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A METHOD FOR THE STUDY OF FULLY DEVELOPED PARALLEL FLOW IN STRAIGHT DUCTS OF ARBITRARY CROSS SECTION

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Abstract

A method is presented for the study of fully developed parallel flow of Newtonian viscous fluid in uniform straight ducts of very general cross-section. The method is based upon the concept of contour lines of constant velocity in a typical cross section of the duct, and uses the function which describes the contour lines as an independent variable to derive the integral momentum equation. The resulting ordinary integro-differential equation is, in principle, much easier to solve than the original momentum equation in partial differential equation form. Several illustrative examples of practical interest are included to explain the method of solution. Some of these solutions are compared with available solutions in the literature. All details are explained by graphs and tables. The method has several interesting features. The study has relevance to biomedical engineering research for blood and urinary tract flow.

1. Introduction

The problem of viscous fluid flow through a duct of uniform cross-section has received considerable attention in the past, especially after the publication of a pioneering work by Osborne Reynolds [15]. However, such a problem requires considerable computational effort in carrying out the details of the solution if the cross-section of the duct is quite arbitrary. Ducts of certain simple shapes like circles, ellipses, and rectangles or when the duct boundary coincides with one of the natural coordinate systems, have been studied extensively by various methods [2, 4, 13, 14], but these ducts

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cannot fulfil many modern engineering requirements. Furthermore, because of the complicated mathematical tools involved in these methods, they are not suitable for arbitrary shaped ducts. However, in some situations the complication caused by the boundary shape can be removed if one uses numerical techniques. Indeed, the most commonly used numerical methods for ducts with geometrically complicated cross sections are finite difference and finite element methods. Although these numerical approximate methods have been found to be quite suitable in most cases, there appears to exist a need for an easier method that will handle problems with arbitrary cross sections in a minumum of computer time. With this in mind, a new method is presented in this study.

The purpose of the present investigation is therefore to provide an alternative way of solving the fully developed laminar duct flow problems and to obtain in an easy manner a highly accurate velocity component parallel to the duct walls. A knowledge of the velocity field permits the calculation of the shear stress, flow rate, etc. In fact, the calculation of such flows is interesting also in biomedical engineering research concerning blood and urinary tract flow.

The method derives the governing equations from first principles. A similar contour lines concept has been used earlier to solve a large class of boundary value problems for linear and nonlinear analysis of plates, shells and membranes as well as for two dimensional heat conduction problems and electromagnetic waveguide problems [7]–[11].

2. An account of the method and derivation of basic equations

Consider a uniform long straight duct of any cross-section and take rectangular axes $oxyz$ with z-axis parallel to the length of the tube and the origin o at the centroid of a right section, as shown in Figure 1. Let the duct be filled with liquid, which is made to oscillate in the direction of the length of the duct due to the application of a pressure gradient and assume that the motion is everywhere axial. It is also assumed that the fluid is incompressible Newtonian and the flow is fully developed with a single velocity component w parallel to the duct walls. Since the flow is assumed to be fully developed, our attention will be focussed far downstream of the duct entrance, so that the initial growing shear layer and acceleration effects have vanished and the velocity is purely axial and independent of its z-coordinate. We are interested here to find the time periodic velocity and pressure fields which can exist inside an infinitely long straight duct.

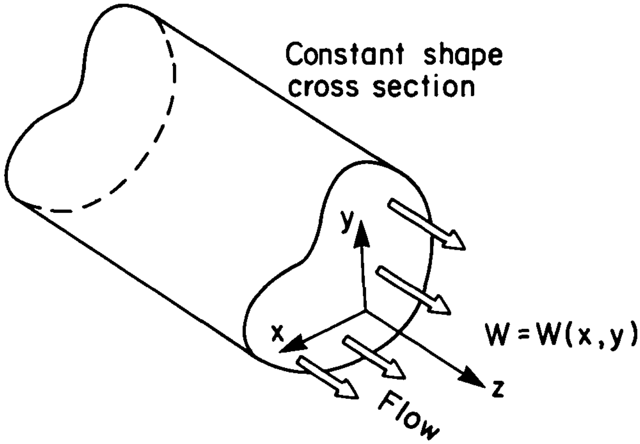

FIGURE 1. Fully developed flow in a duct with an arbitrary cross section.

Imagine the flow pattern is obtained from a representation of velocity distributions and velocity contour lines at any typical cross-section of the duct. A velocity contour line represents the locus of points having the same value of velocity at any fixed instant of time τ_0. Such a curve is called an *isovelocity* curve or an *isotac* for the velocity component $w(x, y, \tau_0)$. These curves may be formed in any way, but no two isovelocity curves cut each other, since no point inside the duct can have two different velocities at the same time. Furthermore, the viscous boundary condition on the duct wall suggests that the velocity component must be zero on the boundary of the cross-section.

Let the family of isovelocity contour lines at any fixed time instant be denoted by $u(x, y)$ =constant. Without loss of generality, one can assume that $u = 0$ on the boundary c of the cross-section. Thus, at time τ these level curves form a contour map of lines of equal velocity—a system of non-intersecting closed curves, with the boundary as one of the lines. Let the family of curves $u(x, y)$ =constant be denoted by c_u, $0 \leq u \leq u^*$, so that $c_0 = c$, the boundary of the cross-section, and c_{u^*} coincides with the point at the centre of the contour pattern at which the maximum $u = u^*$ is attained as shown in Figure 2. It has been assumed here that u increases inwardly. Clearly, steeper velocity increase is distinguished by the closer proximity of lines of equal velocity.

It is to be made clear that in general the mathematical form of $u(x, y)$ will differ from one instant of time, τ_0, to the next. However, it is sufficient to consider a particular instant of time τ_0 that produces a particular form of $u(x, y)$, since initially solutions for w which are separable in space and time variables are being sought.

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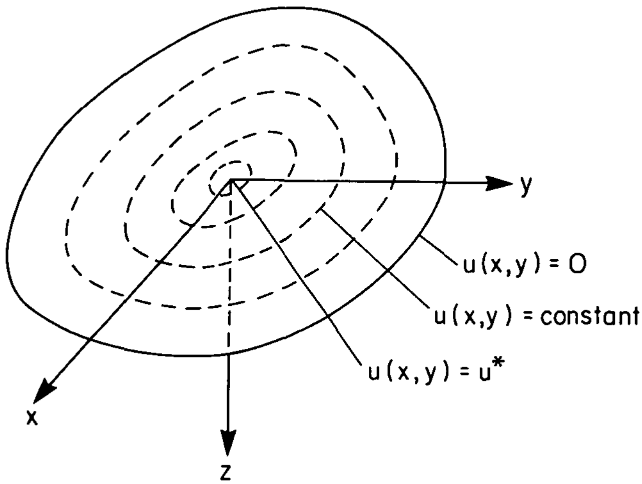

FIGURE 2. Isovelocity contour lines.

To find the velocity distribution, it is necessary to set up the integral form of the momentum equation. Consider the flow of fluid through an element bounded by any closed contour, c_u at any instant τ. According to the theory of fully-developed isothermal viscous flow, the momentum flux of fluid which flows through the element per unit time from the interior of the region to the exterior is equal to [17]

$$\int_{\Omega_u}\int \operatorname{div}\sigma_{i3}\, d\Omega,$$

where σ_{i3} is the z-component of the viscous-stress tensor and Ω_u is the area bounded by the contour c_u. Moreover, for a Newtonian fluid, the viscous stresses are proportional to the element strain-rates. Thus,

$$\sigma_{xz} = \mu\frac{\partial w}{\partial x}$$

$$\sigma_{yz} = \mu\frac{\partial w}{\partial y}$$

$$\text{i.e., } \sigma_{i3} = \mu\vec{\nabla} w$$

where μ is the coefficient of viscosity. Substituting this value of stress tensor in the above expression and using Green's Theorem, one obtains

$$\oint_{c_u} \mu(\vec{\nabla} w \cdot \vec{n})\, ds,$$

where ds is an elemental arc length and the contour integral is taken round the contour c_u, and $\vec{n}$ denotes the unit normal vector to the curve

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$u(x, y)$ =constant in the positive outward direction, i.e., $\vec{n} = -\vec{\nabla} u / |\vec{\nabla} u|$. If however $\vec{\nabla} p$ denotes the driving pressure gradient which is assumed to be purely oscillatory, then the net resultant force in the axial direction is given by

$$\mu \oint_{c_u} (\vec{\nabla} w \cdot \vec{n})\, ds - \int_{\Omega}\int (\vec{\nabla} p \cdot \vec{k})\, d\Omega$$

where $\vec{k}$ is the unit vector in the z-direction. The above expression must be equal to the inertial force, in accordance with the principle of momentum flux of fluid. Hence, using the local balance of surface forces we obtain the fundamental integral momentum equation governing the unsteady laminar motion of a viscous incompressible fluid in a duct in the absence of any external forces, in the form

$$\mu \oint_{c_u} (\vec{\nabla} w \cdot \vec{n})\, ds - \int_{\Omega_u}\int (\vec{\nabla} p \cdot \vec{k})\, d\Omega = \int_{\Omega_u}\int \rho \frac{\partial w}{\partial \tau}\, d\Omega. \tag{1}$$

In the above formulation, the weight of the fluid has been ignored, as the pipe is horizontal. It is to be mentioned here that the governing equation (1) can as well be obtained from the z-component of the Navier-Stokes equations viz.,

$$\mu \nabla^2 w - \frac{\partial p}{\partial z} = \rho \frac{\partial w}{\partial \tau}$$

integrated over the duct area Ω_u bounded by a contour c_u.

