partition (α,β) of n, $S^{(\alpha,\beta)}$ and $S^{(\beta,\alpha)}$ are isomorphic representations of S_n^D.*

(3) *If n is even and α is a partition of $\frac{1}{2}n$, then $S^{(\alpha,\alpha)}$ is the direct sum of two irreducible submodules, denoted by $S^{(\alpha,\alpha)^+}$ and $S^{(\alpha,\alpha)^-}$, of S_n^D, which have the same dimensions.*

(4) *The modules $S^{(\alpha,\beta)}$, where (α,β) runs over all unordered pairs of partitions of n, with $\alpha \neq \beta$, and the modules $S^{(\alpha,\alpha)^+}$ and $S^{(\alpha,\alpha)^-}$, where α runs over all partitions of $\frac{1}{2}n$, form a complete set of non-isomorphic irreducible representations of S_n^D.*

Proposition 3.3 (branching rule; see [10, Theorem 4.18]). *Let (α,β) be a double partition of n and let $S^{(\alpha,\beta)}$ be an irreducible module of S_n^B. Then, the induction rule is*

$$\mathrm{Ind}_{S_n^B}^{S_{n+1}^B}(S^{(\alpha,\beta)}) = \bigoplus_{(\mu,\nu)} S^{(\mu,\nu)},$$

where (μ,ν) ranges over all double partitions of $n+1$ such that the Young diagram of μ is equal to that of α or obtained from that of α by adding one box, and the Young diagram of ν is equal to that of β or obtained from that of β by adding one box.

3.2. Dimensions of certain irreducible representations of S_n^D and S_n^B

Lemma 3.4 (see [5, Lemma 2]). *Let t be fixed. There exists a constant $E(t) > 0$ depending only on t such that if n is sufficiently large depending on t, for any irreducible module S^α of S_n, where α is a partition of n such that none of the rows or columns of α has length at least $n-t$, $\dim(S^\alpha) \geqslant n^{t+1}E(t)$.*

Based on this lemma and (3.4), we can easily get the following result.

Proposition 3.5. *Let t be fixed. There exists a constant $C(t)$ depending only on t such that if n is sufficiently large depending on t, for any irreducible module $S^{(\alpha,\beta)}$ of S_n^B, where (α,β) is a double partition of n such that none of the rows or columns of α or β has length at least $n-t$, $\dim(S^{(\alpha,\beta)}) \geqslant n^{t+1}C(t)$.*

As a quick result of the above proposition and Proposition 3.2 (3), we have the following.

Proposition 3.6. *Let t be fixed. There exists a constant $D(t) > 0$ depending only on t such that if n is sufficiently large depending on t, for any irreducible module V of S_n^D, where $V = S^{(\alpha,\beta)}$, with (α,β) a double partition of n and $\alpha \neq \beta$, or $V = S^{(\alpha,\alpha)^\pm}$, with α a partition of $\frac{1}{2}n$, such that none of the rows or the columns of α or β has length at least $n-t$, $\dim(V) \geqslant n^{t+1}D(t)$.*

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3.3. Lemmas on representations and characters of S_n^D

In this subsection, we will deduce some necessary results on S_n^D for the proof of Theorem 1.2 (2).

Denote by $P_{n,t}$ the set of partitions λ of n such that the first part of λ is not smaller than $n-t$. Given that $\gamma = (b_1, b_2, \ldots, b_h) \in P_{n,t}$, let $C(\gamma)$ be the set of double partitions (α, β) of n such that each part of γ is uniquely assigned as one of the parts of α or β, with b_1 always assigned as the first part of α. Given a double partition $(\alpha, \beta) \in C(\gamma)$, let α' be a partition obtained from α by omitting the first part. Let $X_{(\alpha,\beta)}$ be the set of elements $w \in S_n^D$ such that $\mathrm{Ty}(w\theta_1^{-1}) = (\alpha', \beta)$, where θ_1 is the only cycle of w with $l(\theta_1) = b_1$.

Set $\zeta_t = |P_{n,t}|$. We sort the partitions in $P_{n,t}$ in reverse lexicographical order and define $\gamma_1 = (n) > \gamma_2 > \cdots > \gamma_{\zeta_t} = (n-t, 1^t)$.

Let $\gamma_i, \gamma_j \in P_{n,t}$ and let $(\mu, \nu) \in C(\gamma_i)$, $(\alpha, \beta) \in C(\gamma_j)$. By Lemma 2.1, the eigenvalue of the Cayley graph $\mathrm{Cay}(S_n^D, X_{(\alpha,\beta)})$ (defined on S_n^D and generated by $X_{(\alpha,\beta)}$) corresponding to the irreducible module $S^{(\mu,\nu)}$ is

$$\lambda_{(\mu,\nu)}^{(\alpha,\beta)} = \frac{\sum_{\sigma \in X_{(\alpha,\beta)}} \chi_{(\mu,\nu)}(\sigma)}{\dim(S^{(\mu,\nu)})}.$$

Obviously, $X_{(\alpha,\beta)}$ is a conjugacy class in S_n^B. So $\chi_{(\mu,\nu)}$ is a constant on the set $X_{(\alpha,\beta)}$. Let $\sigma_{(\alpha,\beta)} \in X_{(\alpha,\beta)}$. By (3.3), we have that

$$\chi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)}) = \sum_{(\sigma_{(\alpha,\beta)})^g \in S_{|\mu|}^B \times S_{|\nu|}^B} \chi_\mu(\overline{(\sigma_{(\alpha,\beta)})_1^g})\chi_\nu(\overline{(\sigma_{(\alpha,\beta)})_2^g})(-1)^{\tau((\sigma_{(\alpha,\beta)})_2^g)}.$$

Let ξ_μ, ξ_ν be the characters of the permutation modules M^μ, M^ν for the symmetric groups $S_{|\mu|}$, $S_{|\nu|}$, respectively (see [**5**, § 3] for the definitions of permutation modules). We define

$$\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)}) := \sum_{(\sigma_{(\alpha,\beta)})^g \in S_{|\mu|}^B \times S_{|\nu|}^B} \xi_\mu(\overline{(\sigma_{(\alpha,\beta)})_1^g})\xi_\nu(\overline{(\sigma_{(\alpha,\beta)})_2^g})(-1)^{\tau((\sigma_{(\alpha,\beta)})_2^g)}. \tag{3.5}$$

In the following, we will present a detailed study of the matrix

$$[\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})]_{\substack{(\mu,\nu)\in C(\gamma_i)\\(\alpha,\beta)\in C(\gamma_j)}}.$$

Proposition 3.7 (see [5, Lemma 4]). *Let α be a partition of n and let $\sigma \in S_n$. If $\xi_\alpha(\sigma) \neq 0$, then* cycle-type$(\sigma) \trianglelefteq \alpha$. *Moreover, if* cycle-type$(\sigma) = \alpha = (i_1^{l_1}, i_2^{l_2}, \ldots, i_k^{l_k})$, *then $\xi_\alpha(\sigma) = l_1! l_2! \cdots l_k!$.*

Lemma 3.8. *Let $(\mu, \nu) \in C(\gamma_i)$ and let $(\alpha, \beta) \in C(\gamma_j)$, $1 \leqslant i, j \leqslant \zeta_t$. Then, $\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)}) \neq 0$ only if $\gamma_j \trianglelefteq \gamma_i$. In particular, if $i = j$, (3.5) becomes*

$$\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)}) = \xi_\mu(\sigma_\mu)\xi_\nu(\sigma_\nu) \sum_{(\sigma_{(\alpha,\beta)})^g \in S_{|\mu|}^B \times S_{|\nu|}^B} (-1)^{\tau((\sigma_{(\alpha,\beta)})_2^g)}, \tag{3.6}$$

where $\sigma_\mu, \sigma_\nu \in S_n$ are such that cycle-type$(\sigma_\mu) = \mu$ *and* cycle-type$(\sigma_\nu) = \nu$.

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Proof. Consider the right-hand side of (3.5). By Proposition 3.7,

$$\xi_\mu(\overline{((\sigma_{(\alpha,\beta)})_1^g}) \neq 0$$

only if cycle-type $((\sigma_{(\alpha,\beta)})_1^g) \trianglelefteq \mu$, and

$$\xi_\nu(\overline{((\sigma_{(\alpha,\beta)})_2^g}) \neq 0$$

only if cycle-type $((\sigma_{(\alpha,\beta)})_2^g) \trianglelefteq \nu$. Then,

$$\xi_\mu(\overline{((\sigma_{(\alpha,\beta)})_1^g})\xi_\nu(\overline{((\sigma_{(\alpha,\beta)})_2^g}) \neq 0$$

only if cycle-type $((\sigma_{(\alpha,\beta)})^g) \trianglelefteq \gamma_i$, where γ_i is the merge of μ and ν. Hence, $\gamma_j \trianglelefteq \gamma_i$.

