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The Metric Dimension of the Total Graph of a Finite Commutative Ring

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Abstract. We study the total graph of a finite commutative ring. We calculate its metric dimension in the case when the Jacobson radical of the ring is nontrivial, and we examine the metric dimension of the total graph of a product of at most two fields, obtaining either exact values in some cases or bounds in other, depending on the number of elements in the respective fields.

1 Introduction

In [5], Anderson and Badawi introduced the notion of a total graph of a commutative ring R as the graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y$ is a zero-divisor in R. They studied some graph theoretical parameters of this graph such as diameter and girth. In addition, they studied some special subgraphs of the total graph and the properties of the total graph based on these subgraphs. They also proved that the total graph of a commutative ring is connected if and only if the set of zero-divisors does not form an ideal. In [1] Akbari et al. proved that if the total graph of a finite commutative ring is connected, then it is also a Hamiltonian graph. In [14], Maimani et al. gave the necessary and sufficient conditions for the total graphs of finite commutative rings to be planar or toroidal, and in [18] Tamizh Chelvam and Asir characterized all commutative rings such that their total graphs have genus 2. In [16], Shekarriz et al. studied the total graph of a finite commutative ring and calculated the domination number of such a ring and also found the necessary and sufficient conditions for the graph to be Eulerian. In [8] the authors studied the total graph of a finite non-commutative ring.

The sequence of edges

$$x_0 — x_1, \quad x_1 — x_2, \quad \dots \quad x_{k-1} — x_k$$

in a graph is called *a path of length k*. The *distance* between vertices x and y is the length of the shortest path between them, denoted by $d(x, y)$. The *diameter* $\operatorname{diam}\Gamma$ of the graph Γ is the longest distance between any two vertices of the graph.

A complete graph on m vertices will be denoted by K_m, and a complete bipartite graph with the respective sets of sizes m and n will be denoted by $K_{m,n}$.

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For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of the vertex set of graph G and a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$$

is called the *representation* of v with respect to W. A set W is called a *resolving set* for G if distinct vertices of G have distinct representations with respect to W. A resolving set of minimal cardinality for G is a *basis* of G, and the cardinality of the basis is called the *metric dimension* of G, denoted by $\beta(G)$ [7]. Motivated by the problem of uniquely determining the location of an intruder in a network, Slater [17] introduced the concept of a metric dimension. The metric dimension was then studied by Harary and Melter [10], and it has appeared in various applications of graph theory, for example pharmaceutical chemistry [6,7], robot navigation [12], combinatorial optimization [15], and sonar and coast guard long range navigation [17]. It turns out that determining the metric dimension of a graph remains a NP-complete problem even in special cases like bounded-degree planar graphs [2], or split graphs, bipartite graphs and their complements, line graphs of bipartite graphs [3].

Recently, Faisal et al. [4] studied the metric dimension of the commuting graph of a dihedral group.

Let us recall some basic definitions.

For any commutative ring R, we denote by $Z(R)$ the set of zero-divisors, $Z(R) = \{x \in R;$ there exists $0 \neq y \in R$ such that $xy = 0\}$. In accordance with [5], we define the total graph of a ring R as follows.

Definition The total graph $\tau(R)$ of a commutative ring R is the graph, where

- the set of vertices $V(\tau(R))$ of the graph $\tau(R)$ is the set of all elements in R and
- two distinct vertices x and y are adjacent if and only if $x + y$ is a (left or right) zero-divisor in R.

We limit our study to the total graphs of finite rings because of the following lemma.

Lemma 1.1 ([9,13]) *If R is a ring with m zero divisors, $2 \leq m < \infty$, then R is a finite ring with $|R| \leq m^2$.*

So, if an infinite ring R has more than one zero divisor, then it has infinitely many of them, and so the degree of each vertex in the total graph is infinite, which means that is it difficult (or perhaps even meaningless) to study the graph theoretical properties of the total graph. On the other hand, if an infinite ring R has only one zero divisor, each $a \in R$ is either an isolated vertex or adjacent only to $-a$, so the total graph is a disjoint union of infinitely many graphs isomorphic to K_1 or K_2.

In this paper, we study the metric dimension of the total graph of a finite commutative ring. In the ring setting, the total graph has a special significance, since among all the graphs that are commonly assigned to a ring the total graph takes both ring operations into account. In the next section, we calculate the metric dimension of the total graph of a ring in case the Jacobson radical of the ring is nontrivial and in the last section we turn our attention to studying the metric dimension of the total graph in the remaining case of a finite ring with a trivial Jacobson radical, which implies that the ring is a direct product of fields. We calculate the metric dimension of a total graph of a field. In the case of the product of two fields, we find an exact value for the metric dimension of its total graph when one field is at least twice the size of the other one, and find a bound for the metric dimension in all other cases, whereby we also conjecture about the exact value.

2 The Metric Dimension of the Total Graph of a Finite Ring

We shall make use of the following two lemmas.