When the flow is oscillatory, one may write

$$p(z, \tau) = P(z) e^{i\omega\tau}, \qquad w(u, \tau) = W(x, y) e^{i\omega\tau} \tag{2}$$

where ω is the frequency of oscillation. As an approximation, it is supposed that

$$W(x, y) = W(u) \tag{3}$$

i.e., it is assumed that each of $\mathcal{R}W$ and $\mathcal{I}W$ share the same contour lines u = constant. This is true for $\omega = 0$, because $\mathcal{I}W$ can then be taken to vanish, and is approximately true for large ω, since W is then nearly constant save in a thin boundary layer, where, to a first approximation, it varies with normal distance to the boundary. However, between these two extremes the above approximation is empirical, but appears to result in close agreement with such calculations of velocity by other methods, as is shown in the illustrative examples.

In the above expressions, $w(u, \tau)$ and $p(z, \tau)$ are the real parts of the respective expressions on the right-hand side of (2).

If, however, one considers a pulsating flow, it is possible to assume

$$p = P_s + P_{osc} e^{i\omega\tau} \tag{4}$$

which is composed of steady and oscillatory components. Here P_s is constant and P_{osc} is the complex modulus of the oscillating pressure.

Substituting (2) into (1) and cancelling the factor $e^{i\omega\tau}$, one obtains

$$-\frac{dW}{du}\oint_{c_u}\sqrt{t}\,ds-\frac{1}{\mu}\int_{\Omega_u}\int\frac{dP}{dz}\,d\Omega=i\lambda^2\int_{\Omega_u}\int W\,d\Omega \tag{5}$$

where the notations

$$t=u_x^2+u_y^2,\qquad \nu=\mu/\rho\quad\text{and}\quad\lambda^2=\omega/\nu \tag{6}$$

have been used. Here λ^2 is a reduced frequency. In formulating (5), use has been made of the fact that

$$\vec{\nabla}W\cdot\vec{n}=\frac{dW}{du}\vec{\nabla}u\cdot\vec{n}=-\frac{dW}{du}\left(\frac{\vec{\nabla}u\cdot\vec{\nabla}u}{\sqrt{t}}\right)=-\sqrt{t}\frac{dW}{du}. \tag{7}$$

Now, using the technique of converting the double integral into a contour integral as explained previously [8], which for any continuous function $F(u)$ is given by

$$\int_{\Omega_u}\int F(u)d\Omega=-\int_{u^*}^{u}\oint_{c_u}\frac{F(u)}{\sqrt{t}}\,du\,ds, \tag{8}$$

and then differentiating (5) with respect to u, one finally obtains

$$\frac{d^2W}{du^2}\oint_{c_u}\sqrt{t}\,ds+\frac{dW}{du}\oint_{c}\frac{\nabla^2u}{\sqrt{t}}\,ds-i\lambda^2W\oint_{c_u}\frac{ds}{\sqrt{t}}=\frac{1}{\mu}\frac{dP}{dz}\oint_{c_u}\frac{ds}{\sqrt{t}}. \tag{9}$$

While deriving the second term in the above equation, the following relationship has been used

$$\begin{aligned}\frac{d}{du}\oint_{c_u}\sqrt{t}\,ds&=-\frac{d}{du}\oint_{c_u}\vec{\nabla}u\cdot\vec{n}\,ds\\&=-\frac{d}{du}\int_{\Omega_u}\int\nabla^2u\,d\Omega\\&=\frac{d}{du}\int_{u^*}^{u}\oint_{c_u}\frac{\nabla^2u}{\sqrt{t}}\,du\,ds\\&=\oint_{c_u}\frac{\nabla^2u}{\sqrt{t}}\,ds.\end{aligned} \tag{10}$$

It is interesting to note that (9) is identical to the linear heat conduction equation, in two spatial dimensions with a source term [10]. The pressure gradient can vary only with time, and thus represents a uniformly distributed heat source.

For the oscillatory solution, the third term in (9) requires W to be complex for real P. Hence, both the real and imaginary parts of (9) should be considered together. However, if the flow is steady then $\partial w/\partial\tau = 0$, and w and p are independent of τ. In that case the fundamental integral equation (1) takes the form

$$\frac{d^2w}{du^2}\oint_{c_u}\sqrt{t}\,ds + \frac{dw}{du}\oint_{c_u}\frac{\nabla^2 u}{\sqrt{t}}\,ds = \frac{1}{\mu}\frac{dp}{dz}\oint_{c_u}\frac{ds}{\sqrt{t}}. \tag{11}$$

Although in the present study we are concerned with unsteady flows, it is clear that the steady flow equation can be obtained from the unsteady flow equation (9) by simply putting $\lambda = 0$. Thus, the pulsatile flows comprising both steady and oscillatory components may be obtained by linear superposition of the two solutions. It is clear that the analytical solution predicted by (9) and (11) should be exact in nature, at least in those cases where the assumption (3) is valid for both $\mathcal{R}W$ and $\mathcal{I}W$. In fact, for a straight duct of uniform cross-section, this assumption seems to be true, which is evident from the examples discussed below.

3. Illustrations

In order to test the accuracy of the method, a number of illustrative examples will now be discussed. These examples will describe the fluid flow characteristics of non-circular ducts, which have become a subject of considerable practical interest in connection with compact heat exchange equipment.

Our first step is to find the appropriate equation for the family of isovelocity contour lines at any time τ_0. Although in principle the equation of isovelocity contour lines can be determined both in steady as well as in unsteady cases, one can still solve the duct problems without a priori knowledge of the contour equations. This will be explained in one of the illustrative examples. It is however to be mentioned that the equation of isovelocity contours can be obtained using a few well-known analogies of laminar flow in conduits viz., the membrane analogy, the soap-bubble analogy, the torsion analogy, the vorticity flow analogy, etc. These analogies are usually true for steady-flow problems, but one can still consider them to be approximately true for oscillatory-flow problems, especially when the duct geometry has symmetry about the coordinate lines. This will be evident from the first two illustrative examples.

(a) Flow through a duct of elliptical cross section

Flow in an elliptical duct is selected as a useful test of the method, since for the limiting case, i.e. for a circular duct, the exact solution is known. This problem, however, has been initially studied by Khamrui [4].

Consider the fluid in a long elliptical duct which is made to oscillate under the influence of a periodic pressure gradient. Assume the motion is everywhere axial and gradually approaches the fully developed flow. Since in this case W is symmetric about both axes of the ellipse, it will be assumed that the isovelocity curves form similar and similarly situated ellipses with the boundary of the duct as one of those lines. Hence we will have

$$u(x, y) = 1 - x^2/a^2 - y^2/b^2 \tag{12}$$

It is to be mentioned here that for the case of a circular duct, this is no longer an assumption. Calculations of the contour integrals appearing in (9) yield

$$\oint_{C_u} \sqrt{t}\, ds = \frac{2\pi(a^2+b^2)}{ab}(1-u), \tag{13}$$

$$\oint_{C_u} \frac{1}{\sqrt{t}}\, ds = \pi ab, \tag{14}$$

$$\oint_{C_u} \frac{\nabla^2 u}{\sqrt{t}}\, ds = -\frac{2\pi(a^2+b^2)}{ab} \tag{15}$$

where $t = 4(x^2/a^4 + y^2/b^4)$, and the contour integrals (13)–(15) are evaluated along the boundary of the ellipse $x^2/a^2 + y^2/b^2 = 1 - u$ as explained in [7]. Substituting these values in (9) one obtains

$$(1-u)\frac{d^2W}{du^2} - \frac{dW}{du} - \frac{i\lambda^2 a^2 b^2}{2(a^2+b^2)}W = \frac{a^2b^2}{2\mu(a^2+b^2)}\frac{dP}{dz} \tag{16}$$

which in terms of a new variable f reduces to a Bessel's equation

$$\frac{d^2W}{df^2} + \frac{1}{f}\frac{dW}{df} - iK^2 W = \frac{K^2}{\mu\lambda^2}\frac{dP}{dz} \tag{17}$$

where

$$f^2 = 1 - u \quad \text{and} \quad K^2 = \frac{2a^2b^2\lambda^2}{(a^2+b^2)}. \tag{18}$$

The solution to (17) is given by

$$W = A_1 I_0(\sqrt{i}Kf) + A_2 K_0(\sqrt{i}Kf)) + \frac{i}{\mu\lambda^2}\frac{dP}{dz} \tag{19}$$

where the third term which represents a particular solution to (17) can be interpreted as a uniform oscillating "Plug" velocity $(-\frac{dP}{dz}\frac{e^{-i\pi/2}}{\mu\lambda^2})$ (see [13]).