If $\gamma_i = \gamma_j$, then cycle-type $((\sigma_{(\alpha,\beta)})_1^g) = \mu$ and cycle-type $((\sigma_{(\alpha,\beta)})_2^g) = \nu$. Let $\sigma_\mu, \sigma_\nu \in S_n$ such that cycle-type$(\sigma_\mu) = \mu$ and cycle-type$(\sigma_\nu) = \nu$. Hence, (3.6) follows. □

Lemma 3.9. *The matrix*

$$[\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})]_{\substack{(\mu,\nu)\in C(\gamma_{\zeta_t})\\(\alpha,\beta)\in C(\gamma_{\zeta_t})}} \tag{3.7}$$

is invertible.

Proof. Recall that $\zeta_t = (n-t, 1^t)$. Let $\mu = (n-t, 1^{t-p})$ and let $\nu = (1^p)$, where $0 \leqslant p \leqslant t$. Let $\alpha = (n-t, 1^{t-q})$ and let $\beta = (1^q)$, where $0 \leqslant q \leqslant t$. We will use (3.6) to compute $\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})$. Due to Proposition 3.7, $\xi_\mu(\sigma_\mu) = (t-p)!$, $\xi_\nu(\sigma_\nu) = p!$ and

$$\sum_{(\sigma_{(\alpha,\beta)})^g \in S^B_{|\mu|}\times S^B_{|\nu|}} (-1)^{\tau((\sigma_{(\alpha,\beta)})_2^g)} = \sum_{i=0}^{\min\{p,q\}} (-1)^i \binom{q}{i}\binom{t-q}{p-i}.$$

Due to a result given in [**2**], we know that the matrix

$$\left[\sum_{i=0}^{\min\{p,q\}} (-1)^i \binom{q}{i}\binom{t-q}{p-i}\right]_{\substack{0\leqslant p\leqslant t\\0\leqslant q\leqslant t}}$$

is invertible. Therefore, the matrix

$$[\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})]_{\substack{(\mu,\nu)\in C(\gamma_{\zeta_t})\\(\alpha,\beta)\in C(\gamma_{\zeta_t})}} = \left[(t-p)!p!\sum_{i=0}^{\min\{p,q\}} (-1)^i \binom{q}{i}\binom{t-q}{p-i}\right]_{\substack{0\leqslant p\leqslant t\\0\leqslant q\leqslant t}}$$

is invertible. □

Lemma 3.10. *The matrix*

$$[\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})]_{\substack{(\mu,\nu)\in C(\gamma_i)\\(\alpha,\beta)\in C(\gamma_i)}} \tag{3.8}$$

is invertible, where $1 \leqslant i \leqslant \zeta_t$.

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Proof. Assume that $\gamma_i = (n-s, i_1^{l_1}, \ldots, i_r^{l_r}) \in P_{n,t}$, where $0 \leqslant s \leqslant t$, $i_1 > i_2 > \cdots > i_r \geqslant 1$ and

$$\sum_{j=1}^{r} i_j l_j = s.$$

Let $\mu = (n-s, i_1^{l_1-p_1}, \ldots, i_r^{l_r-p_r})$ and let $\nu = (i_1^{p_1}, \ldots, i_r^{p_r})$, where $0 \leqslant p_j \leqslant l_j$ and $j \in [r]$. Let $\alpha = (n-s, i_1^{l_1-q_1}, \ldots, i_r^{l_r-q_r})$ and let $\beta = (i_1^{q_1}, \ldots, i_r^{q_r})$, where $0 \leqslant q_j \leqslant l_j$ and $j \in [r]$. We use (3.6) to compute $\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})$. Note that

$$\sum_{(\sigma_{(\alpha,\beta)})^g \in S^B_{|\mu|} \times S^B_{|\nu|}} (-1)^{\tau((\sigma_{(\alpha,\beta)})^g_2)} = \prod_{i=1}^{r} \left(\sum_{j_i=0}^{\min\{p_i,q_i\}} (-1)^{j_i} \binom{q_i}{j_i} \binom{l_i-q_i}{p_i-j_i} \right).$$

Then, (3.8) becomes

$$\left[\xi_\mu(\sigma_\mu)\xi_\nu(\sigma_\nu) \prod_{i=1}^{r} \left(\sum_{j_i=0}^{\min\{p_i,q_i\}} (-1)^{j_i} \binom{q_i}{j_i} \binom{l_i-q_i}{p_i-j_i} \right) \right]_{\substack{(p_1,\ldots,p_r),\ \text{with } 0\leqslant p_i \leqslant l_i \text{ for any } i\in[r] \\ (q_1,\ldots,q_r),\ \text{with } 0\leqslant q_j \leqslant l_j \text{ for any } j\in[r]}}, \tag{3.9}$$

where $\xi_\mu(\sigma_\mu) = (l_1-p_1)!\cdots(l_r-p_r)!$, $\xi_\nu(\sigma_\nu) = p_1!\cdots p_r!$ and the rows and columns of (3.9) are indexed by r-tuples $(p_1,\ldots,p_r)$ and $(q_1,\ldots,q_r)$, respectively.

Set

$$D := \left[\prod_{i=1}^{r} \left(\sum_{j_i=0}^{\min\{p_i,q_i\}} (-1)^{j_i} \binom{q_i}{j_i} \binom{l_i-q_i}{p_i-j_i} \right) \right]_{\substack{(p_1,\ldots,p_r),\ \text{with } 0\leqslant p_i \leqslant l_i \text{ for any } i\in[r] \\ (q_1,\ldots,q_r),\ \text{with } 0\leqslant q_j \leqslant l_j \text{ for any } j\in[r]}}$$

and

$$D_i := \left[\sum_{j_i=0}^{\min\{p_i,q_i\}} (-1)^{j_i} \binom{q_i}{j_i} \binom{l_i-q_i}{p_i-j_i} \right]_{\substack{0\leqslant p_i\leqslant l_i \\ 0\leqslant q_i\leqslant l_i}}.$$

Then, $D = D_1 \otimes \cdots \otimes D_r$ and D is invertible, since $D_1, \ldots, D_r$ are invertible (see [**2**]). Therefore, (3.8) is invertible. □

Lemma 3.11. *Let $(\mu,\nu) \in C(\gamma_i)$ and let $(\alpha,\beta) \in C(\gamma_j)$, $1 \leqslant j \leqslant i \leqslant \zeta_t$. Then, $\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})$ does not depend on n.*

Proof. Assume that $\mu = (n-s, b_1, \ldots, b_{l_1})$ and that $\nu = (b_{l_1+1}, \ldots, b_l)$. Assume that $\alpha = (n-k, c_1, \ldots, c_{h_1})$ and that $\beta = (c_{h_1+1}, \ldots, c_h)$. Then, $s \leqslant k \leqslant t$.

Recall (3.5) for $\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})$:

$$\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)}) = \sum_{(\sigma_{(\alpha,\beta)})^g \in S^B_{|\mu|} \times S^B_{|\nu|}} \xi_\mu(\overline{(\sigma_{(\alpha,\beta)})^g_1})\xi_\nu(\overline{(\sigma_{(\alpha,\beta)})^g_2})(-1)^{\tau((\sigma_{(\alpha,\beta)})^g_2)}.$$

We will show that both the number of the coset representatives g such that $(\sigma_{(\alpha,\beta)})^g \in S^B_{|\mu|} \times S^B_{|\nu|}$ and the value $\xi_\mu(\overline{(\sigma_{(\alpha,\beta)})^g_1})\xi_\nu(\overline{(\sigma_{(\alpha,\beta)})^g_2})$ do not depend on n.

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Since $(\sigma_{(\alpha,\beta)})_2^g \in S_{|\nu|}^B$ and $|\nu| \leqslant s \leqslant t < n-t \leqslant n-k$ (recall that $n > 3t+1$), the only cycles of $\sigma_{(\alpha,\beta)}$ that can be permuted by g to lie in $S_{|\nu|}^B$ are those short cycles of $\sigma_{(\alpha,\beta)}$ whose lengths are smaller than t. Since the sum of the length of short cycles is

$$\sum_{i=1}^{h} c_i \leqslant t,$$

there are no more than $\binom{t}{|\nu|}$ ways to choose g, which does not depend on n.