Lemma 2.1 *If R is a finite ring, then every $a \in R$ is either invertible or zero-divisor.*

Proof Choose any $a \in R$. Since R is finite, there exist integers $k, l > 0$ such that $a^k = a^{k+l}$. Choose the smallest such k. Then $a^k(a^l - 1) = 0$, and either $a^l = 1$ (so a is invertible) or $aa^{k-1}(a^l - 1) = 0$ (so a is a zero divisor). ∎

Lemma 2.2 ([5]) *If R is a finite ring, then*

$$\operatorname{diam} \tau(R) = \begin{cases} \infty, & \textit{if } R \textit{ is local,} \\ 2, & \textit{if } R \textit{ is not local.} \end{cases}$$

In this section we investigate the metric dimension of the total graph of a non-semisimple finite ring. We start with the following lemma. Here, $J(R)$ denotes the Jacobson radical of R.

Lemma 2.3 *For every finite ring R we have $a + b \in Z(R)$ if and only if*

$$(a + J(R)) + (b + J(R)) \in Z(R/J(R)).$$

Proof Let J denote $J(R)$. If $a + b \in Z(R)$, there exists a nonzero $c \in R$ such that $(a + b)c = 0$, and thus $((a + J) + (b + J))(c + J) = J$. So, either $(a + J) + (b + J)$ is a zero-divisor in R/J or it is invertible by Lemma 2.1. If $(a+J)+(b+J)$ is invertible, there exists $u \in R$ such that $(u+J)((a+J)+(b+J)) = 1+J$, or, equivalently, $u(a+b) \in 1+J$. Since R is finite, J is nilpotent, and thus all elements in $1+J$ are invertible, so $u(a+b)$ is an invertible element in R, which contradicts the assumption $a + b \in Z(R)$. Thus, $(a + J) + (b + J) \in Z(R/J)$.

If $(a+J)+(b+J) \in Z(R/J)$, there exists $c \in R \setminus J$ such that $((a+J)+(b+J))(c+J) = J$, and thus $(a+b)c \in J$. If $a+b$ is invertible in R, there exists u such that $u(a+b) = 1$, and therefore $c = u(a + b)c \in J$, a contradiction. Thus, $a + b$ is not invertible, so $a + b \in Z(R)$. ∎

Next, we need the following definitions.

Definition Let v be a vertex of a graph G. Then the *open neighbourhood* of v is $N(v) = \{u \in V(G);$ there exists an edge uv in G$\}$ and the *closed neighbourhood* of v us $N[v] = N(v) \cup \{v\}$. Two distinct vertices u and v of G are *twins* if $N(u) = N(v)$ or $N[u] = N[v]$.

The following lemma can be found in [11].

Lemma 2.4 ([11]) *Suppose u and v are twins in a connected graph G, and the set W is a resolving set for G. Then u or v is in W.*

We will need the following lemma on metric dimensions of some special graphs. Here, the graph $G_1 \cup G_2$ denotes the disjoint union of graphs G_1 and G_2.

Lemma 2.5 (i) *If $|G_1|, |G_2| \geq 2$, then $\beta(G_1 \cup G_2) = \beta(G_1) + \beta(G_2)$;*
(ii) *$\beta(K_n) = n - 1$;*
(iii) *For $m + n \geq 3$, $\beta(K_{m,n}) = m + n - 2$.*

Proof The first statement is obvious; the second and third follow from [7, Theorems 3 and 4]. ∎

We can now examine the metric dimension of the total graph of a ring with respect to the total graph of the ring modulo its Jacobson radical. The following theorem now describes the metric dimension of the total graph of a finite non-semisimple ring.

Theorem 2.6 *Let R be a finite commutative ring with its Jacobson radical J. If $J \neq 0$, then the metric dimension of $\tau(R)$ equals $\beta(\tau(R)) = |J - 1||R/J|$.*

Proof Let us first assume that R is not local and suppose that W is a resolving set for $\tau(R)$. Let $a - b \in J$ and $c \in R$. Then by Lemma 2.3, $a + c \in Z(R)$ if and only if $a + c + J$ is a zero divisor in R/J, and $b + c \in Z(R)$ if and only if $b + c + J$ is a zero divisor in R/J. Since $a + c + J = b + c + J$, this implies that $N(a) = N(b)$. R is not local, so the total graph of R is connected by Lemma 2.2, and by Lemma 2.4 we get that all except perhaps one element of every coset from R/J is in W. This now yields $\beta(\tau(R)) \geq |J - 1||R/J|$.