Here A_1 and A_2 are arbitrary constants and I_0 and K_0 are modified Bessel functions of the first and second kind. These constants can however be determined by imposing the viscous no slip boundary condition of zero velocity at the duct wall together with the regularity condition at the centre of the duct. The centre, of course, is the point where u attains the value u^*, which in this case is the origin of coordinates. Using the above conditions, one obtains

$$A_1 = \frac{-i}{\mu\lambda^2}\frac{dP}{dz}/I_0(\sqrt{i}K), \qquad A_2 = 0. \tag{20}$$

Thus,

$$W = \frac{i}{\mu\lambda^2}\frac{dP}{dz}\left(1 - \frac{I_0(\sqrt{i}Kf)}{I_0(\sqrt{i}K)}\right). \tag{21}$$

The Bessel functions appearing in the above expression may be split up into their real and imaginary parts by means of the relations

$$I_0(\sqrt{i}x) = J_0(\sqrt{-i}x) = ber x + i bei x \tag{22}$$

and if W is written in the form

$$W = W_1 + iW_2, \tag{23}$$

it is found that

$$W_1 = \frac{1}{\mu\lambda^2}\frac{dP}{dz}\left(\frac{ber K bei K f - bei K ber K f}{ber^2 K + bei^2 K}\right) \tag{24}$$

and

$$W_2 = \frac{1}{\mu\lambda^2}\frac{dP}{dz}\left(1 - \frac{ber K ber K f + bei K bei K f}{ber^2 K + bei^2 K}\right). \tag{25}$$

In the limiting case, when $a = b$, the ellipse reduces to a circle. In this case one obtains

$$f = \gamma/a, \quad K = a\sqrt{\omega/\nu} \tag{26}$$

and (21) reduces to

$$W = \frac{i}{\rho\omega}\frac{dP}{dz}\left(1 - \frac{J_0(\sqrt{-i\omega/\nu}\,r)}{J_0(\sqrt{-i\omega/\nu}\,a)}\right), \tag{27}$$

which coincides with the exact solution for a circular duct [2].

If, however, one considers the flow to be steady, then the governing equation (11) for steady flow yields

$$\frac{d^2w}{df^2} + \frac{1}{f}\frac{dw}{df} = \frac{K_1^2}{\mu}\frac{dp}{dz}, \tag{28}$$

where

$$K_1^2 = 2a^2b^2/(a^2+b^2). \tag{29}$$

The solution to (28) with associated boundary conditions can be written as

$$w = -\frac{\frac{dp}{dz}a^2b^2}{2\mu(a^2+b^2)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{30}$$

which again is the exact solution of the problem [1].

Let us now examine the unsteady situation for very small and very large frequencies.

In the case of very small frequency oscillations, the parameter K will be small, and to a first order approximation, one obtains

$$berK \simeq 1, \qquad beiK \simeq K^2/4. \tag{31}$$

Thus, remembering that $0 \leq f \leq 1$, expressions for W_1 and W_2 given in (24) and (25) reduce to

$$W_1 = -\frac{4K^2}{\mu\lambda^2}\frac{dP}{dz}\frac{1-f^2}{k^4+16}, \tag{32}$$

$$W_2 = \frac{1}{\mu\lambda^2}\left(1-\frac{1+\frac{K^4f^2}{16}}{1+\frac{K^4}{16}}\right), \tag{33}$$

which, for $K^4 \ll 16$, ultimately take the form,

$$W_1 = -\frac{a^2b^2}{2\mu(a^2+b^2)}\frac{dP}{dz}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right), \tag{34}$$

$$W_2 \simeq 0. \tag{35}$$

Hence, considering the real part in expression (2) the velocity w becomes

$$w = -\frac{a^2b^2}{2\mu(a^2+b^2)}\frac{dP}{dz}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\cos\omega\tau, \tag{36}$$

indicating that when the frequency of the pressure gradient is small the flow is laminar under a steady pressure gradient which varies slowly with time. This result is also shown by Khamrui [4].

In the case when the frequency of oscillations ω is very large, and hence K is also very large, one can use the following asymptotic forms of ber and bei functions

$$\begin{gathered} berK = (2\pi K)^{-\frac{1}{2}}e^{\alpha}\cos\beta, \quad beiK = (2\pi K)^{-\frac{1}{2}}e^{\alpha}\sin\beta, \\ \text{and} \quad ber^2K + bei^2K = \frac{e^{2\alpha}}{2\pi K} \end{gathered} \tag{37}$$

where

$$\alpha = \frac{K}{\sqrt{2}}, \ \beta = \frac{K}{\sqrt{2}} - \frac{\pi}{8}. \tag{38}$$

Therefore,

$$\frac{berK}{ber^2K + bei^2K} = \frac{\sqrt{2\pi K}}{e^{K/\sqrt{2}}}\cos\beta \tag{39}$$

and

$$\frac{beiK}{ber^2K + bei^2K} = \frac{\sqrt{2\pi K}}{e^{K/\sqrt{2}}}\sin\beta\,. \tag{40}$$

Substituting these asymptotic forms in (24) and (25), one obtains

$$W_1 = -\frac{1}{\mu\lambda^2}\frac{dP}{dz}\frac{1}{\sqrt{f}}e^{-\gamma}\sin\gamma\,, \tag{41}$$

$$\text{and } W_2 = \frac{1}{\mu\lambda^2}\frac{dP}{dz}\left(1 - \frac{1}{\sqrt{f}}e^{-\gamma}\cos\gamma\right), \tag{42}$$

where

$$\gamma = (1-f)K/\sqrt{2}\,. \tag{43}$$

Thus,

$$W = W_1 + iW_2 = \frac{i}{\mu\lambda^2}\frac{dP}{dz}\left[1 - \frac{1}{\sqrt{f}}e^{-(1+i)\gamma}\right] \tag{44}$$

which for large ω (suitably scaled) can be expressed as

$$W \sim \frac{i}{\omega\rho}\frac{dP}{dz}\left[1 - \exp\left(1 - \sqrt{\frac{i\omega\rho}{\mu}}n\right)\right], \tag{45}$$

where n denotes a suitably chosen geometrical parameter.

The fluid velocity is given by

$$\begin{aligned} w &= \mathrm{Re}(We^{i\omega\tau}) \\ &= -\frac{1}{\mu\lambda^2}\frac{dP}{dz}\left[\sin\omega\tau - \frac{1}{\sqrt{f}}e^{-\gamma}\sin(\omega\tau-\gamma)\right] \end{aligned} \tag{46}$$

which agrees with Sexl's [15] results for a circle if we put $f = r/a$. Consequently, when the frequency of oscillation is large, one obtains

$$W_1 \simeq 0\,, \tag{47}$$

$$W_2 = \frac{1}{\mu\lambda^2}\frac{dP}{dz} \tag{48}$$

and hence the velocity w is given by

$$w = -\frac{1}{\mu\lambda^2}\frac{dP}{dz}\sin\omega\tau\,, \tag{49}$$

indicating that there is a phase lag of 90° behind the pressure gradient.

Let us now examine the velocity components W_1 and W_2 given in (24) and (25). Using the calculated values of *ber* and *bei* functions, one can obtain the ratios W_1/W^* and W_2/W^* where W^* is the centreline velocity which, for steady flow, becomes the maximum velocity. Since in many instances one is concerned with an oscillation of definite magnitude rather than with the forces which produce this oscillation, the variations of W_1/W^* and W_2/W^* are shown in Figure 3(a), (b) for periods of oscillation given by various values of the parameter $\eta = b\lambda$ and for $a/b = 2$. The parameter η is usually referred to as Stokes number. In Figure 4, the velocity profiles for oscillatory flows in an elliptical duct of aspect ratio 2: 1 have been shown for a set of values of η^2. As η^2 increases from the steady case $(\eta^2 = 0)$, the velocity profile changes markedly from the parabolic profile.