Note that $\xi_\mu(\overline{(\sigma_{(\alpha,\beta)})_1^g})$ is the number of μ-tabloids fixed by $\overline{(\sigma_{(\alpha,\beta)})_1^g}$. To count these, first note that the numbers in the $(n-k)$-cycle of $\overline{(\sigma_{(\alpha,\beta)})_1^g}$ must all lie in the first row of the μ-tabloid. Then, we are left with a $(k-s, b_1, \dots, b_{l_1})$-'tabloid', which we need to fill with the remaining $|\mu| - (n-k)$ elements in such a way that $\overline{(\sigma_{(\alpha,\beta)})_1^g}$ fixes it. It is easy to see that the number of ways of doing this are independent of n, as is $\xi_\nu(\overline{(\sigma_{(\alpha,\beta)})_2^g})$. □

Following [**5**, Definition 11], for $\gamma = (b_1, b_2, \dots, b_h) \in P_{n,t}$, define $\operatorname{split}(\gamma) = (b_1 - t - 1, t+1, b_2, \dots, b_h)$. Let $C(\operatorname{split}(\gamma))$ be the set of double partitions (α, β) of n such that each part of $\operatorname{split}(\gamma)$ is uniquely assigned as one of the parts of α or β, with $b_1 - t - 1$ and $t+1$ always assigned as the first two parts of α. Given that $(\alpha,\beta) \in C(\operatorname{split}(\gamma))$, let α' be a partition obtained from α by omitting the first two parts. Let $X_{(\alpha,\beta)}$ be the set of elements $w \in S_n^D$ such that $\operatorname{Ty}(w\theta_1^{-1}\theta_2^{-1}) = (\alpha', \beta)$, where θ_1, θ_2 are the only two cycles of w such that $l(\theta_1) = b_1 - t - 1$, $l(\theta_2) = t+1$.

Lemma 3.12. *Let $\gamma_i, \gamma_j \in P_{n,t}$ and let $(u,v) \in C(\gamma_i)$, $(\alpha,\beta) \in C(\operatorname{split}(\gamma_j))$. Then, $\chi_{(\mu,\nu)}$ is a constant on the set $X_{(\alpha,\beta)}$.*

Proof. By the construction of $X_{(\alpha,\beta)}$, the elements in $X_{(\alpha,\beta)}$ are in two conjugacy classes of S_n^B, which are distinguished by the signs of the two long cycles whose lengths are greater than t. For $w \in X_{(\alpha,\beta)}$, recall that

$$\chi_{(\mu,\nu)}(w) = \sum_{w^g \in S_{|\mu|}^B \times S_{|\nu|}^B} \chi_\mu(\overline{w_1^g})\chi_\nu(\overline{w_2^g})(-1)^{\tau(w_2^g)}.$$

Since $|\nu| \leqslant t$, both the two long cycles of w should be permuted by a coset representative g to lie in $S_{|\mu|}^B$ and their signs do not affect the right-hand side of the formula. □

Lemma 3.13. *Let $\gamma_i, \gamma_j \in P_{n,t}$ and let $(u,v) \in C(\gamma_i)$, $(\alpha,\beta) \in C(\gamma_j)$. Let $\alpha = (b_1, b_2, \dots, b_k)$ and let $\beta = (b_{k+1}, \dots, b_h)$, where $b_1 \geqslant n-t$. Let $\alpha' = (b_1 - t - 1, t + 1, b_2, \dots, b_k)$. Let $\sigma \in X_{(\alpha,\beta)}$ and let $\sigma' \in X_{(\alpha',\beta)}$. Then, $\xi_{(\mu,\nu)}(\sigma) = \xi_{(\mu,\nu)}(\sigma')$.*

Proof. For an element $w \in S_n^D$, recall that

$$\xi_{(\mu,\nu)}(w) = \sum_{w^g \in S_{|\mu|}^B \times S_{|\nu|}^B} \xi_\mu(\overline{w_1^g})\xi_\nu(\overline{w_2^g})(-1)^{\tau(w_2^g)}.$$

Since $|\nu| \leqslant t$, the only cycles of σ that can be permuted by a coset representative to lie in $S_{|\nu|}^B$ are the short cycles, i.e. those with lengths smaller than t. The same thing happens to σ'. Furthermore, if $\overline{(\sigma')_1^g}$ fixes a μ-tabloid, the numbers in the $(b_1 - t - 1)$-cycle and the $(t+1)$-cycle must all lie in the first row of this μ-tabloid. It follows that $\overline{\sigma_1^g}$, produced by merging these two cycles of $\overline{(\sigma')_1^g}$, fixes exactly the same tabloids as $\overline{(\sigma')_1^g}$ does. □

3.4. 'Fat' irreducible representations of S_n^D

In this subsection, we give a proof of Theorem 2.4 following that of [**4**, Theorem 18].

Let $U_{(\alpha,\beta)}$ be the direct sum of all submodules of group algebra $\mathbb{C}S_n^D$ isomorphic to the irreducible module $S^{(\alpha,\beta)}$, where (α,β) is a double partition of n. Let

$$U_t = \bigoplus_{\alpha_1 \geqslant n-t} U_{(\alpha,\beta)},$$

where α_1 is the first part of α.

A coset of the stabilizer of t points in $[n]$ is called a *t-coset*. Let $\mathcal{B}_t$ be the set of ordered t-tuples $(i_1,\dots,i_t)$, where $i_1,\dots,i_t \in [\pm n]$ and $|i_1|,\dots,|i_t|$ are pairwise distinct. Then, a t-coset in S_n^D can be denoted by $T_{(a,b)} = \{\sigma \in S_n^D\colon \sigma(a) = b\}$, where $a = (i_1,\dots,i_t)$, $b = (j_1,\dots,j_t)$ are in $\mathcal{B}_t$ and $\sigma(a) = b$ means $\sigma(i_1) = j_1,\dots,\sigma(i_t) = j_t$.

Note that we compose permutations from right to left, i.e. if $\sigma,\pi \in S_n^D$ and $i \in [\pm n]$, then $(\sigma\pi)(i) = \sigma(\pi(i))$.

Consider the subspace M^t of the group algebra $\mathbb{C}S_n^D$, which takes the following vectors as a basis set:

$$\Bigg\{ \sum_{\substack{\sigma(n-t+i)=j_i\\ i\in[t]}} \sigma\colon (j_1,\dots,j_t) \in \mathcal{B}_t \Bigg\}.$$

Then, M^t is a permutation module of S_n^D under natural left multiplication. We have the following proposition.

Proposition 3.14 (see [9]). *$M^t \cong \mathrm{Ind}_{S_{n-t}^D}^{S_n^D}(S^{((n-t),\emptyset)})$, where $S^{((n-t),\emptyset)}$ is the trivial module of S_{n-t}^D. Furthermore, by the branching rule (Proposition 3.3), we have that*

$$M^t \cong \bigoplus_{\alpha_1 \geqslant n-t} c_{(\alpha,\beta)} S^{(\alpha,\beta)},$$

where the sum is over all double partitions (α,β) of n such that the first part α_1 of α is not smaller than $n-t$, and $c_{(\alpha,\beta)}$ is a positive integer.

Theorem 3.15. *We have that*

$$U_t = \mathrm{span}\{1_{T_{(a,b)}}\colon a,b \in \mathcal{B}_t\}.$$

Proof. We follow the argument of [**4**, Theorem 18] to prove Theorem 3.15. Let W be the sum of right translates of M^t, i.e. the subspace of $\mathbb{C}S_n^D$ spanned by $\{1_{T_{(a,b)}}\colon a,b \in \mathcal{B}_t\}$. We will show that $W = U_t$.

It is obvious that $M^t \subseteq U_t$. Since U_t is a two-sided ideal of $\mathbb{C}S_n^D$, it is closed under right multiplication of elements of $\mathbb{C}S_n^D$. So, $W \subseteq U_t$.