Now, suppose $X = \{a_1, \dots, a_{|R/J|}\}$ is a complete set of representatives of the cosets from R/J and let $W = R \setminus X$. Obviously, $|W| = |J - 1||R/J|$. To prove that W is a resolving set for $\tau(R)$ we only have to check that no two elements from X have the same representations with respect to W. Suppose the contrary, so that for some $i \neq j$ we have $d(a_i, w) = d(a_j, w)$ for all $w \in W$. Since $J \neq 0$, W contains a representative of every coset from R/J. If $w \in N(a_i)$, then $w + J \subseteq N(a_i)$ by Lemma 2.3, and this implies that either $N[a_i] = N[a_j]$ (if a_i and a_j are connected in $\tau(R)$) or $N(a_i) = N(a_j)$ (if they are not connected). However, $a_i + (Z(R) - a_i) \subseteq Z(R)$. If $N[a_i] = N[a_j]$, then $a_j + (Z(R) - a_i) \setminus \{a_j\} \subseteq Z(R)$, and if $N(a_i) = N(a_j)$, then $a_j + (Z(R) - a_i) \setminus \{a_i\} \subseteq Z(R)$. In both cases $a_j - a_i$ is connected in $\tau(R)$ with all except perhaps one element of $Z(R)$. We can decompose R as a direct product of (finite commutative) local rings, $R = R_1 \times R_2 \times \cdots \times R_k$, with R_i local for each i and let $a_j - a_i = (x_1, x_2, \dots, x_k)$ for some $x_l \in R_l$, $l = 1, 2, \dots, k$. Since $a_j - a_i \notin J$, there exists an integer s such that $x_s \notin J(R_s)$. But R_s is a local ring, so x_s is invertible in R_s. Because of the assumption that $J \neq 0$, we also know that there exists some integer t such that $J(R_t) \neq 0$. Now, let $b = (1 - x_1, \dots, 1 - x_{s-1}, 0, 1 - x_{s+1}, \dots, 1 - x_k)$ and observe that $a_j - a_i + b + J(R_t) = (1, \dots, 1, x_s, 1, \dots, 1) + J(R_t)$ is an invertible element of R. Since $|J(R_t)| \geq 2$, at least two elements of $b + J(R_t)$ lie in $Z(R)$, so $a_j - a_i$ is not connected to at least two elements from $Z(R)$, which is a contradiction.

Now, assume that R is local. By [5, Theorems 2.1 and 2.2], we know that $\tau(R)$ is a disconnected graph isomorphic to $|R/J|$ copies of $K_{|J|}$ if $2 \in J$ and isomorphic to a copy $K_{|J|}$ and $\frac{|R/J|-1}{2}$ copies of $K_{|J|,|J|}$ if $2 \notin J$. The result now follows from Lemma 2.5. ■

Now, it is left for us to investigate the metric dimension of the total graph of a semisimple ring. Since any commutative Artinian (and hence also finite) ring can be written as a direct product of local rings, this means the rings we are left to investigate are the products of fields. This is the topic of the next section.

3 The Metric Dimension of the Total Graph of a Product of Fields

In this section, we study the metric dimension of a finite semisimple ring. Since any commutative finite ring is a direct product of local rings, we can limit our study to the case where the ring is a product of finite fields.

Let us first consider the case of one field.

Lemma 3.1 *If F is a field, then*

$$\beta(\tau(F)) = \begin{cases} |F| - 1, & \text{if } \operatorname{char}(F) = 2, \\ \dfrac{|F|+1}{2}, & \text{otherwise.} \end{cases}$$

Proof If $\operatorname{char}(F) = 2$, then $\tau(F)$ is equal to $|F|$ disconnected copies of K_1 and thus $\beta(\tau(F)) = |F| - 1$. If $\operatorname{char}(F) \neq 2$, then $\tau(F)$ is equal to a copy of K_1 and $\frac{|F|-1}{2}$ disconnected copies of $K_{1,1}$, and thus $\beta(\tau(F)) = 1 + \frac{|F|-1}{2} = \frac{|F|+1}{2}$. ■

Next, we move on to the case of the product of two fields. It is not entirely clear how the methods we use could be extended to the product of more than two fields, as the total graphs in those cases turn out to be a level of degree more complex.

In general, we can state the following proposition.

Proposition 3.2 *Let $R = F_1 \times F_2$, where F_1 and F_2 are finite fields. Then $\beta(\tau(R)) \leq |F_1| + |F_2| - 2$.*

Proof We have to examine the case $F_1 = F_2 = \mathbb{Z}_2$ separately. Observe that $W = \{(0,0),(1,0)\}$ is a resolving set for the total graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$, so in this case the statement holds.

Suppose now that at least one of the fields F_1, F_2 has at least 3 elements. Denote $F_1 \setminus \{0\} = \{x_1, x_2, \ldots, x_{|F_1|-1}\}$ and $F_2 \setminus \{0\} = \{y_1, y_2, \ldots, y_{|F_2|-1}\}$. Define the set $W = \{(x_1,0),(x_2,0),\ldots,(x_{|F_1|-1},0)\} \cup \{(0,y_1),(0,y_2),\ldots,(0,y_{|F_2|-1})\}$. Obviously, $|W| = |F_1| + |F_2| - 2$. We have to prove that W is a resolving set for $\tau(R)$. Suppose there exist $a = (a_1,a_2), b = (b_1,b_2) \in R \setminus W$ such that $d(a,w) = d(b,w)$ for all $w \in W$. Since R is not local, by Lemma 2.2 we have $d(x,y) \in \{1,2\}$ for all $x,y \in R$. Also, for $x = (x_1,x_2), x' = (x_1',x_2') \in R$, we have $d(x,x') = 1$ if and only if $x_1 = -x_1'$ or $x_2 = -x_2'$. If $a_1 = b_1 = 0$, then $a_2 = b_2 = 0$, since $a,b \notin W$. If $a_1 = b_1 \neq 0$,

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then $a_2, b_2 \neq 0$ and $1 = d(a,(0,-a_2)) = d(b,(0,-a_2))$ implying that $a_2 = b_2$. If $a_1 \neq b_1$ then $a_1 \neq 0$ or $b_1 \neq 0$. Suppose without loss of generality that $a_1 \neq 0$. Then $1 = d(a,(-a_1,0)) = d(b,(-a_1,0))$, so $b_2 = 0$, and thus $b = 0$ (since $b \notin W$). Now, $1 = d(b,w) = d(a,w)$ for all $w \in W$, thus $d(a,(0,y)) = 1$ for all $y \in F_2 \setminus \{0\}$, which yields $|F_2| = 2$. Since $a_2 \neq 0$, we also have $d(a,(x,0)) = 1$ for all $x \in F_1 \setminus \{0\}$, and thus $|F_1| = 2$, but we have already verified this case in the first part. ∎

In many cases, this bound can be lowered. We need the following technical lemmas.