Also shown in Figure 5 are velocity profiles for elliptical ducts of different aspect ratio corresponding to $\eta = 3$. Centre-plane oscillatory flows in an elliptical duct of aspect ratio 2: 1 have also been computed for $\eta^2 = 0$, 1 and 2. The numerical values of the velocity profiles (real and imaginary parts) are calculated and displayed in Table 1. The results show excellent agreement with that of O'Brien [14] calculated on the same dimensional form from her Figure 6.

From the foregoing analysis it can be concluded that the practical importance of the above method stems not so much from the fact that it constitutes an experimental verification for the determination of the lines of constant velocity for oscillatory flow, but from the fact that it furnishes the basis for an intuitive, qualitative discussion of isovelocity contour lines in cases where the exact determination is difficult.

(b) Flow in a straight duct where cross-section is an equilateral triangle

As a second example, consider the case of a flow in a duct with cross-section in the form of an equilateral triangle as shown in Figure 6. If the flow is steady, then the equation of isovelocity contours can be obtained, as mentioned earlier, using well-known analogies of laminar flow in conduits in the form

$$u(x, y) = 4a^3 - 3a(x^2 + y^2) + 3xy^2 - x^3 \tag{50}$$

which, in fact, is the equation of the boundary of the plane section. Consider the above equation to be approximately true for the oscillatory flow. Clearly $u = 0$ on the boundary of the duct and $u = u^* = 4a^3$ at the centroid of the section.

The value of the contour integrals appearing in (9) are obtained with the

(a)

(b)

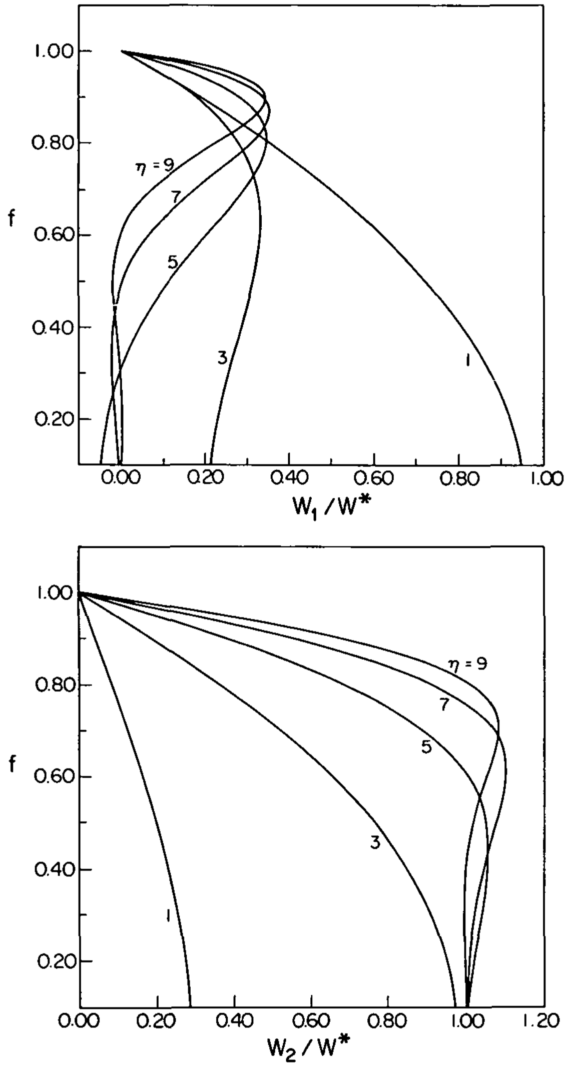


FIGURE 3. Velocity profiles in an elliptical duct for different values of η:
(a) Normalised real part W_1/W^*.
(b) Normalised imaginary part W_2/W^*.

help of Green's Theorem as follows:

$$\oint_{C_u} \sqrt{t}\, ds = -\int_{\Omega_u}\int \nabla^2 u \, d\Omega = 12aA(u) = 12aA_0\left(1 - \frac{u}{u^*}\right), \tag{51}$$

$$\oint_{C_u} \frac{ds}{\sqrt{t}} = -\frac{1}{12a}\frac{d}{du}\oint_{C_u} \sqrt{t}\, dS = \frac{A_0}{u^*}, \tag{52}$$

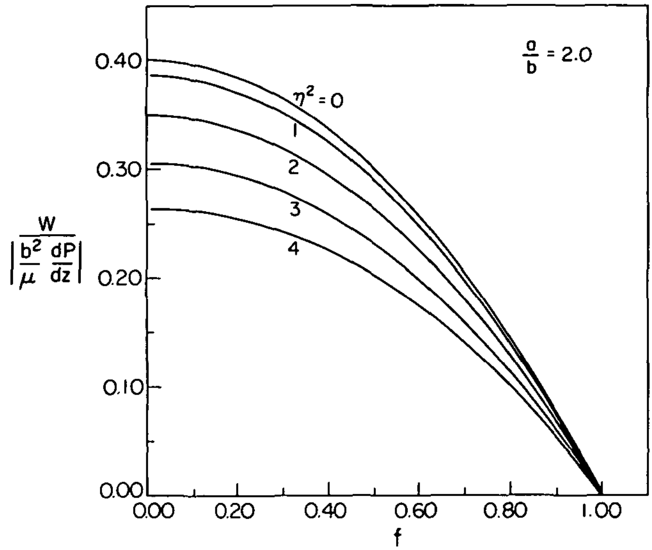


FIGURE 4. Velocity profiles in an elliptical duct of aspect ratio 2:1.

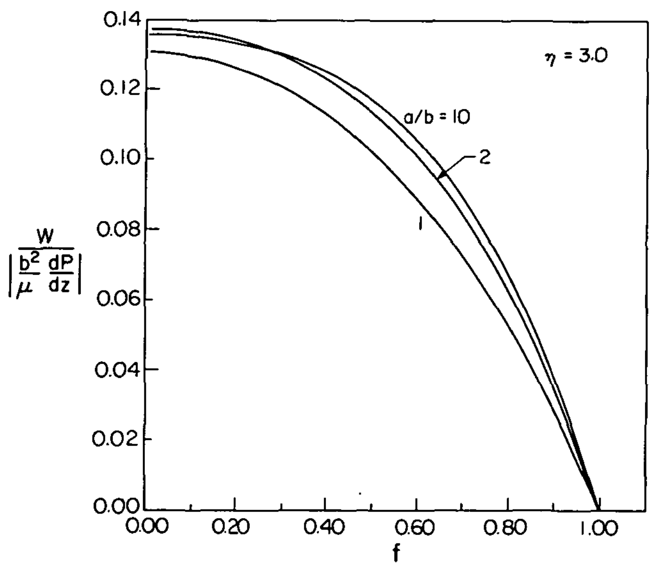


FIGURE 5. Velocity profiles in an elliptical duct for different aspect ratio.

TABLE 1. Centre plane velocity profiles for oscillatory flow in an elliptical duct of aspect ratio $a/b = 2/1$

	x		0.00	0.25	0.50	0.75
$\eta^2 = 0$	$\widehat{W}_1$	A	0.4	0.375	0.3	0.175
		B	0.4	0.375	0.3	0.175
	$\widehat{W}_2$	A	0.0	0.0	0.0	0.0
	$\widehat{W}_2$	B	0.0	0.0	0.0	0.0
$\eta^2 = 1$	$\widehat{W}_1$	A	0.397	0.373	0.299	0.174
		B	0.399	0.374	0.300	0.175
	$\widehat{W}_2$	A	0.030	0.027	0.021	0.010
		B	0.029	0.025	0.018	0.010
$\eta^2 = 2$	$\widehat{W}_1$	A	0.392	0.367	0.294	0.172
		B	0.394	0.365	0.298	0.175
	$\widehat{W}_2$	A	0.059	0.054	0.040	0.021
		B	0.059	0.053	0.041	0.021

$$\widehat{W}_1 = |W_1| / \left| \frac{b^2}{\mu} \frac{dp}{dz} \right|, \qquad \widehat{W}_2 = |W_2| / \left| \frac{b^2}{\mu} \frac{dp}{dz} \right|$$

A = Present results

B = O'Brien [14]

where $A(u)$ is the area enclosed by the region Ω_u and A_0 is the total area of the triangular section. With the values of these integrals introduced (9) finally reduces to

$$(u^* - u)\frac{d^2W}{du^2} - \frac{dW}{du} - \frac{i\lambda^2}{12a}W = \frac{1}{12a\mu}\frac{dP}{dz} \tag{53}$$

which in terms of a new independent variable f given by

$$u^* - u = f^2 \tag{54}$$

assumes the form

$$\frac{d^2W}{df^2} + \frac{1}{f}\frac{dW}{df} - iK^2W = \frac{K^2}{\mu\lambda^2}\frac{dP}{dz} \tag{55}$$