On the other hand, by a well-known fact from the general representation theory (see [**4**, Lemma 17]), the sum of all right translates of M^t contains all submodules of $\mathbb{C}S_n^D$ isomorphic to $S^{(\alpha,\beta)}$, with $\alpha_1 \geqslant n-t$, i.e. $U_{(\alpha,\beta)} \subseteq W$ for each double partition (α,β), with $\alpha_1 \geqslant n-t$. So, $U_t \subseteq W$. $\square$

4. Proof of the main theorem for S_n^D

Now, we begin to prove Theorem 1.2 for S_n^D. Recall that

$$\varpi = -\frac{1}{2^t n(n-1)\cdots(n-t+1)-1}.$$

Lemma 4.1. *For $1 \leqslant j \leqslant \zeta_t$, let $\tilde{\gamma}_j \in \{\gamma_j, \operatorname{split}(\gamma_j)\}$ and let $(\alpha,\beta) \in C(\tilde{\gamma}_j)$. Let $\Gamma_{(\alpha,\beta)} = \operatorname{Cay}(S_n^D, X_{(\alpha,\beta)})$. Let $\lambda_{(\mu,\nu)}^{(\alpha,\beta)}$ be the eigenvalue of $\Gamma_{(\alpha,\beta)}$ corresponding to the irreducible module $S^{(\mu,\nu)}$, where $(\mu,\nu) \in C(\gamma_i)$, with $1 \leqslant i \leqslant \zeta_t$. Then, there exist constant values of $d_{(\alpha,\beta)}$, with $d_{(\tilde{\gamma}_{\zeta_t},\emptyset)} = 0$, such that*

$$\sum_{(\alpha,\beta)} d_{(\alpha,\beta)}\lambda_{(\mu,\nu)}^{(\alpha,\beta)} = \begin{cases} 1, & (\mu,\nu) = ((n),\emptyset), \\ \varpi, & (\mu,\nu) \in C(\gamma_i), \text{ with } 1 < i \leqslant \zeta_t, \end{cases} \tag{4.1}$$

where the sum is over all $(\alpha,\beta) \in C(\tilde{\gamma}_j)$ and $1 \leqslant j \leqslant \zeta_t$. Furthermore, there exists a constant $E(t)$ depending only on t such that for any (α,β), $|d_{(\alpha,\beta)}| < E(t)/2^{n-1}(n-1)!$.

Proof. Without loss of generality, let $\tilde{\gamma}_j = \gamma_j$ for all $1 \leqslant j \leqslant \zeta_t$ (Lemmas 3.12 and 3.13 ensure that the following proof also works for other choices of $\tilde{\gamma}_j$). Then, we have that

$$\lambda_{(\mu,\nu)}^{(\alpha,\beta)} = \frac{|X_{(\alpha,\beta)}|\chi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})}{\dim(S^{(\mu,\nu)})},$$

where $\sigma_{(\alpha,\beta)} \in X_{(\alpha,\beta)}$. Define $d'_{(\alpha,\beta)} = d_{(\alpha,\beta)}|X_{(\alpha,\beta)}|$. Then, (4.1) becomes

$$\sum_{(\alpha,\beta)} d'_{(\alpha,\beta)}\chi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)}) = \begin{cases} 1, & (\mu,\nu) = ((n),\emptyset), \\ \varpi\dim(S^{(\mu,\nu)}), & (\mu,\nu) \in C(\gamma_i), \text{ with } 1 < i \leqslant \zeta_t. \end{cases} \tag{4.2}$$

Using Young's rule (see [**5**, Theorem 15]), we rewrite (4.2) as

$$\sum_{(\alpha,\beta)} d'_{(\alpha,\beta)}\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)}) \\ = \begin{cases} 1 + \varpi(\xi_\mu(\mathrm{id}) - 1), & (\mu,\nu) \in C(\gamma), \text{ with } \nu = \emptyset \text{ and } \gamma \in P_{n,t}, \\ \varpi\xi_{(\mu,\nu)}(\mathrm{id}), & (\mu,\nu) \in C(\gamma), \text{ with } \nu \neq \emptyset \text{ and } \gamma \in P_{n,t}. \end{cases} \tag{4.3}$$

The coefficient matrix of (4.3),

$$[\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})]_{\substack{(\mu,\nu)\in C(\gamma_i) \text{ and } 1\leqslant i\leqslant\zeta_t \\ (\alpha,\beta)\in C(\gamma_j) \text{ and } 1\leqslant j\leqslant\zeta_t}}, \tag{4.4}$$

is a $\zeta_t \times \zeta_t$ block upper-triangle matrix, with blocks indexed by $\gamma_1,\ldots,\gamma_{\zeta_t}$ (see Lemma 3.8). Due to Lemma 3.10, each block on the diagonal of (4.4) is invertible, and so is (4.4). Thus, there indeed exist values of $d_{(\alpha,\beta)}$ such that (4.1) holds.

Consider the equations in (4.3) that are labelled by the partitions $(\mu,\nu) \in C(\gamma_{\zeta_t})$. Recall that $\gamma_{\zeta_t} = (n-t, 1^t)$. Set $\mu_p = (n-t, 1^{t-p})$ and $\nu_p = (1^p)$, $0 \leqslant p \leqslant t$. Set $\alpha_q = (n-t, 1^{t-q})$ and $\beta_q = (1^q)$, $0 \leqslant q \leqslant t$. From (4.3), we have that

$$\sum_{q=0}^{t} d'_{(\alpha_q,\beta_q)} \sum_{i=0}^{\min\{p,q\}} (-1)^i \binom{q}{i}\binom{t-q}{p-i} = \begin{cases} \dfrac{1}{t!}(1+\varpi(\chi_{\bar{\mu}_0}(1)-1)), & p=0,\\ \dfrac{1}{(t-p)!p!}\varpi\chi_{(\bar{\mu}_p,\bar{\nu}_p)}(1), & 0<p\leqslant t. \end{cases}$$

A computation shows that $d'_{(\alpha_0,\beta_0)} = 0$, which means that $d_{((n-t,1^t),\emptyset)} = 0$.

Due to Lemma 3.11, the coefficients $\xi_{(\mu,\nu)}(\sigma_{(\alpha,\beta)})$ on the left-hand side of (4.3) do not depend on n. It can be shown by a direct computation that the coefficients on the right-hand side of (4.3) also do not depend on n. Then, there exists a constant $E'(t)$ independent of n such that, for any (α,β), $|d'_{(\alpha,\beta)}| < E'(t)$. Also, by a direct computation, we can show that

$$|X_{(\alpha,\beta)}| > (n-1)!2^{n-1}(1/2^t(t!)^2).$$

Let $E(t) = E'(t)2^t(t!)^2$. Then, $d_{(\alpha,\beta)} < E(t)/2^{n-1}(n-1)!$, as desired. □

Following [**5**], an irreducible module $S^{(\alpha,\beta)}$ of S_n^D is called *fat* if $\alpha_1 \geqslant n-t$, where α_1 is the first part of α; it is called *thin* if $S^{(\alpha^t,\beta)}$ is fat, where α^t is the transpose of α (see [**5**, Definition 9]). All the other irreducible modules of S_n^D are called *medium*.

Based on the results displayed in Lemma 4.1 and Proposition 3.6, we can follow the argument of the proof of [**5**, Theorem 23] to prove the following.

Lemma 4.2. *In the setup of Lemma 4.1, let V be a medium irreducible module of S_n^D and let $\lambda_V^{(\alpha,\beta)}$ be the eigenvalue of $\Gamma_{(\alpha,\beta)}$ corresponding to V. Let*

$$\lambda_V = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)}\lambda_V^{(\alpha,\beta)},$$

where the sum is over all $(\alpha,\beta) \in C(\tilde{\gamma}_j)$ and $1 \leqslant j \leqslant \zeta_t$. Then, $|\lambda_V| = o(\varpi)$ is an infinitesimal of higher order than ϖ when $n \to \infty$.