Lemma 3.3 *Let $R = F_1 \times F_2$, where F_1 and F_2 are finite fields and let W be a resolving set for $\tau(R)$. If there exist $t \geq 2$ and $y_1, y_2, \dots y_t \in F_2$ such that for any $(x_1,x_2) \in W$ we have $x_2 \neq y_l$ for all $l = 1,2,\dots,t$ and for any $y \notin \{y_1, y_2, \dots, y_t\}$ there exists $x_1 \in F_1$ such that $(x_1, y) \in W$, then $|W| \geq |F_1|(t-1) + |F_2| - 2t$.*

Proof Choose $x_1 \in F_1$. For $l = 1,2,\dots,t$ define elements $r_l = (x_1, -y_l) \in R$ and observe that for $1 \leq i,j \leq t$ we have $d(r_i,w) = d(r_j,w)$ for all $w \in W$. Since W is a resolving set, this implies that either r_i or r_j lies in W for all $i \neq j$. The element x_1 was an arbitrary element from F_1, so W has to contain at least $|F_1|(t-1)$ elements with the second component equal to one of elements $-y_1, -y_2, \dots, -y_t$. The elements in W have at least $|F_2| - t$ different second components, so there are at least $|F_2| - 2t$ elements in W with the second component not equal to any of $\pm y_1, \pm y_2, \dots, \pm y_t$, and we thus get $|W| \geq |F_1|(t-1) + |F_2| - 2t$. ∎

Lemma 3.4 *Let $R = F_1 \times F_2$, where F_1, F_2 are finite fields, $|F_1| \leq |F_2|$, $f_1 \in F_1$, and $f_2 \in F_2$. Let $W \subseteq R$ be such a set that for each $x_2 \in F_2 \setminus \{f_2\}$ there exists a unique $z = (z_1,z_2) \in W$ such that $z_2 = x_2$ and for each $x_1 \in F_1 \setminus \{f_1\}$ there exist $s = (s_1,s_2) \neq r = (r_1,r_2) \in W$ such that $s_1 = r_1 = x_1$. Suppose further that $(f_1,r_2),(r_1,f_2) \notin W$ for all $r_1 \in F_1, r_2 \in F_2$. If for any $w \in W$ also $-w \in W$, then W is a resolving set for $\tau(R)$.*

Proof Suppose there exist $a = (a_1,a_2) \neq b = (b_1,b_2) \in R \setminus W$ such that $d(a,w) = d(b,w)$ for all $w \in W$.

Examine first the case when $a_1 \neq -f_1$. There exist $s = (s_1,s_2) \neq r = (r_1,r_2) \in W$ such that $s_1 = r_1 = -a_1$. Obviously, $d(a,r) = d(a,s) = 1$, therefore $d(b,r) = d(b,s) = 1$. Since $r_2 \neq s_2$, this yields $b_1 = a_1$. Therefore, $b_2 \neq a_2$, so $\{f_2\}$ is a proper subset of $\{-a_2,-b_2\}$. Suppose without loss of generality that $f_2 \neq -a_2$. Then there exists a unique $z = (z_1,z_2) \in W$ such that $z_2 = -a_2$. Now, $1 = d(a,z) = d(b,z)$, so $z_1 = -b_1 = -a_1$. However, this implies that $a = -(-a_1,-a_2) = -z \in W$, since $z \in W$, but this is a contradiction.

It remains to check the case $a_1 = -f_1$. If $a_2 \neq -f_2$, then $1 = d(a,w)$ for a unique element $w = (w_1,-a_2) \in W$. If $b_2 = a_2$ (and $b_1 \neq a_1$), then there exist $r = (-b_1,r_2) \neq s = (-b_1,s_2) \in W$ such that $d(b,r) = d(b,s) = 1$, which is a contradiction since a is connected to only one element in W. On the other hand, if $b_2 \neq a_2$, then the fact that b is connected to only one element in W yields $b_2 = -f_2$ and $b_1 = -w_1$. Since $w_1 \neq f_1$ we can find an element $w' = (w_1,w_2) \in W$ with $w' \neq w$ and $d(b,w') = 1$, which is again a contradiction. Lastly, if $a_2 = -f_2$, then $d(a,w) = 2$ for all $w \in W$. Since a is the only element in R with this property, we again have $a = b$, a contradiction. ∎

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We can now calculate the metric dimension in the case $|F_2| \geq 2|F_1| - 1$.