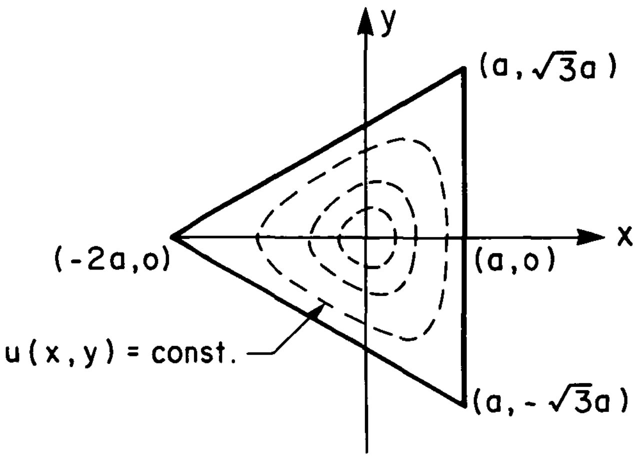


FIGURE 6. Equilateral triangular duct geometry.

where

$$K^2 = \lambda^2/3a. \tag{56}$$

It is interesting to note that (55) is identical in form to (17). Hence the method of solution used in the previous illustration can be used, giving

$$W = \frac{i}{\mu\lambda^2}\frac{dP}{dz}\left(1 - \frac{I_0(\sqrt{i}Kf)}{I_0(\sqrt{iu^*}K)}\right). \tag{57}$$

If, however, one considers the case of steady flow, then the governing equation (11) yields

$$\frac{d^2w}{df^2} + \frac{1}{f}\frac{dw}{df} = \frac{K_1^2}{\mu}\frac{dp}{dz} \tag{58}$$

where

$$K_1^2 = 1/(3a). \tag{59}$$

The solution to (58) is given by

$$\begin{aligned} w &= -\frac{K_1^2}{4\mu}\frac{dp}{dz}u \\ &= -\frac{\frac{dp}{dz}}{12a\mu}(4a^3 - 3a(x^2+y^2) + 3xy^2 - x^3) \end{aligned} \tag{60}$$

which in fact is the exact solution for steady flow [1].

Now returning to the unsteady case, it can be shown, as in the previous example, that

$$W_1 = \frac{1}{\mu\lambda^2}\frac{dP}{dz}\left(\frac{berK\sqrt{u^*}beiKf - beiK\sqrt{u^*}berKf}{ber^2K\sqrt{u^*} + bei^2K\sqrt{u^*}}\right),$$

$$W_2 = \frac{1}{\mu\lambda^2}\frac{dP}{dz}\left(1 - \frac{berK\sqrt{u^*}beiKf + beiK\sqrt{u^*}berKf}{ber^2K\sqrt{u^*} + bei^2K\sqrt{u^*}}\right). \qquad (61)$$

In Figure 7(a), (b) velocity profiles (normalised real and imaginary parts) are shown for various values of the parameter η^2 where $\eta = a\lambda = a\sqrt{\omega/\nu}$. As η^2 increases from its zero value when the flow is fully steady, the deviations from parabolic profile become quite apparent. Also shown in Figure 8 are graphs for W/W^* against u/u^* where $W = \sqrt{W_1^2 + W_2^2}$ and $W^* = |\frac{a^2}{\mu}\frac{dP}{dz}|$. The parameter η^2 sometimes denoted by ω^* is called the kinetic Reynolds number and is a measure of viscous effects on oscillating flow. The flow may be turbulent when ω^* exceeds a certain value.

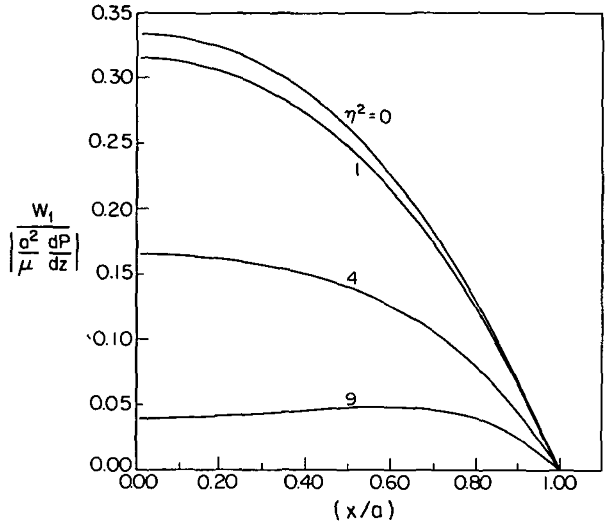
(c) Flow in regular polygonal ducts

It has been made clear in the previous two examples that the accuracy of the solution of oscillatory duct problems depends largely on knowledge of the form of the contour function $u(x, y)$, and attention was focussed on a possible approximation for one such function. However, it will be shown in this example that the oscillatory flow problems can be solved without a priori knowledge of the contour function for any simply connected section that can be mapped conformally onto a unit circle, for which isovelocity contours are concentric circles. Thus, the method of conformal transformations can be used to obtain an accurate approximation for the solution to the problem. It is the purpose of this illustration to explain the procedure involved and to show that the proposed method is quite general and can be used for any arbitrary cross-section.

Method

Consider the duct geometry as shown in Figure 2. It is required to obtain the conformal mapping function that maps the contour c onto the unit circle in the complex plane $\zeta = \xi + i\eta$, Figure 9. The existence of such a conformal transformation between the simply connected region R and the unit circle $|\zeta| = 1$ is guaranteed by the well-known Riemann mapping theorem. The mapping function can be expressed in a general form as $z = \psi(\zeta)$ or $\zeta = \phi(z)$. If the family of isovelocity contours in the ζ-plane is denoted by, say,

(a)



(b)

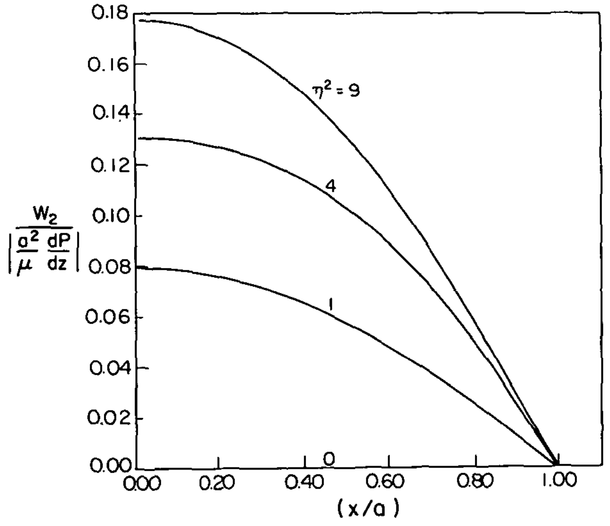


FIGURE 7. Velocity profiles in an equilateral triangular duct:
(a) Normalised real part.
(b) Normalised imaginary part.

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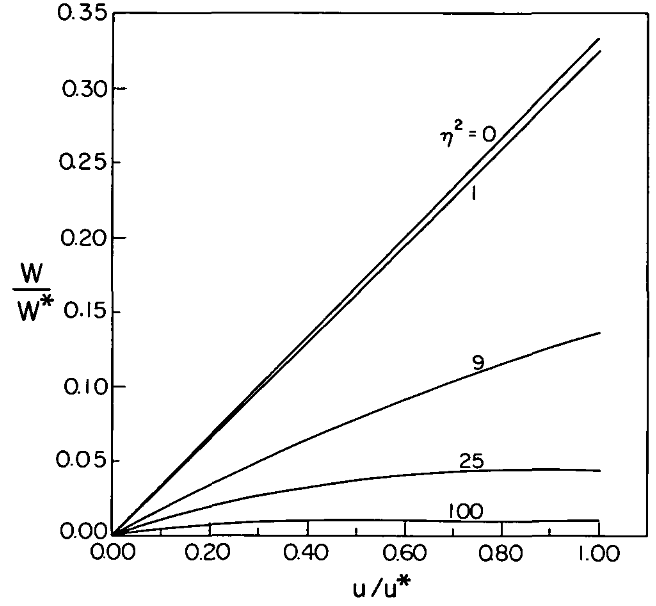

FIGURE 8. Velocity profiles in an equilateral triangular duct for a range of values of η.