For $\gamma \in P_{n,t}$, define

$$e(\gamma) = \begin{cases} \gamma & \text{if a permutation with cycle-type } \gamma \text{ is even},\\ \operatorname{split}(\gamma) & \text{if a permutation with cycle-type } \gamma \text{ is odd}, \end{cases}$$

$$o(\gamma) = \begin{cases} \gamma & \text{if a permutation with cycle-type } \gamma \text{ is odd},\\ \operatorname{split}(\gamma) & \text{if a permutation with cycle-type } \gamma \text{ is even}. \end{cases}$$

Theorem 4.3. *There exists a linear combination Γ_{e} of Cayley graphs on S_n^D, each of which is generated by a union of conjugacy classes included in FPF_t such that its eigenvalues are as described in the first line of Table 1.*

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Table 1. *Eigenvalues.*

	$S^{((n),\emptyset)}$	fat, $\neq S^{((n),\emptyset)}$	$S^{((1^n),\emptyset)}$	thin, $\neq S^{((1^n),\emptyset)}$	medium
Γ_{e}	1	ϖ	1	ϖ	$o(\varpi)$
Γ_{o}	1	ϖ	-1	$-\varpi$	$o(\varpi)$
Γ	1	ϖ	0	0	$o(\varpi)$

Proof. Take

$$\Gamma_{\mathrm{e}} = \sum_{\gamma \in P_{n,t}} \sum_{(\alpha,\beta) \in C(e(\gamma))} d_{(\alpha,\beta)} \operatorname{Cay}(S_n^D, X_{(\alpha,\beta)}),$$

where the $d_{(\alpha,\beta)}$ are as described in Lemma 4.1. Then, we have that $\lambda_{((n),\emptyset)} = 1$ and $\lambda_V = \varpi$ for each fat $V \neq S^{((n),\emptyset)}$. By Lemma 4.2, for each medium V, we have that $|\lambda_V| = o(\varpi)$. Assume that $V = S^{(\mu^t,\nu^t)}$ is thin; then $S^{(\mu,\nu)}$ is fat. Since $\chi_{(\mu^t,\nu^t)}(\sigma) = \operatorname{sgn}(\bar\sigma)\chi_{(\mu,\nu)}(\sigma) = \chi_{(\mu,\nu)}(\sigma)$ (note that $\bar\sigma$ is even), we have that

$$\lambda_V = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\mu^t,\nu^t)} = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\mu,\nu)} = \begin{cases} 1 & \text{if } V = S^{((1^n),\emptyset)}, \\ \varpi & \text{if } V \text{ is thin but } \neq S^{((1^n),\emptyset)}, \end{cases}$$

completing the proof. □

Theorem 4.4. *There exists a linear combination Γ_{o} of Cayley graphs on S_n^D, each of which is generated by a union of conjugacy classes included in* FPF_t *such that its eigenvalues are as described in the second line of Table 1.*

Proof. Take

$$\Gamma_{\mathrm{o}} = \sum_{\gamma \in P_{n,t}} \sum_{(\alpha,\beta) \in C(o(\gamma))} d_{(\alpha,\beta)} \operatorname{Cay}(S_n^D, X_{(\alpha,\beta)}),$$

where the $d_{(\alpha,\beta)}$ are as described in Lemma 4.1. Assume that $V = S^{(\mu^t,\nu^t)}$ is thin; then $S^{(\mu,\nu)}$ is fat. This time, we have that $\chi_{(\mu^t,\nu^t)}(\sigma) = \operatorname{sgn}(\bar\sigma)\chi_{(\mu,\nu)}(\sigma) = -\chi_{(\mu,\nu)}(\sigma)$, since $\bar\sigma$ is odd. Thus,

$$\lambda_V = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\mu^t,\nu^t)} = -\sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\mu,\nu)} = \begin{cases} -1 & \text{if } V = S^{((1^n),\emptyset)}, \\ -\varpi & \text{if } V \text{ is thin but} \neq S^{((1^n),\emptyset)}. \end{cases}$$

□

Theorem 4.5. *There exists a linear combination Γ of Cayley graphs on S_n^D, each of which is generated by a union of conjugacy classes included in* FPF_t *such that its eigenvalues are as described in the last line of Table 1.*

Proof. Take $\Gamma = \frac{1}{2}\Gamma_{\mathrm{e}} + \frac{1}{2}\Gamma_{\mathrm{o}}$. □

Now, we can find the bound of a maximal $(2t)$-intersecting subset of S_n^D, by applying Lemma 2.2 and Theorem 4.5, which is $(-\varpi/(1-\varpi))|S_n^D| = 2^{n-t-1}(n-t)!$. This proves Theorem 2.3. Meanwhile, if I is a maximum-sized $(2t)$-intersecting subset of S_n^D, we have that $1_I \in \mathrm{span}(\{v_1\} \cup \{v_i \colon \lambda_i = \varpi\}) = U_t$. Since we have proved in Theorem 3.15 that $U_t = \mathrm{span}\{1_{T_{(a,b)}} \colon a, b \in \mathcal{B}_t\}$, we have that

$$1_I \in \mathrm{span}\{1_{T_{(a,b)}} \colon a, b \in \mathcal{B}_t\}. \tag{4.5}$$

Let $\mathcal{A}_t$ be the set of ordered t-tuples $(i_1, \ldots, i_t)$ such that $i_1, \ldots, i_t \in [n]$ are pairwise distinct. For any $a = (i_1, \ldots, i_t) \in \mathcal{B}_t$, let $-a = (-i_1, \ldots, -i_t) \in \mathcal{B}_t$. Then, for any $b = (j_1, \ldots, j_t) \in \mathcal{B}_t$, we have that $T_{(-a,b)} = T_{(a,-b)}$. Therefore, due to (4.5), we can assume that

$$1_I = \sum_{a \in \mathcal{A}_t, b \in \mathcal{B}_t} r_{(a,b)} 1_{T_{(a,b)}}, \quad \text{where } r_{a,b} \in \mathbb{R}. \tag{4.6}$$

In the following, we will deduce from (4.6) that I must be a t-coset $T_{(a,b)}$. Without loss of generality, we assume that $\mathrm{id} \in I$.

Theorem 4.6. *Assume that*

$$1_I = \sum_{a \in \mathcal{A}_t, b \in \mathcal{B}_t} r_{(a,b)} 1_{T_{(a,b)}},$$

where I is a maximal $(2t)$-intersecting subset of S_n^D and $\mathrm{id} \in I$. Then, there exists $e \in \mathcal{A}_t$ such that $I = T_{(e,e)}$.

Proof. We first divide S_n^D into 2^{n-1} pairwise disjoint subsets, each of which has size $n!$.

Let $\mathcal{F}$ be the family of subsets Y of $[-n] = \{-1, \ldots, -n\}$ such that $|Y|$ is even. Given that $Y = \{p_1, \ldots, p_{2k}\} \in \mathcal{F}$, let $\mathcal{F}_{\{p_1,\ldots,p_{2k}\}}$ be the set of elements $\sigma \in S_n^D$ such that $p_1, \ldots, p_{2k}$ are exactly the $2k$ negative numbers among its images $\{\sigma(1), \ldots, \sigma(n)\}$. For example, $\mathcal{F}_\emptyset = S_n$. Then, all elements in $\mathcal{F}_{\{p_1,\ldots,p_{2k}\}}$ have the same image set

$$\mathrm{Im}\,\mathcal{F}_{\{p_1,\ldots,p_{2k}\}} = \{p_1, \ldots, p_{2k}\} \cup ([n] \setminus \{|p_1|, \ldots, |p_{2k}|\}).$$

Then,

$$S_n^D = \bigcup_{\{p_1,\ldots,p_{2k}\} \in \mathcal{F}} \mathcal{F}_{\{p_1,\ldots,p_{2k}\}}.$$

Thus, as a function on S_n^D,

$$1_I = \sum_{\{p_1,\ldots,p_{2k}\} \in \mathcal{F}} 1_I \downarrow_{\mathcal{F}_{\{p_1,\ldots,p_{2k}\}}}.$$

Now, restrict both sides of (4.6) to $\mathcal{F}_{\{p_1,\ldots,p_{2k}\}}$:

$$1_I \downarrow_{\mathcal{F}_{\{p_1,\ldots,p_{2k}\}}} = \sum_{a \in \mathcal{A}_t, b \in \mathcal{B}_t} r_{(a,b)} 1_{T_{(a,b)}} \downarrow_{\mathcal{F}_{\{p_1,\ldots,p_{2k}\}}}. \tag{4.7}$$

Given $\{p_1,\dots,p_{2k}\}\in\mathcal{F}$, let $\mathcal{B}_t^{p_1,\dots,p_{2k}}$ be the set of t-tuples $(j_1,\dots,j_t)\in\mathcal{B}_t$ such that $j_1,\dots,j_t\in\operatorname{Im}\mathcal{F}_{\{p_1,\dots,p_{2k}\}}$. Then, (4.7) is

$$1_I\downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}=\sum_{a\in\mathcal{A}_t,b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}} r_{(a,b)}1_{T_{(a,b)}}\downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}. \tag{4.8}$$

We will now follow the argument of the proof of [**5**, Theorem 27] to show that there exist non-negative numbers $\{m_{(a,b)}^{p_1,\dots,p_{2k}}: a\in\mathcal{A}_t,\ b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}\}$ such that

$$1_I\downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}=\sum_{a\in\mathcal{A}_t,b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}} m_{(a,b)}^{p_1,\dots,p_{2k}}1_{T_{(a,b)}}\downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}. \tag{4.9}$$

The idea is for $\mathcal{F}_{\{p_1,\dots,p_{2k}\}}$ to fulfil the role of S_n in [**5**, Theorem 27].