Theorem 3.5 *Let $R = F_1 \times F_2$, where F_1, F_2 are finite fields with $|F_1| \leq |F_2|$ and either* $\operatorname{char}(F_2) \neq 2$ *or* $\operatorname{char}(F_1) = \operatorname{char}(F_2) = 2$. *If* $|F_2| \geq 2|F_1| - 1$, *then* $\beta(\tau(R)) = |F_2| - 1$.

Proof Denote $F_1 \setminus \{0\} = \{x_0, x_1, \dots, x_{|F_1|-2}\}$ and $F_2 \setminus \{0\} = \{y_0, y_1, \dots, y_{|F_2|-2}\}$. If F_1 and F_2 are not of characteristic 2, then sort the elements of $F_1 \setminus \{0\}$ in such a way that $x_{2i} = -x_{2i+1}$ for all $i = 0, 1, \dots, \frac{|F_1|-3}{2}$ and the elements of $F_2 \setminus \{0\}$ in such a way that $y_{2i} = -y_{2i+1}$ for all $i = 0, 1, \dots, \frac{|F_2|-3}{2}$. Then define $z_i = (x_{i \bmod (|F_1|-1)}, y_i)$ for $0 \leq i \leq |F_2| - 2$. Now, let $W = \{z_0, z_1, \dots, z_{|F_2|-2}\}$.

If $\operatorname{char}(F_1) = 2$ and $\operatorname{char}(F_2) \neq 2$, then sort elements of $F_2 \setminus \{0\}$ in such a way that $y_i = -y_{i+(|F_1|-1)}$ for all $i = 0, 1, \dots, |F_1| - 2$ and $y_{2i} = -y_{2i+1}$ for all $i = |F_1| - 1, |F_1|, \dots, \frac{|F_2|-3}{2}$. Then define $z_i = (x_{i \bmod (|F_1|-1)}, y_i)$ for $0 \leq i \leq 2|F_1| - 3$, and if $|F_2| \geq 2|F_1| + 1$ also define $z_i = (0, y_i)$ for $i = 2|F_1| - 2, 2|F_1| - 1, \dots, |F_2| - 3$. Again, let $W = \{z_0, z_1, \dots, z_{|F_2|-2}\}$ and note that in both cases W is a set with $|F_2| - 1$ elements.

Since $|F_2| \geq 2|F_1| - 1$, W satisfies the conditions of Lemma 3.4 (for $f_1 = 0$, $f_2 = 0$), so W is a resolving set, and this implies that $\beta(\tau(R)) \leq |F_2| - 1$.

Now, choose an arbitrary resolving set W for $\tau(R)$. We will prove that $|W| \geq |F_2| - 1$. Suppose otherwise, $|W| \leq |F_2| - 2$. Then by Lemma 3.3,

$$|W| \geq |F_1|(t-1) + |F_2| - 2t$$

for some $t \geq 2$, which implies $|W| \geq |F_1| + |F_2| - 4$ and thus $|F_1| \leq 2$. We can conclude that $F_1 = \mathbb{Z}_2$ and $|W| = |F_1| + |F_2| - 4$. There exist $y_1 \neq y_2 \in F_2$ such that $(x, y_1), (x, y_2) \notin W$ for all $x \in F_1$. If $\operatorname{char}(F_2) = 2$, then for any $x \in F_1$ we have $d((x, y_1), w) = d((x, y_2), w)$ for all $w \in W$, which is a contradiction. Otherwise if $(x, -y_1) \notin W$ for all $x \in F_1$, then $d((x, -y_1), w) = d((x, y_1), w)$ for all $x \in X$ and $w \in W$, a contradiction. So, there exists an element $x_1 \in F_1$ such that $(x_1, -y_1) \in W$. Similarly, we get $x_2 \in F_1$ such that $(x_2, -y_2) \in W$. This also implies that F_2 is a field of odd characteristic with at least 5 elements. By the proof of Lemma 3.3, $|W| = |F_1| + |F_2| - 4$ implies that x_1 and x_2 are unique elements in F_1 with this property. If $x_1 = x_2$, then $(x_1, y_1), (x_1, y_2) \notin W$, and thus $d((x_1, y_1), w) = d((x_1, y_2), w)$ for all $w \in W$, which is a contradiction. So, $x_1 \neq x_2$. If there exists $y_3 \in F_2$ such that $y_3 \neq y_1, y_2$ and $(x, y_3) \notin W$ for all $x \in F_1$, then by the above there exists $x_3 \in F_1$ such that $(x_3, -y_3) \in W$, and since $x_3 = x_1$ or $x_3 = x_2$, this again leads to a contradiction. Now, choose an $y_0 \in F_2 \setminus \{y_1, y_2, -y_1, -y_2\}$. There exists a unique $x_0 \in F_1$ such that $(x_0, -y_0) \in W$. If $x_0 = x_1$, then (x_0, y_0) and (x_1, y_1) are both connected to $(x_0, -y_0), (x_1, -y_1) \in W$, which are the only elements in W with the second component equal to $-y_0$ or $-y_1$. This implies that $(x_0, y_0) \in W$, and we can reach the same conclusion in the case where $x_0 = x_2$. Set $x_0' = x_0 + 1$. This implies that $(x_0', y_0), (x_0', -y_0) \notin W$. We either have $x_0' = x_1$ or $x_0' = x_2$, and we can suppose without loss of generality that $x_0' = x_1$. However, both $(x_0', -y_0)$ and (x_1, y_1) are connected to (x_0, y_0) and $(x_1, -y_1)$, which are the only elements in W with the second component equal to y_0 or $-y_1$. This implies $d((x_0', -y_0), w) = d((x_1, y_1), w)$ for all $w \in W$, which is a contradiction, so the theorem holds. ∎

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Next, we study the case $|F_2| \le 2|F_1| - 2$. Again, we start with a few technical lemmas and a definition.