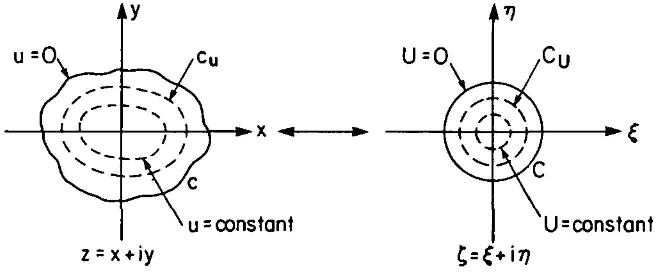

FIGURE 9. Conformal mapping geometry.

$U(\xi, \eta) = \text{constant}$, then one can write

$$U(\xi, \eta) = 1 - \xi^2 - \eta^2 \tag{62}$$

and clearly, $U = 0$ on the boundary C, and $U^* = 1$ at the centre of the unit circle. It is assumed here that there is a one-to-one correspondence between the contours $U(\xi, \eta) = \text{constant}$ in the ζ-plane and the contours $u(x, y) = \text{constant}$ in the z-plane. Calculating the values of the various integrals appearing in the integro-differential equation (9), in the transformed plane, one obtains

$$t = u_x^2 + u_y^2 = 4u_z u_{\overline{z}} = 4U_\zeta U_{\overline{\zeta}} \left| \frac{d\zeta}{dz} \right|^2 , \tag{63}$$

$$\nabla^2 u = 4u_{z\overline{z}} = 4U_{\zeta\overline{\zeta}} \left| \frac{d\zeta}{dz} \right|^2 , \tag{64}$$

and for any continuous function $f(x, y)$

$$\oint_{c_u} f(x, y)\, ds = \oint_{c_u} f(z, \overline{z})\, ds = \oint_{C_U} f(\zeta, \overline{\zeta}) \left| \frac{dz}{d\zeta} \right| dS , \tag{65}$$

where the Jacobian of the transformation $J = dz/d\zeta$ and dS is the elemental arc length of the appropriate circle $U =$ constant in the ζ-plane. Thus, the values of the integrals in (9) are

$$\oint_{c_u} \sqrt{t}\, ds = 2 \oint_{C_U} \sqrt{U_\zeta U_{\overline{\zeta}}}\, dS , \tag{66}$$

$$\oint_{c_u} \frac{\nabla^2 u}{\sqrt{t}}\, ds = 2 \oint_{C_U} \frac{U_{\zeta\overline{\zeta}}}{\sqrt{U_\zeta U_{\overline{\zeta}}}}\, dS , \tag{67}$$

$$\oint_{c_u} \frac{ds}{\sqrt{t}} = \oint_{C_U} \frac{1}{2\sqrt{U_\zeta U_{\overline{\zeta}}}} \left| \frac{dz}{d\zeta} \right|^2 dS. \tag{68}$$

The above integrals can further be simplified using the facts that

$$4U_\zeta U_{\overline{\zeta}} = U_\xi^2 + U_\eta^2 = 4(1 - U), \tag{69}$$

$$4U_{\zeta\overline{\zeta}} = U_{\xi\xi} + U_{\eta\eta} = -4, \tag{70}$$

$$\oint_{C_U} dS = 2\pi\sqrt{1 - U} \tag{71}$$

yielding

$$\oint_{c_u} \sqrt{t}\, ds = 4\pi(1 - U), \tag{72}$$

$$\oint_{c_u} \frac{\nabla^2 u}{\sqrt{t}}\, ds = -4\pi , \tag{73}$$

$$\oint_{c_u} \frac{ds}{\sqrt{t}} = \frac{\perp}{2\sqrt{1 - U}} \oint_{C_U} \left| \frac{dz}{d\zeta} \right|^2 dS . \tag{74}$$

The contour integral appearing above in (74) can be evaluated once the mapping function $z = \psi(\zeta)$ is known. The governing equation (9) can now be written in the transformed plane as

$$(1-U)\frac{d^2W}{dU^2} - \frac{dW}{dU} - i\lambda^2 H(U)W = \frac{1}{\mu}\frac{dP}{dz}H(U) \tag{75}$$

where

$$H(U) = \frac{\oint_{C_U} \left|\frac{dz}{d\zeta}\right|^2 dS}{8\pi\sqrt{1-U}}. \tag{76}$$

The differential equation (75) is now solved by the method of collocation assuming a power series of the form

$$W(U) = \sum_{n=1}^{N} a_n U^n. \tag{77}$$

Substituting this expression into (75) and requiring that it holds for N different values of U, one finally arrives at the following system of algebraic equations

$$\sum_{n=1}^{N}\left[n(n-1)(1-U_j)U_j^{n-2} - nU_j^{n-1}\right]a_n - i\lambda^2\sum_{n=1}^{N} H(U_j)U_j^n a_n = \frac{1}{\mu}\frac{dP}{dz}H(U_j) \tag{78}$$

for $j = 1, 2, \dots, N$ and $0 < U_j < 1$.

Thus, once the H-function is known, the above system of equations can be solved numerically and hence values of real and imaginary parts of velocity components can be obtained.

If, however, one considers the steady case, then the governing equation (11) for steady flow can be written in the transformed plane as

$$(1-U)\frac{d^2W}{dU^2} - \frac{dW}{dU} = \frac{1}{\mu}\frac{dp}{dz}H(U) \tag{79}$$

which again can be solved easily by the method outlined above. For a detailed discussion of the method, see Mazumdar and Hill [12].

As an illustration of the proposed method, consider first the case of a flow through a polygonal cross-section as shown in Figure 10. The mapping function that transforms a regular polygonal region onto the unit circle is given by the well-known Schwartz-Christoffel transformation [5].

$$z = \psi(\zeta) = a\Gamma_n \int_0^{\zeta} (1+t^n)^{-\frac{2}{n}}\, dt, \tag{80}$$

where n is the number of sides, a is the apothem of the regular polygon, and Γ_n is the mapping coefficient given by

$$\Gamma_n = 1\Big/\int_0^1 (1+t^n)^{-\frac{2}{n}}\, dt. \tag{81}$$

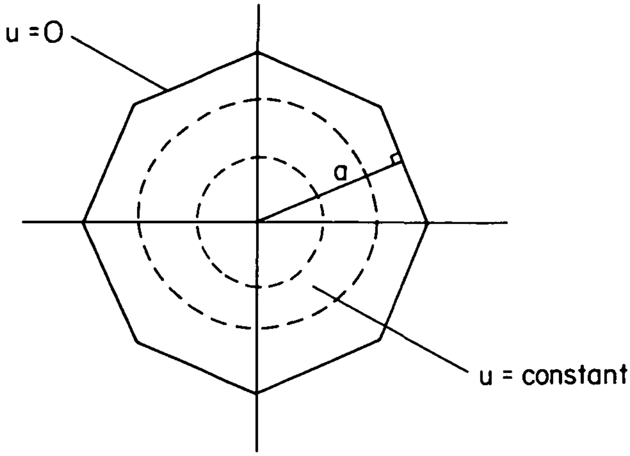

FIGURE 10. Regular polygonal duct.

The numerical values of the mapping coefficients for some regular polygons are given in Table 2.

TABLE 2. Numerical Values for Regular Polygonal Transformations.

Shape Parameter	Mapping Coefficient Γ_n
3 (triangle)	1.135
4 (square)	1.079
5 (pentagon)	1.052
6 (hexagon)	1.038
7 (heptagon)	1.028
8 (octagon)	1.022

Also shown in Figures 11 and 12 are the variations of real and imaginary velocity components for different values of λa for $n = 3$ and 4, compared with known results. It is interesting to note that for $n = 3$ i.e., for an equilateral triangular duct the graphs are exactly the same as those obtained previously in Figures 7(a) and 7(b). Furthermore, the numerical values for centre-plane velocity profile for a square duct $(n = 4)$ given in Table 3 agree very well with those of O'Brien [14].