From (4.8), we represent $1_I\downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}$ by a matrix $R^{p_1,\dots,p_{2k}}$, whose entry in the (a,b)-position is $r_{(a,b)}$, where $a\in\mathcal{A}_t$ and $b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}$. Let $\mathcal{L}_t^{p_1,\dots,p_{2k}}$ be the set of all t-lines of $R^{p_1,\dots,p_{2k}}$ (see [**5**, Definition 13] for the definition of *t-lines*). For every t-line $L\in\mathcal{L}_t^{p_1,\dots,p_{2k}}$, we add a variable x_L to each entry on L, and then obtain a new matrix $M^{p_1,\dots,p_{2k}}$, which is still needed to represent the function $1_I\downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}$. This means that

$$\sum_{L\in\mathcal{L}_t^{p_1,\dots,p_{2k}}} x_L=0. \tag{4.10}$$

All the entries of $M^{p_1,\dots,p_{2k}}$ are required to be non-negative, i.e.

$$\sum_{(a,b)\in L} x_L+r_{(a,b)}\geqslant 0\quad\text{for each } a\in\mathcal{A}_t \text{ and } b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}. \tag{4.11}$$

We see that if (4.10) and (4.11) are both satisfied, then (4.9) is proved by taking

$$m_{(a,b)}^{p_1,\dots,p_{2k}}=\sum_{(a,b)\in L} x_L+r_{(a,b)},\quad\text{where } a\in\mathcal{A}_t,\ b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}.$$

By the duality theorem of linear programming, any solution of (4.11) must satisfy the condition

$$\sum_{L\in\mathcal{L}_t^{p_1,\dots,p_{2k}}} x_L\geqslant-\sum_{a\in\mathcal{A}_t,b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}} c_{(a,b)}r_{(a,b)}, \tag{4.12}$$

where $C=(c_{(a,b)})_{a\in\mathcal{A}_t,b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}}$ is a t-bistochastic matrix (see [**5**, Definition 14]).

We see that if

$$-\sum_{a\in\mathcal{A}_t,b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}} c_{(a,b)}r_{(a,b)}>0, \tag{4.13}$$

then (4.12) will contradict (4.10) and we will fail. So we only need to show that (4.13) cannot happen.

Since C is a t-bistochastic matrix, by the generalized Birkhoff theorem (see [**5**, Theorem 29]), there exist non-negative constants $q_1,\dots,q_h$, with $\sum_{i=1}^h q_i=1$, and elements $\sigma_1,\dots,\sigma_h\in\mathcal{F}_{\{p_1,\dots,p_{2k}\}}$ such that

$$C=\sum_{i=1}^h\sum_{a\in\mathcal{A}_t} q_iE_{a,\sigma_i(a)},$$

where $E_{a,\sigma_i(a)}$ is an $|\mathcal{A}_t| \times |\mathcal{A}_t|$ matrix unit, whose only non-zero element is 1 in the $(a, \sigma_i(a))$-position.

Then,

$$-\sum_{a\in\mathcal{A}_t,b\in\mathcal{B}_t^{p_1,\dots,p_{2k}}} c_{(a,b)}r_{(a,b)} = -\sum_{i=1}^{h} q_i \sum_{a\in\mathcal{A}_t} r_{(a,\sigma_i(a))} = -\sum_{i=1}^{h} q_i 1_I(\sigma_i) \leqslant 0.$$

Therefore, (4.9) is proved. In the following, we will determine the exact values of $\{m_{(a,b)}^{p_1,\dots,p_{2k}} : a \in \mathcal{A}_t,\ b \in \mathcal{B}_t^{p_1,\dots,p_{2k}}\}$ in (4.9). We claim that

$$m_{(a,b)}^{p_1,\dots,p_{2k}} = 0 \quad \text{for any } a \in \mathcal{A}_t,\ b \in \mathcal{B}_t^{p_1,\dots,p_{2k}},\ \text{with } a \neq b. \tag{4.14}$$

We explain (4.14) as follows. Assume that $a = (i_1, \dots, i_t)$ and $b = (j_1, \dots, j_t)$. Then, there exists $\sigma \in \mathcal{F}_{\{p_1,\dots,p_{2k}\}}$ such that $\sigma(a) = b$ and σ does not stabilize any element in $[n] \setminus \{i_1, \dots, i_t\}$. Since $a \neq b$, then σ has fewer than t fixed points in $[n]$, which cannot be in a $(2t)$-intersecting subset containing id. Therefore, $1_I(\sigma) = 0$, which means that the right-hand side of (4.9) is zero when acting on σ, i.e.

$$m_{(a,b)}^{p_1,\dots,p_{2k}} + \sum_{a'\in\mathcal{A}_t,a'\neq a} m_{(a',\sigma(a'))}^{p_1,\dots,p_{2k}} = 0. \tag{4.15}$$

Since each term on the left-hand side of (4.15) is non-negative, then $m_{(a,b)}^{p_1,\dots,p_{2k}} = 0$.

We also claim that

$$\exists \text{ at most one element } a \in \mathcal{A}_t \cap \mathcal{B}_t^{p_1,\dots,p_{2k}} \text{ such that } m_{(a,a)}^{p_1,\dots,p_{2k}} \neq 0. \tag{4.16}$$

We explain (4.16) as follows. Assume that $a = (i_1, \dots, i_t) \in \mathcal{B}_t^{p_1,\dots,p_{2k}} \cap \mathcal{A}_t$, $b = (j_1, \dots, j_t) \in \mathcal{B}_t^{p_1,\dots,p_{2k}} \cap \mathcal{A}_t$ and $a \neq b$. Since n is sufficiently large depending on t, there exist $\sigma_1, \sigma_2 \in \mathcal{F}_{\{p_1,\dots,p_{2k}\}}$ such that $\sigma_1(a) = a$, $\sigma_2(b) = b$ and $\sigma_1\sigma_2^{-1}$ has fewer than t fixed points in $[n]$. Therefore, at least one of σ_1 and σ_2 is not in I. Assume that $\sigma_2 \notin I$. Then, $1_I(\sigma_2) = 0$, which means that the right-hand side of (4.9) is zero when acting on σ_2:

$$m_{(b,b)}^{p_1,\dots,p_{2k}} + \sum_{b'\in\mathcal{A}_t,b'\neq b} m_{(b',\sigma_2(b'))}^{p_1,\dots,p_{2k}} = 0. \tag{4.17}$$

Since each term on the left-hand side of (4.17) is non-negative, then $m_{(b,b)}^{p_1,\dots,p_{2k}} = 0$. Apply this analysis to another pair a, a' in $\mathcal{B}_t^{p_1,\dots,p_{2k}} \cap \mathcal{A}_t$, and we see that at least one of a and a', say a', satisfies $m_{(a',a')}^{p_1,\dots,p_{2k}} = 0$. Continue in this way, and at last we obtain (4.16).

Now consider $1_I \downarrow_{S_n}$. Since id $\in I$ and $1_I(\text{id}) = 1$, (4.14) and (4.16) show that there exists exactly one element $e \in \mathcal{A}_t$ such that

$$1_I \downarrow_{S_n} = 1_{T_{(e,e)}} \downarrow_{S_n}. \tag{4.18}$$

Assume that $e = (q_1, \dots, q_t)$. Then, for any element $\sigma \in S_n$ such that $\sigma(e) = e$, we see by (4.18) that $1_I(\sigma) = 1_{T_{(e,e)}}(\sigma) = 1$. This means that $\sigma \in I$.

For any $\{p_1, \dots, p_{2k}\} \in \mathcal{F}$, consider $1_I \downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}$. Choose $\sigma_1 \in S_n$ such that $\sigma_1(e) = e$ and σ_1 does not stabilize any element in $[n] \setminus \{q_1, \dots, q_t\}$. For any $a = (i_1, \dots, i_t) \in$

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$\mathcal{B}_t^{p_1,\dots,p_{2k}} \cap \mathcal{A}_t$ and $a \neq e$, there exists $\sigma_2 \in \mathcal{F}_{\{p_1,\dots,p_{2k}\}}$ such that $\sigma_2(a) = a$ and $\sigma_2\sigma_1^{-1}$ has fewer than t fixed points in $[n]$. Since $\sigma_1 \in I$, we have that $\sigma_2 \notin I$ and $1_I(\sigma_2) = 0$. This forces $m_{(a,a)}^{p_1,\dots,p_{2k}} = 0$. Therefore, we claim that the only possible non-zero term on the right-hand side of (4.9) is $m_{(e,e)}^{p_1,\dots,p_{2k}} 1_{T_{(e,e)}} \downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}$.