Definition Let $R = F_1 \times F_2$, where F_1, F_2 are finite fields and let $W \subseteq R$. We say that an element $(x_0, y_0) \in W$ such that and $(x_0, y), (x, y_0) \notin W$ for all $x \in F_1 \setminus \{x_0\}$, $y \in F_2 \setminus \{y_0\}$ is *a unique resolving element* for $\tau(R)$.

Lemma 3.6 *Let $R = F_1 \times F_2$, where F_1, F_2 are finite fields with* $\operatorname{char}(F_1), \operatorname{char}(F_2) \ne 2$*, let $W_1 \subseteq (F_1 \setminus \{-1,0,1\}) \times (F_2 \setminus \{-1,0,1\})$ and $W = \{(1,0),(0,1),(-1,-1)\} \cup W_1 \subseteq R$ be such sets that the following conditions are met:*

(i) *for every $x \in F_1$ there exists $y' \in F_2$ such that $(x, y') \in W$;*
(ii) *for every $y \in F_2$ there exists $x' \in F_1$ such that $(x', y) \in W$;*
(iii) *no element from W_1 is a unique resolving element for $\tau(R)$;*
(iv) *for every $x \in F_1, y_1, y_2 \in F_2$ such that $(x, y_1), (x, y_2) \in W$ we also have either $(-x, -y_1) \in W$ or $(-x, -y_2) \in W$;*
(v) *for every $y \in F_2, x_1, x_2 \in F_1$ such that $(x_1, y), (x_2, y) \in W$ we also have either $(-x_1, -y) \in W$ or $(-x_2, -y) \in W$.*

Then W is a resolving set for R.

Proof Suppose there exist $(a,b) \ne (c,d) \in R \setminus W$ such that $d\big((a,b), w\big) = d\big((c,d), w\big)$ for all $w \in W$.

First consider the case when $a \notin \{-1,0,1\}$ and $b \notin \{-1,0,1\}$. Since $(a,b) \ne (c,d)$, there exist either $w_1 = (-a, y_1) \ne w_2 = (-a, y_2) \in W$ or $w_1' = (x_1, -b) \ne w_2' = (x_2, -b) \in W$ (otherwise there would be a unique element if $R \setminus W$ connected to the only two elements in W that have $-a$ on the first and $-b$ on the second component). Suppose without loss of generality that we have w_1 and w_2. This implies that $d((c,d), w_1) = d((c,d), w_2) = 1$, which gives us $c = a$. But by (ii) there also exist $u_1, u_2 \in F_1$ such that $(u_1, -b), (u_2, -d) \in W$, and this yields $d((a,d),(u_1,-b)) = d((a,b),(u_1,-b)) = 1$. Thus, either $b = d$ or $u_1 = -a$. But we also have $d\big((a,b),(u_2,-d)\big) = d\big((a,d),(u_2,-d)\big) = 1$, so either $b = d$ or $u_2 = -a$. But $b \ne d$ now has a consequence that $(-a,-b), (-a,-d) \in W$, and thus by (iv) either $(a,b) \in W$ or $(c,d) = (a,d) \in W$, a contradiction.

If $a = 0$, then (a,b) is connected to $(0,1)$, so $d((c,d),(0,1)) = 1$, which implies that $c = 0$ or $d = -1$. If $c = 0$, then $d = b$, since otherwise by (ii) the element $(0,b)$ would be connected to some $(x,-b) \in W$ and $(0,d)$ would be connected to some $(x',-d) \in W$, and thus $x = x' = 0$ which is a contradiction since 0 appears only once as the first component of an element from W. If $c \ne 0$, then $d = -1$. Now, if $b = 0$, then (a,b) is connected to $(1,0)$, so $c = -1$, and thus $(c,d) \in W$, a contradiction. If $b = -1$, then (a,b) is connected only to $(0,1)$, so $c \notin \{-1,0,1\}$ and thus (c,d) is connected to some element in W_1 and (a,b) is not, a contradiction. Since $(a,b) \notin W$, the last remaining case is that $b \notin \{-1,0,1\}$. Then (a,b) is connected to some $(x,-b) \in W_1$, and by (iii) there exists either some $x \ne x' \in F_1 \setminus \{-1,0,1\}$ such that $(x',-b) \in W_1$ or some $-b \ne y \in F_2 \setminus \{-1,0,1\}$ such that $(x,y) \in W_1$. But then $c = -x$ and either $(c,d) = (-x,-1)$ is connected to (x,y) while $(a,b) = (0,b)$ is not, or

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$(a,b) = (0,b)$ is connected to $(x',-b)$ while $(c,d) = (-x,-1)$ is not, which again leads to a contradiction.