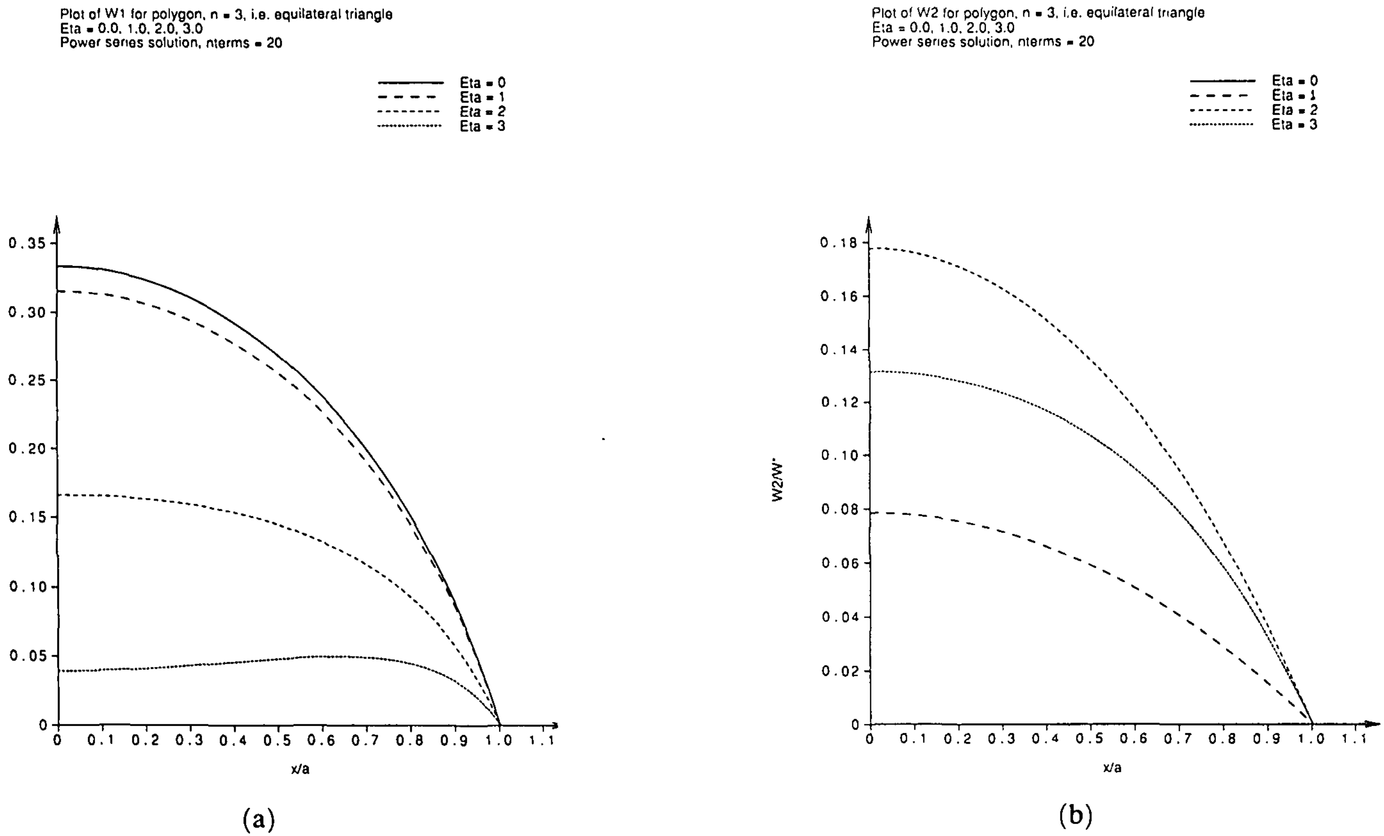

FIGURE 11. Velocity profiles for polygonal duct for $n = 3$ (equilateral triangle):
(a) Normalised real part.
(b) Normalised imaginary part.

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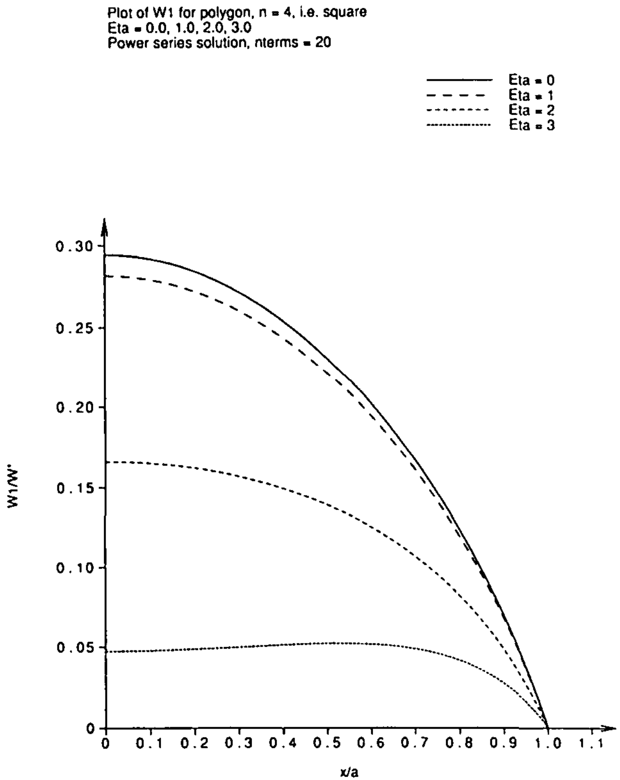
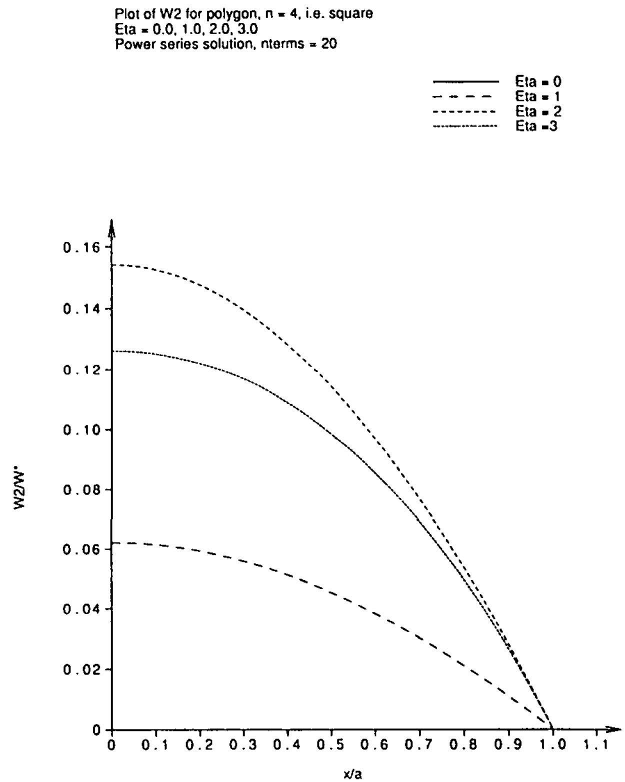


(a)



(b)

FIGURE 12. Velocity profiles for polygonal duct for $n = 4$ (square):
(a) Normalised real part.
(b) Normalised imaginary part.

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TABLE 3. Centre plane velocity profiles for oscillatory flow in a square duct: $\eta = 1$. The figures in brackets are due to O'Brien [14].

x/a	Real part		Imaginary part	
0	0.2815	(0.2817)	−0.0621	(−0.0623)
0.125	0.2778	(0.2780)	−0.0610	(−0.0612)
0.25	0.2668	(0.2669)	−0.0577	(−0.0578)
0.375	0.2482	(0.2478)	−0.0525	(−0.0525)
0.5	0.2208	(0.2201)	−0.0452	(−0.0451)
0.625	0.1865	(0.1828)	−0.0359	(−0.0358)
0.75	0.1400	(0.1347)	−0.0250	(−0.0259)
0.875	0.0810	(0.0743)	−0.0132	(−0.0128)
1.0	0	(0)	0	(0)

(d) Flow in cardioidal ducts

As a final example of fully developed laminar flow cases, let us consider the flow in a pipe whose cross-section is that of a cardioid, Figure 13. The

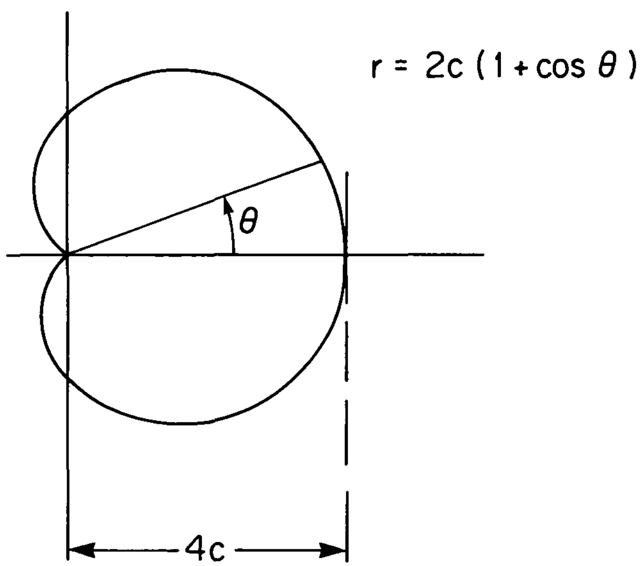

FIGURE 13. Cardioidal duct.

mapping function in this case is well known [6] and is given by

$$z = \frac{a}{1+m}(\zeta + m\zeta^2), \tag{82}$$

where the parameter a is the radius of the circumscribing circle and the parameter m may have values ranging from 0 to 0.5. For $m = 0$, one gets the case of a circle duct whereas for $m = 0.5$, the case of a duct in the actual cardioidal shape is obtained. Velocity profiles for $m = 0$ and $m = 0.5$ are shown in Figures 14 and 15.