According to the above analysis, we obtain that

$$1_I = \sum_{\substack{\{p_1,\dots,p_{2k}\}\in\mathcal{F} \\ \{q_1,\dots,q_t\}\subseteq \operatorname{Im}\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}} m_{(e,e)}^{p_1,\dots,p_{2k}} 1_{T_{(e,e)}} \downarrow_{\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}. \tag{4.19}$$

Since $|I| = 2^{n-t-1}(n-t)!$ and

$$\sum_{\substack{\{p_1,\dots,p_{2k}\}\in\mathcal{F} \\ \{q_1,\dots,q_t\}\subseteq \operatorname{Im}\mathcal{F}_{\{p_1,\dots,p_{2k}\}}}} |\{\sigma \in \mathcal{F}_{\{p_1,\dots,p_{2k}\}} : \sigma(e) = e\}| = 2^{n-t-1}(n-t)!,$$

then each term on the right-hand side of (4.19) is non-zero and we have that $1_I = 1_{T_{(e,e)}}$. □

5. Proof of the main theorem for S_n^B

The proof of Theorem 1.2 for S_n^B is quite similar to that for S_n^D.

Let $\gamma \in P_{n,t}$ and let $(\alpha,\beta) \in C(\gamma)$ or $C(\operatorname{split}(\gamma))$. Let $Y_{(\alpha,\beta)} \subset S_n^B$ be the analogue of $X_{(\alpha,\beta)} \subset S_n^D$. Let $Y_{(\alpha,\beta)} = Y_{(\alpha,\beta)}^+ \cup Y_{(\alpha,\beta)}^-$ with $\tau(w) \equiv 0 \pmod 2$ for each $w \in Y_{(\alpha,\beta)}^+$ and $\tau(w) \equiv 1 \pmod 2$ for each $w \in Y_{(\alpha,\beta)}^-$. Note that $Y_{(\alpha,\beta)}^+ = X_{(\alpha,\beta)} \subset S_n^D$.

By an argument as in the proof of Lemma 3.12, we have the following.

Lemma 5.1. *Let $\gamma_i, \gamma_j \in P_{n,t}$ and let $(u,v) \in C(\gamma_i)$, $(\alpha,\beta) \in C(\gamma_j)$ or $C(\operatorname{split}(\gamma))$. Then, $\chi_{(\mu,\nu)}$ is a constant on the set $Y_{(\alpha,\beta)}$.*

By an argument similar to that in the proof of Lemma 3.13, we have the following.

Lemma 5.2. *Let $\gamma_i, \gamma_j \in P_{n,t}$ and let $(u,v) \in C(\gamma_i)$, $(\alpha,\beta) \in C(\gamma_j)$. Let $\alpha = (b_1, b_2, \dots, b_k)$ and let $\beta = (b_{k+1}, \dots, b_h)$, where $b_1 \geqslant n-t$. Let $\alpha' = (b_1 - t - 1, t+1, b_2, \dots, b_k)$. Let $\sigma \in Y_{(\alpha,\beta)}^{\varepsilon}$ and let $\sigma' \in Y_{(\alpha',\beta)}^{\varepsilon'}$, where $\varepsilon, \varepsilon' \in \{+,-\}$. Then, $\xi_{(\mu,\nu)}(\sigma) = \xi_{(\mu,\nu)}(\sigma')$.*

Lemma 5.3. *Let $\gamma_i, \gamma_j \in P_{n,t}$ and let $(u,v) \in C(\gamma_i)$. Let $\tilde{\gamma}_j \in \{\gamma_j, \operatorname{split}(\gamma_j)\}$ and let $(\alpha,\beta) \in C(\tilde{\gamma}_j)$. Then, $\operatorname{Cay}(S_n^B, Y_{(\alpha,\beta)}^+)$, $\operatorname{Cay}(S_n^B, Y_{(\alpha,\beta)}^-)$ and $\operatorname{Cay}(S_n^D, X_{(\alpha,\beta)})$ have the same eigenvalue corresponding to the irreducible module $S^{(\mu,\nu)}$, which is denoted by $\lambda_{(\mu,\nu)}^{(\alpha,\beta)}$, as described in Lemma 4.1.*

Based on the above lemmas, we have the following.

Lemma 5.4. *Lemma 4.1 still holds if we replace the Cayley graph $\Gamma_{(\alpha,\beta)} = \operatorname{Cay}(S_n^D, X_{(\alpha,\beta)})$ by $\Gamma_{(\alpha,\beta)}^+ = \operatorname{Cay}(S_n^B, Y_{(\alpha,\beta)}^+)$ or $\Gamma_{(\alpha,\beta)}^- = \operatorname{Cay}(S_n^B, Y_{(\alpha,\beta)}^-)$.*

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Table 2. *Eigenvalues.*

	$S^{((n),\emptyset)}$	left-fat, $\neq S^{((n),\emptyset)}$	$S^{((1^n),\emptyset)}$	left-thin, $\neq S^{((1^n),\emptyset)}$	$S^{(\emptyset,(n))}$	right-fat, $\neq S^{(\emptyset,(n))}$	$S^{(\emptyset,(1^n))}$	right-thin, $\neq S^{(\emptyset,(1^n))}$	medium
Γ_{e}^{+}	1	ϖ	1	ϖ	1	ϖ	1	ϖ	$o(\varpi)$
Γ_{e}^{-}	1	ϖ	1	ϖ	-1	$-\varpi$	-1	$-\varpi$	$o(\varpi)$
Γ_{o}^{+}	1	ϖ	-1	$-\varpi$	1	ϖ	-1	$-\varpi$	$o(\varpi)$
Γ_{o}^{-}	1	ϖ	-1	$-\varpi$	-1	$-\varpi$	1	ϖ	$o(\varpi)$
Γ	1	ϖ	0	0	0	0	0	0	$o(\varpi)$

Let (α,β) be a double partition of n. We call an irreducible module $S^{(\alpha,\beta)}$ of S_n^B *left-fat* if $\alpha_1 \geqslant n-t$, where α_1 is the first part of α; *left-thin* if $S^{(\alpha^t,\beta)}$ is left-fat; *right-fat* if $\beta_1 \geqslant n-t$, where β_1 is the first part of β; *right-thin* if $S^{(\alpha,\beta^t)}$ is right-fat; and *medium* for all other cases.

Similarly to Lemma 4.2, we have the following.

Lemma 5.5. *Let $\tilde{\gamma}_j \in \{\gamma_j, \mathrm{split}(\gamma_j)\}$ and let $(\alpha,\beta) \in C(\tilde{\gamma}_j)$, where $1 \leqslant j \leqslant \zeta_t$. Let $\lambda_V^{(\alpha,\beta)}$ be the eigenvalue of $\Gamma^{+}_{(\alpha,\beta)}$ or $\Gamma^{-}_{(\alpha,\beta)}$ corresponding to a medium module V of S_n^B. Set*

$$\lambda_V = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda_V^{(\alpha,\beta)},$$

where the $d_{(\alpha,\beta)}$ are as described in Lemma 4.1. Then, $|\lambda_V| = o(\varpi)$ is an infinitesimal of higher order than ϖ when $n \to \infty$.

In the following, let FPF_t be the set of elements in S_n^B that fix fewer than t points in $[n]$.

Theorem 5.6. *There exists a linear combination Γ_{e}^{+} of Cayley graphs on S_n^B, each of which is generated by a union of conjugacy classes included in FPF_t such that its eigenvalues are as described in the first line of Table 2.*

Proof. Take

$$\Gamma_{\mathrm{e}}^{+} = \sum_{\gamma \in P_{n,t}} \sum_{(\alpha,\beta) \in C(e(\gamma))} d_{(\alpha,\beta)} \operatorname{Cay}(S_n^B, Y^{+}_{(\alpha,\beta)}),$$

where the $d_{(\alpha,\beta)}$ are as described in Lemma 4.1. Then, $\lambda_{((n),\emptyset)} = 1$ and $\lambda_V = \varpi$ for each left-fat $V \neq S^{((n),\emptyset)}$ by Lemma 5.4. By Lemma 5.5, for each medium V, we have that $|\lambda_V| = o(\varpi)$.