Similarly, we can check the cases $a = 1$ and $a = -1$, since $0, 1$, and -1 all appear exactly once at each component of an element from W. ∎

We now prove the theorem on the metric dimension for the case of the ring $F_1 \times F_2$ with $|F_1| \leq |F_2|$ and $|F_2| \leq 2|F_1| - 2$, where both F_1 and F_2 are either fields of an odd characteristic or both have characteristic equal to 2.

Theorem 3.7 *Let $R = F_1 \times F_2$, where F_1, F_2 are finite fields with $|F_1| \leq |F_2|$. If $|F_2| \leq 2|F_1| - 2$. Then*

$$\beta(\tau(R)) \leq \begin{cases} \dfrac{2|F_1| + |F_2| - 2}{2}, & \text{if } \operatorname{char}(F_1) = \operatorname{char}(F_2) = 2, \\ 2|F_1| - 2, & \text{if } \operatorname{char}(F_1) = 2 \text{ and } \operatorname{char}(F_2) \neq 2, \\ \dfrac{2|F_1| + |F_2| - 3}{2}, & \text{if } \operatorname{char}(F_1) \neq 2 \text{ and } \operatorname{char}(F_2) \neq 2. \end{cases}$$

Proof Let us first consider the case $\operatorname{char}(F_1) \neq 2 \operatorname{char}(F_2) \neq 2$. Denote

$$F_1' = F_1 \setminus \{0,1,-1\} = \{x_1, x_2, \dots, x_{|F_1|-3}\},$$
$$F_2' = F_2 \setminus \{0,1,-1\} = \{y_1, y_2, \dots, y_{|F_2|-3}\}.$$

Sort elements of F_1' in such a way that

$$x_1 = -x_{|F_1|-3}, x_2 = -x_{|F_1|-4}, \dots, x_{\frac{|F_1|-3}{2}} = -x_{\frac{|F_1|-3}{2}+1}$$

and the elements of F_2' in such a way that

$$y_1 = -y_{|F_2|-3}, y_2 = -y_{|F_2|-4}, \dots, y_{\frac{|F_2|-3}{2}} = -y_{\frac{|F_2|-3}{2}+1}.$$

Now, define the following sets

$$W_1 = \{(1,0),(0,1),(-1,-1)\},$$
$$W_2 = \left\{(x_1,y_1),(x_2,y_2),\dots,\left(x_{\frac{|F_1|-3}{2}}, y_{\frac{|F_1|-3}{2}}\right)\right\},$$
$$W_3 = -W_2 = \left\{(-x_1,-y_1),(-x_2,-y_2),\dots,\left(-x_{\frac{|F_1|-3}{2}}, -y_{\frac{|F_1|-3}{2}}\right)\right\}.$$

Observe that since F_2 is a field of an odd characteristic, we have $|F_2| \leq 2|F_1| - 3$, so $|F_2| - |F_1| \leq |F_1| - 3$. If $|F_1| = |F_2|$, then define $W_4 = \emptyset$; otherwise, define

$$W_4 = \left\{\left(x_1, y_{\frac{|F_1|-1}{2}}\right), \left(x_2, y_{\frac{|F_1|+1}{2}}\right), \dots, \left(x_{|F_2|-|F_1|}, y_{|F_2|-\frac{|F_1|+3}{2}}\right)\right\}.$$

Let $W' = (W_2 \cup W_3) \setminus \{(x,y) \in W_2 \cup W_3;\ \text{there exists } y' \in F_2 \text{ such that } (x,y') \in W_4\}$. Observe that $|W'| = 2|F_1| - |F_2| - 3$, so W' is a (possibly empty) set with an even number of elements. Denote the elements of $W' = \{(u_1,v_1),(u_2,v_2),\dots,(u_{|W'|}, v_{|W'|})\}$. Define

$$W_5 = \{(u_1,v_2),(u_3,v_4),\dots,(u_{|W'|-1}, v_{|W'|})\}.$$

By the construction, no element in $W_2 \cup W_3 \cup W_4 \cup W_5$ is a unique resolving element. It is obvious that W_1, W_2, W_3, W_4, and W_5 are disjoint sets, and one can verify that the

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set $W = W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$ satisfies the conditions of Lemma 3.6 and is hence a resolving set for $\tau(R)$. Because

$$|W| = 3 + \big(|F_1| - 3\big) + \big(|F_2| - |F_1|\big) + \frac{2|F_1| - |F_2| - 3}{2},$$

we get $\beta(\tau(R)) \le \frac{2|F_1|+|F_2|-3}{2}$.

