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Distribution Algebras on p-Adic Groups and Lie Algebras

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Abstract. When F is a p-adic field, and $G = \mathbb{G}(F)$ is the group of F-rational points of a connected algebraic F-group, the complex vector space $\mathcal{H}(G)$ of compactly supported locally constant distributions on G has a natural convolution product that makes it into a $\mathbb{C}$-algebra (without an identity) called the Hecke algebra. The Hecke algebra is a partial analogue for p-adic groups of the enveloping algebra of a Lie group. However, $\mathcal{H}(G)$ has drawbacks such as the lack of an identity element, and the process $G \mapsto \mathcal{H}(G)$ is not a functor. Bernstein introduced an enlargement $\mathcal{H}^{\wedge}(G)$ of $\mathcal{H}(G)$. The algebra $\mathcal{H}^{\wedge}(G)$ consists of the distributions that are left essentially compact. We show that the process $G \mapsto \mathcal{H}^{\wedge}(G)$ is a functor. If $\tau\colon G \to H$ is a morphism of p-adic groups, let $F(\tau)\colon \mathcal{H}^{\wedge}(G) \to \mathcal{H}^{\wedge}(H)$ be the morphism of $\mathbb{C}$-algebras. We identify the kernel of $F(\tau)$ in terms of $\mathrm{Ker}(\tau)$. In the setting of p-adic Lie algebras, with $\mathfrak{g}$ a reductive Lie algebra, $\mathfrak{m}$ a Levi, and $\tau\colon \mathfrak{g} \to \mathfrak{m}$ the natural projection, we show that $F(\tau)$ maps G-invariant distributions on $\mathcal{G}$ to $N_G(\mathfrak{m})$-invariant distributions on $\mathfrak{m}$. Finally, we exhibit a natural family of G-invariant essentially compact distributions on $\mathfrak{g}$ associated with a G-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ and in the case of $SL(2)$ show how certain members of the family can be moved to the group.

1 Introduction

Suppose that G is a connected Lie group. Let $\mathrm{Lie}(G)$ denote the Lie algebra of G, and $\mathfrak{g} = \mathrm{Lie}_{\mathbb{C}}(G) := \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathrm{Lie}(G)$. An indispensable tool in the representation theory of G is the complex universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. Under suitable conditions, a complex representation $\pi\colon G \to \mathrm{GL}(V)$ of G can be differentiated to yield a representation of $\mathfrak{U}(\mathfrak{g})$ into $\mathrm{End}_{\mathbb{C}}(V)$. Furthermore, the process $G \mapsto \mathfrak{U}(\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C})$ is a functor from the category of connected Lie groups to the category of algebras over $\mathbb{C}$.

Suppose that F is a non-archimedean local field of characteristic zero, *i.e.*, a p-adic field, and $G = \mathbb{G}(F)$ is the group of F-rational points of an algebraic group $\mathbb{G}$. The discovery of constructions in the theory of complex representations of G which are reasonable analogues of the Lie group enveloping algebra and its center have taken a long time to emerge. A straightforward generalization of passing from a representation π of G to a representation of its Lie algebra is not possible. The best initial substitute is to replace the enveloping algebra of a Lie group, which arises from differential operators, by the Hecke algebra $\mathcal{H}(G)$, which is the space of locally

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constant compactly supported distributions on G. Two drawbacks to $\mathcal{H}(G)$ are the following:

(i) The process $G \mapsto \mathcal{H}(G)$ is not functorial. If $\tau\colon G \to H$ is a homomorphism of algebraic groups, there is no natural candidate for a map $F(\tau)\colon \mathcal{H}(G) \to \mathcal{H}(H)$.

(ii) In the situation of Lie groups, for a suitable abelian category $\mathcal{C}$ of representations of a Lie group G, the center of the category $\mathcal{C}$ is isomorphic to the center $\mathfrak{Z}(\mathfrak{U}(\mathrm{Lie} \otimes_{\mathbb{R}} \mathbb{C}))$ of the enveloping algebra $\mathfrak{U}(\mathrm{Lie} \otimes_{\mathbb{R}} \mathbb{C})$. In the p-adic situation, the natural abelian category $\mathcal{C}$ to take is that of smooth representations. Unfortunately, the center of the Hecke algebra $\mathcal{H}(G)$ is often the zero function, for example, when G is a simple, and so not equal to the center of the category $\mathcal{C}$.