4. Conclusions

In the preceding analysis, a new approximate method of solution for fully developed fluid flow in a duct of arbitrary shape has been developed. Both steady and unsteady flows have been considered. It has been demonstrated that the proposed method produces results which are sufficiently accurate for engineering purposes.

It is not claimed here that the contour function method will replace powerful numerical techniques like finite element and finite difference methods, since these techniques are becoming much more sophisticated and capable, especially with the ever increasing power of computers. However, the proposed method is a relatively simple, quasi-analytical approach which requires very little computer time and computer memory. These features make the method superior to available numerical methods where the solution involves all interior points. In addition, the results produced by its use may be utilised as useful checks on those solutions produced by finite element and other more intricate numerical methods, although there are not many measurements of velocity in fully developed unsteady flow available in the literature. Finally, because of the assumption involved in selecting the contour function in the first two examples, the method yields exact solutions for steady flow problems and highly accurate approximate solutions in oscillatory flow problems.

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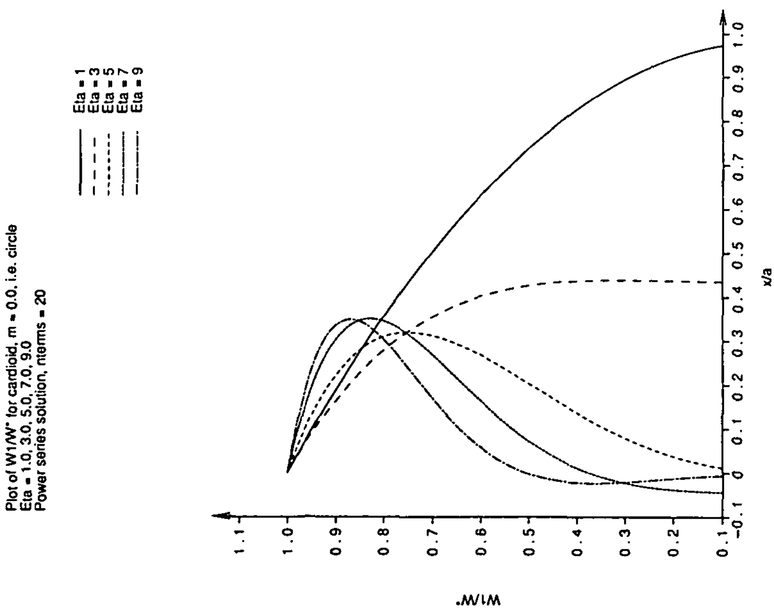
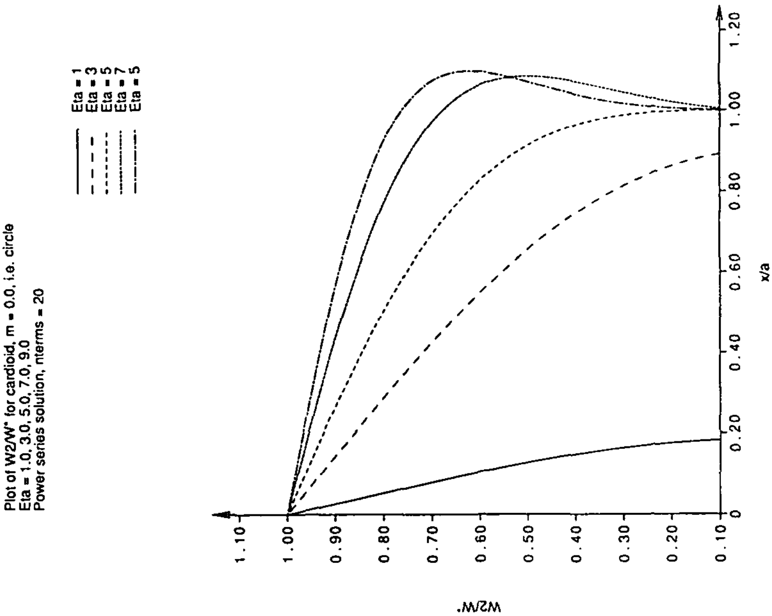
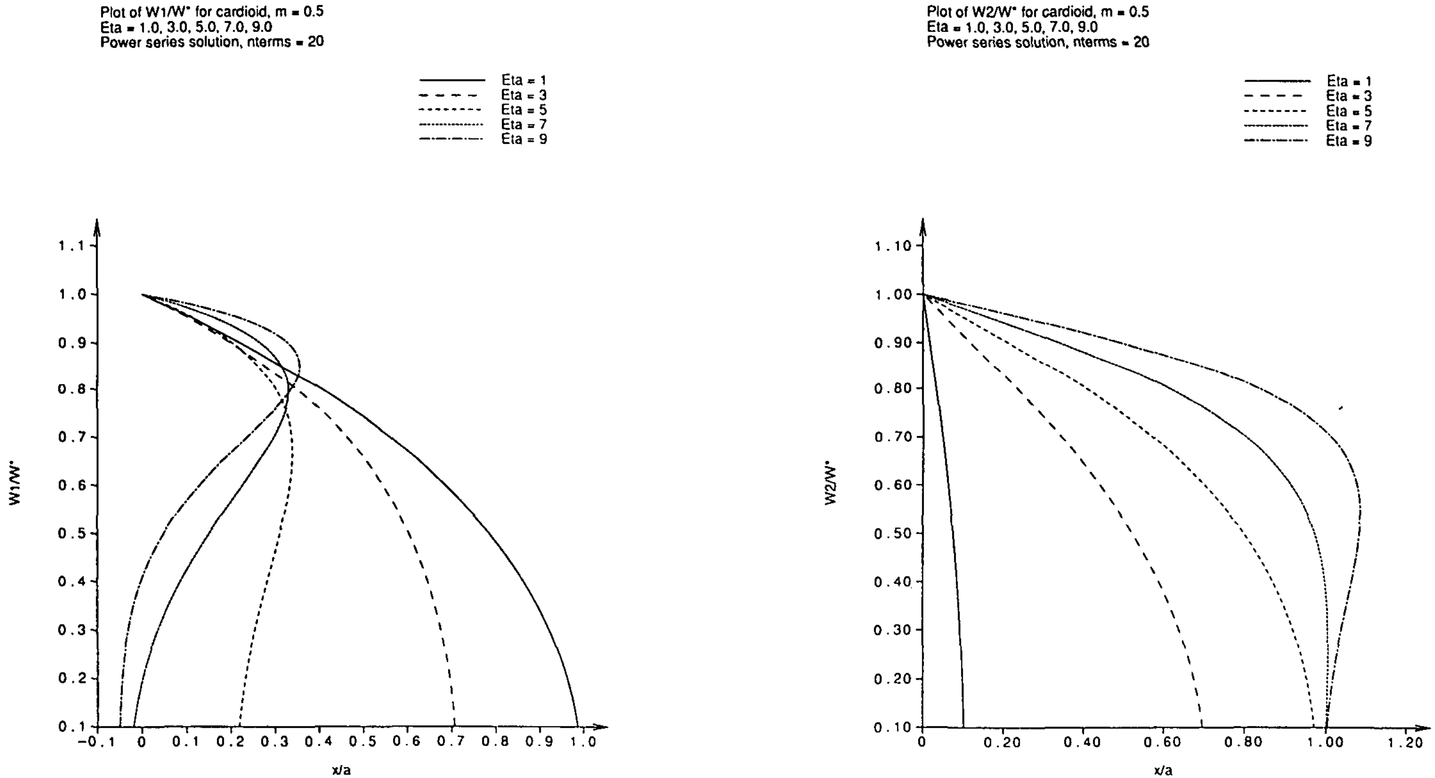


(a)



(b)

FIGURE 14. Velocity profiles in a cardioidal duct for $m = 0$ (circle):
(a) Normalised real part.
(b) Normalised imaginary part.



(a) (b)

FIGURE 15. Velocity profiles in a cardioidal duct for $m = 0.5$:
(a) Normalised real part.
(b) Normalised imaginary part.

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Visiting Research Professor in the Department of Mechanical Engineering, University of Waterloo, where part of this work was undertaken.

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