Suppose next that $\operatorname{char}(F_1) = 2$ and $\operatorname{char}(F_2) \neq 2$. Denote

$$F_1 \setminus \{0\} = \{x_0, x_1, \dots, x_{|F_1|-2}\} \quad \text{and} \quad F_2 \setminus \{0\} = \{y_0, y_1, \dots, y_{|F_2|-2}\}.$$

Sort elements of $F_2 \setminus \{0\}$ in such a way that $y_{2i} = -y_{2i+1}$ for all $i = 0, 1, \dots, \frac{|F_2|-3}{2}$. Then define $z_{2i} = (x_i, y_{2i \bmod (|F_2|-1)})$ and $z_{2i+1} = (x_i, y_{2i+1 \bmod (|F_2|-1)})$ for $0 \le i \le |F_1| - 2$. Now, let $W = \{z_0, z_1, \dots, z_{2|F_1|-3}\}$ be a set with $2|F_1| - 2$ elements. Let us prove that W is a resolving set for $\tau(R)$. Suppose there exist $(a, b) \neq (c, d) \in R \setminus W$ such that $d((a, b), w) = d((c, d), w)$ for all $w \in W$. If $a = x_i$ for some $i \in \{0, 1, \dots, |F_1| - 2\}$, then $d((c, d), (x_i, y_{2i})) = d((c, d), (x_i, y_{2i+1})) = 1$, implying that $c = x_i$. If $b = y_{2j}$ for some $j \in \{0, 1, \dots, \frac{|F_2|-3}{2}\}$, then $d((c, d), (x_j, y_{2j+1})) = 1$ as well, which together with the fact that $i \neq j$ yields $d = y_{2j}$, a contradiction. Similarly, we treat the case $b = y_{2j+1}$ for some $i \neq j \in \{0, 1, \dots, \frac{|F_2|-3}{2}\}$. If $b = 0$, then (a, b) is not connected to any element from W, so $d = 0$ as well. Lastly, if $a = 0$, then (a, b) is connected to at most one element from W, which implies that $c = 0$ as well. Since $(0, b)$ is connected to $(0, -b) \in W$ if and only if $b \neq 0$ (and not connected to any element from W if $b = 0$) it must follow that $b = d$, which again leads us to a contradiction.

Finally, suppose that $\operatorname{char}(F_1) = \operatorname{char}(F_2) = 2$. Since $|F_2| \le 2|F_1| - 2$, we have $|F_1| = |F_2|$, and thus $F_1 = F_2$ by the uniqueness of finite fields. Denote $F_1 = \{x_1, x_2, \dots, x_{|F_1|}\}$ and define

$$W_1 = \big\{(x_1, x_1), (x_2, x_2), \dots, (x_{|F_1|-1}, x_{|F_1|-1})\big\},$$
$$W_2 = \big\{(x_1, x_2), (x_3, x_4), \dots, (x_{|F_1|-1}, x_{|F_1|})\big\}.$$

Again, let $W = W_1 \cup W_2$ and suppose there exist $(a, b) \neq (c, d) \in R \setminus W$ such that $d((a, b), w) = d((c, d), w)$ for all $w \in W$. If $a = x_i$ for some $i \in \{1, 3, \dots, |F_1| - 1\}$, then $d((a, b), (x_i, x_i)) = d((a, b), (x_i, x_{i+1})) = 1$, so $c = x_i$, and since $(a, b) \notin W$ we have $b \neq x_i, x_{i+1}$. Thus, $1 = d((a, b), (b, b)) = d((c, d), (b, b))$, which implies $b = d$. If $a = x_i$ for some $i = 2, 4, \dots, |F_1| - 2$, then $d((a, b), (x_i, x_i)) = 1$. If $d = x_i$, then $d((c, d), (x_{i-1}, x_i)) = 1$, which is a contradiction, since $b \neq x_i$. Thus, $c = x_i$ and $1 = d((a, b), (b, b)) = d((c, d), (b, b))$ which implies $b = d$ (since $b \neq x_i$). Since we can swap (a, b) and (c, d), this only leaves us to check the case $a = c = x_{|F_1|}$. If $d = x_{|F_1|}$ then $d((c, d), (x_{|F_1|-1}, x_{|F_1|})) = 1$ which implies $b = x_{|F_1|}$, a contradiction since $(a, b) \neq (c, d)$. So, $d \neq x_{|F_1|}$ and thus $d((a, b), (d, d)) = d((c, d), (d, d)) = 1$, which again leads us to the contradicting fact that $b = d$. This implies that W is a resolving set for $\tau(R)$. Because

$$|W| = (|F_1| - 1) + \frac{|F_1|}{2} = \frac{3|F_1| - 2}{2} = \frac{2|F_1| + |F_2| - 2}{2},$$

we get $\beta(\tau(R)) \le \frac{2|F_1|+|F_2|-2}{2}$. ∎

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We have reasons to believe that the metric dimensions are actually equal to the ones stated in the cases of the above theorem. We therefore conjecture the following.

Conjecture *Let $R = F_1 \times F_2$, where F_1, F_2 are finite fields with $|F_1| \leq |F_2|$. If $|F_2| \leq 2|F_1| - 2$, then*

$$\beta(\tau(R)) = \begin{cases} \dfrac{2|F_1| + |F_2| - 2}{2}, & \textit{if } \operatorname{char}(F_1) = \operatorname{char}(F_2) = 2, \\ 2|F_1| - 2, & \textit{if } \operatorname{char}(F_1) = 2 \textit{ and } \operatorname{char}(F_2) \neq 2, \\ \dfrac{2|F_1| + |F_2| - 3}{2}, & \textit{if } \operatorname{char}(F_1) \neq 2 \textit{ and } \operatorname{char}(F_2) \neq 2. \end{cases}$$

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