Assume that $V = S^{(\mu^t,\nu^t)}$ is left-thin; then $S^{(\mu,\nu)}$ is left-fat. For any $\sigma \in Y^{+}_{(\alpha,\beta)}$, we have that $\chi_{(\mu^t,\nu^t)}(\sigma) = \mathrm{sgn}(\bar{\sigma})\chi_{(\mu,\nu)}(\sigma) = \chi_{(\mu,\nu)}(\sigma)$ (note that $\bar{\sigma}$ is even). Hence,

$$\lambda_V = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\mu^t,\nu^t)} = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\mu,\nu)} = \begin{cases} 1 & \text{if } V = S^{((1^n),\emptyset)}, \\ \varpi & \text{if } V \text{ is left-thin but } \neq S^{((1^n),\emptyset)}. \end{cases}$$

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Assume that $V = S^{(\mu,\nu)}$ is right-fat; then $S^{(\nu,\mu)}$ is left-fat. For any $\sigma \in Y^{+}_{(\alpha,\beta)}$, we have that

$$\begin{aligned}\chi_{(\mu,\nu)}(\sigma) &= \sum_{\sigma^g \in S_{|\mu|} \times S_{|\nu|}} \chi_\mu(\overline{\sigma_1^g})\chi_\nu(\overline{\sigma_2^g})(-1)^{\tau(\sigma_2^g)} \\ &= \sum_{\sigma^g \in S_{|\mu|} \times S_{|\nu|}} \chi_\mu(\overline{\sigma_1^g})\chi_\nu(\overline{\sigma_2^g})(-1)^{\tau(\sigma_1^g)} \\ &= \chi_{(\nu,\mu)}(\sigma).\end{aligned}$$

Note that $(-1)^{\tau(\sigma_1^g)} = (-1)^{\tau(\sigma_2^g)}$, since $\tau(\sigma) \equiv 0 \pmod 2$. Then,

$$\lambda_V = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\mu,\nu)} = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\nu,\mu)} = \begin{cases} 1 & \text{if } V = S^{(\emptyset,(n))}, \\ \varpi & \text{if } V \text{ is right-fat but } \neq S^{(\emptyset,(n))}. \end{cases}$$

Assume that $V = S^{(\mu^t,\nu^t)}$ is right-thin; then $S^{(\nu,\mu)}$ is left-fat. For any $\sigma \in Y^{+}_{(\alpha,\beta)}$, since $\bar{\sigma}$ is even and $\tau(\sigma) \equiv 0 \pmod 2$, we have that

$$\lambda_V = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\mu^t,\nu^t)} = \sum_{(\alpha,\beta)} d_{(\alpha,\beta)} \lambda^{(\alpha,\beta)}_{(\nu,\mu)} = \begin{cases} 1 & \text{if } V = S^{(\emptyset,(1^n))}, \\ \varpi & \text{if } V \text{ is right-thin but } \neq S^{(\emptyset,(1^n))}. \end{cases}$$

□

Theorem 5.7. *There exists a linear combination Γ_{e}^{-} of Cayley graphs on S_n^B, each of which is generated by a union of conjugacy classes included in* FPF_t *such that its eigenvalues are as described in the second line of Table 2.*

Proof. Take

$$\Gamma_{\mathrm{e}}^{-} = \sum_{\gamma \in P_{n,t}} \sum_{(\alpha,\beta) \in C(e(\gamma))} d_{(\alpha,\beta)} \operatorname{Cay}(S_n^B, Y^{-}_{(\alpha,\beta)}),$$

where the $d_{(\alpha,\beta)}$ are as described in Lemma 4.1. We omit the proof since it is similar to that of Theorem 5.6. □

Theorem 5.8. *There exists a linear combination Γ_{o}^{+} of Cayley graphs on S_n^B, each of which is generated by a union of conjugacy classes included in* FPF_t *such that its eigenvalues are as described in the third line of Table 2.*

Proof. Take

$$\Gamma_{\mathrm{o}}^{+} = \sum_{\gamma \in P_{n,t}} \sum_{(\alpha,\beta) \in C(o(\gamma))} d_{(\alpha,\beta)} \operatorname{Cay}(S_n^B, Y^{+}_{(\alpha,\beta)}),$$

where the $d_{(\alpha,\beta)}$ are as described in Lemma 4.1. We omit the proof. □

Theorem 5.9. *There exists a linear combination Γ_{o}^{-} of Cayley graphs on S_n^B, each of which is generated by a union of conjugacy classes included in* FPF_t *such that its eigenvalues are as described in the fourth line of Table 2.*

Proof. Take
$$\Gamma_{\mathrm{o}}^{-}=\sum_{\gamma\in P_{n,t}}\sum_{(\alpha,\beta)\in C(o(\gamma))} d_{(\alpha,\beta)}\,\mathrm{Cay}(S_n^B, Y_{(\alpha,\beta)}^{-}),$$
where the $d_{(\alpha,\beta)}$ are as described in Lemma 4.1. We omit the proof here. □

Theorem 5.10. *There exists a linear combination Γ of Cayley graphs on S_n^B, each of which is generated by a union of conjugacy classes included in* FPF_t *such that its eigenvalues are as described in the last line of Table 2.*

Proof. Take $\Gamma=\frac{1}{4}\Gamma_{\mathrm{e}}^{+}+\frac{1}{4}\Gamma_{\mathrm{e}}^{-}+\frac{1}{4}\Gamma_{\mathrm{o}}^{+}+\frac{1}{4}\Gamma_{\mathrm{o}}^{-}$. □

Based on Theorem 5.10, we can proceed as in § 4 to give the corresponding versions of Theorems 3.15 and 4.6 for S_n^B, and finally prove Theorem 1.2 for S_n^B. Since we need make no important modifications to previous working to achieve this, we omit the proof.

6. Some comments on imprimitive reflection groups

Let m, n be positive integers and let δ be a fixed mth primitive root of unity. Let $G(m,1,n)$ be a group consisting of all permutations w on the set $\{\delta^k i\colon k\in[m],\ i\in[n]\}$ such that $w(\delta^k i)=\delta^k w(i)$. Let p be a positive integer such that $p\mid m$. Let $G(m,p,n)$ be a normal subgroup of $G(m,1,n)$ which consists of all permutations $w\in G(m,1,n)$ such that
$$\sum_{i=1}^{m} k_i\equiv 0 \pmod p.$$
In [**12**], Shephard and Todd proved that any irreducible imprimitive reflection group is isomorphic to some $G(m,p,n)$.

Note that, when $m\leqslant 2$, $G(m,p,n)\in\{S_n,S_n^B,S_n^D\}$. In the preceding sections of this paper, we have proved that if n is sufficiently large depending on t, a maximal $(2t)$-intersecting subset of S_n^D(respectively, S_n^B) is a coset of the stabilizer of t points in $[n]$, which generalizes the result on S_n given in [**5**]. It is natural to ask whether we can further generalize this result to any imprimitive reflection group $G(m,p,n)$.

It seems that this should be possible, but a definite answer depends on whether the non-singularity of a matrix K can be proved. Let t be fixed as before. Let Λ be the set of ordered m-tuples $u=(u_1,\dots,u_m)$ such that $u_1,\dots,u_m$ are non-negative integers and
$$\sum_{i=1}^{m} u_i=t.$$

Let K be a matrix whose rows and columns are both indexed by elements in Λ. For $u=(u_1,\dots,u_m)\in\Lambda$ and $v=(v_1,\dots,v_m)\in\Lambda$, the entry in the (u,v)-position of K is
$$K_{u,v}=\sum_{\substack{j_l^k\geqslant 0\ \forall k\in[m],\,l\in[m]\\ j_l^1+j_l^2+\cdots+j_l^m=v_l\ \forall l\in[m]\\ j_1^k+j_2^k+\cdots+j_m^k=u_k\ \forall k\in[m]}}\ \prod_{l=1}^{m}\binom{v_l}{j_l^1,j_l^2,\dots,j_l^m}\prod_{k,l=1}^{m}\delta^{klj_l^k}.$$

When $m = 2$ and $\delta = -1$, K is reduced to

$$\left[\sum_{i=0}^{\min\{p,q\}} (-1)^i \binom{q}{i}\binom{t-q}{p-i}\right]_{\substack{0\leqslant p\leqslant t\\ 0\leqslant q\leqslant t}},$$

which has been proved invertible in [**2**]. However, we make the following statements.

Conjecture 6.1. *The matrix K described above is invertible.*

Conjecture 6.2. *Provided n is sufficiently large depending on t, the maximal (mt)-intersecting subsets of $G(m,p,n)$ are cosets of stabilizers of t points in $[n]$.*

The case $t = 1$ of Conjecture 6.2 has been proved in [**13**].

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