In seminal work in the 1980's (see [BD]), Bernstein introduced an analogue of the center of the universal enveloping algebra and the enveloping algebra itself. Bernstein's center has several realizations. In category terms, it is the center of the category of smooth representations of $G = \mathbb{G}(F)$. More concretely, it is the algebra of G-invariant distributions on G that have the property of essential compactness, *i.e.,* for any $f \in C_c^\infty(G)$, the convolution $D \star f$ has compact support, *i.e.,* belongs to $C_c^\infty(G)$. Note the G-invariance hypothesis on D means $D \star f$ equals $f \star D$. The realization of the Bernstein center as G-invariant essentially compact distributions is its geometrical realization. A simple argument to show the categorical and geometric definitions of the Bernstein center are equivalent is in [MT]. Under the additional hypothesis that $\mathbb{G}$ is reductive, a characterization of G-invariant essentially compact distribution D can be given spectrally. Let $(\widehat{G})_{\text{temp}}$ denote the tempered dual of G and $\Theta(G)$ the space of infinitesimal characters, *i.e.,* cuspidal data. Recall that

(i) the space $\Theta(G)$ is a countable disjoint union of irreducible, complex, affine, algebraic varieties;

(ii) there is a canonical map $\kappa\colon (\widehat{G})_{\text{temp}} \to \Theta(G)$.

Then D is essentially compact if and only if there is a regular function r on $\Theta(G)$ such that the Fourier transform of D equals Plancherel measure multiplied by $r \circ \kappa$.

Bernstein (see [BD, §1.4]) also gave a ring theoretical realization of the Bernstein center. One defines a left completion $\mathcal{H}\hat{}(G)$ of the Hecke algebra $\mathcal{H}(G)$ as follows: A distribution D on G is left essentially compact if for all $f \in C_c^\infty(G)$, the convolution $D \star f$ has compact support, and so belongs to $C_c^\infty(G)$. Then $\mathcal{H}\hat{}(G)$ is defined as the space of all left essentially compact distributions. Two left essentially compact distributions can be convoluted to produce another left essentially compact distribution, and this product makes $\mathcal{H}\hat{}(G)$ into a $\mathbb{C}$-algebra. The distribution on G that is the delta function δ_{1_G} at the identity of G is the identity element of $\mathcal{H}\hat{}(G)$. The center of $\mathcal{H}\hat{}(G)$ is the Bernstein center. One can also define a $\mathbb{C}$-algebra of right essentially compact distributions. A more consistent notation might be to denote the algebra of left (resp. right) essentially compact distributions as $\hat{}\mathcal{H}(G)$ (resp. $\mathcal{H}\hat{}(G)$), but here we follow the pre-existing notation in [BD, § 1.4].

The algebra $\mathcal{H}\hat{}(G)$ is a good analogue for p-adic groups of the universal enveloping algebra for Lie groups. It contains the Hecke algebra $\mathcal{H}(G)$ as a two-sided ideal. A smooth representation of the p-adic group G yields a non-degenerate representation of $\mathcal{H}(G)$ that can then be naturally extended to $\mathcal{H}\hat{}(G)$. See [MT] for an exposition of the latter. In analogy with the enveloping algebra associated with a Lie group being a functor, our first result here, Theorem 3.1, is to show that the process $G \mapsto \mathcal{H}^{\wedge}(G)$ is a functor. Let F denote this functor. So, if $\tau\colon G \to H$ is a morphism of p-adic algebraic groups, then $F(\tau)\colon \mathcal{H}^{\wedge}(G) \to \mathcal{H}^{\wedge}(H)$ is a morphism of $\mathbb{C}$-algebras. Our second result, Theorem 4.3, gives a geometric interpretation of the algebra homomorphism $F(\tau)$. Roughly speaking, a left essentially compact distribution D on G can be integrated along the cosets of the kernel $\operatorname{Ker}(\tau)$ of τ to produce a left essentially compact distribution on H. Here, one must do a weighted integration based on the modular functions of G and the quotient (G/K). In particular, the kernel of the algebra homomorphism $F(\tau)\colon \mathcal{H}^{\wedge}(G) \to \mathcal{H}^{\wedge}(H)$ is the space of left essentially compact distributions whose weighted integral along any coset of $\operatorname{Ker}(\tau)$ is zero.

The functor $G \mapsto \mathcal{H}^{\wedge}(G)$ can be applied in the context of p-adic Lie algebras. Take $\mathfrak{g} = \operatorname{Lie}(\mathbb{G})(F)$ to be the F-rational points of the Lie algebra $\operatorname{Lie}(\mathbb{G})$ of an F-group $\mathbb{G}$. The group structure on $\mathfrak{g}$ is its additive vector space structure. Convolution of two functions in $C_c^\infty(\mathfrak{g})$ is commutative, as is the convolution of a function f and a distribution D, as well as the convolution of two essentially compact distributions. We use the notation $\mathcal{D}(\mathfrak{g})$ to denote the commutative algebra of essentially compact distributions on $\mathfrak{g}$. It seems fruitful to view $\mathcal{D}(\mathfrak{g})$ as a p-adic analogue of the algebra of constant coefficient differential operators on the Lie algebra of a Lie group, and, in particular, to consider the subalgebra of $\operatorname{Ad}(G)$-invariant distributions $\mathcal{D}(\mathfrak{g})^G$. In this context, suppose that $\mathfrak{g}$ is reductive, and $\mathfrak{m}$ is a Levi subalgebra of $\mathfrak{g}$. There is a canonical vector space map $\tau\colon \mathfrak{g} \to \mathfrak{m}$. Let $N_G(\mathfrak{m})$ denote the normalizer of $\mathfrak{m}$ in G. In Theorem 6.2 we show the algebra map $F(\tau)\colon \mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{m})$ takes the space of G-invariant distributions $\mathcal{D}(\mathfrak{g})^G$ to the space $\mathcal{D}(\mathfrak{m})^{N_G(\mathfrak{m})}$ of $N_G(\mathfrak{m})$-invariant essentially compact distributions on $\mathfrak{m}$. The analogue with the Harish-Chandra homomorphism for complex Lie algebras is obvious. Suppose that B is an $\operatorname{Ad}(G)$-invariant, non-degenerate, symmetric, bilinear form on $\mathfrak{g}$, and $\psi\colon F \to \mathbb{C}$ is a non-trivial additive character. In Theorem 6.3, we show the $\operatorname{Ad}(G)$-invariant distribution on $\mathfrak{g}$ associated to the $\operatorname{Ad}(G)$-invariant function $x \mapsto \psi(B(x,x))$ is essentially compact. The analogue of this with a second order $\operatorname{Ad}(G)$-invariant differential operator on a complex semisimple Lie algebra is clear. In Theorem 7.1, for the case of $\mathrm{SL}(2)$, we show how certain of these G-invariant essentially compact distributions on the Lie algebra can be truncated to the set of topologically nilpotent elements and then moved to the group by the Cayley transform. In (7.2) we calculate the scalars by which these distributions on the group act on irreducible cuspidal representations.

In Section 2, we introduce notation and recall some results about the algebra $\mathcal{H}^{\wedge}(G)$. In particular, we recall the equivalence of a left (resp. right) essentially compact distribution with a compatible system of elements $f_J \in \mathcal{H}(G) \star e_J$ (resp. $f_J \in e_J \star \mathcal{H}(G)$) for a (countable) fundamental system $\mathcal{J}$ of open compact subgroups of the identity of G. In Section 3, for a group homomorphism $\tau\colon G \to H$, we recall the module action of $C_c^\infty(G)$ on $C_c^\infty(H)$ and then show the process $G \mapsto \mathcal{H}^{\wedge}(G)$ is a functor. In Section 4, we determine the kernel of the $\mathbb{C}$-algebra map $F(\tau)\colon \mathcal{H}^{\wedge}(G) \to \mathcal{H}^{\wedge}(H)$ in terms of the kernel $\operatorname{Ker}(\tau)$. Section 5 has results on the essential compactness of distributions on a vector space, in particular, associated with a non-degenerate symmetric bilinear form. In Section 6, we apply the results of section 5 to the case of a p-adic Lie algebra. In Section 7, we consider the case of the group $\mathrm{SL}(2)$.

2 Notation, Setup, and Review

Suppose that F is a non-archimedean local field of characteristics zero, *i.e.*, a p-adic field. Let AlgG_F denote the category of F-groups. Our assumption that F is p-adic means an F-group $\mathbb{G}$ is completely determined by the group $G = \mathbb{G}(F)$ of its F-rational points. In all that follows we use this correspondence and sometimes do not distinguish between $\mathbb{G}$ and G. Suppose that $\mathbb{N}$ is normal F-group of $\mathbb{G}$. Recall from [S, §12.3] that the quotient group $\mathbb{G}(F)/\mathbb{N}(F)$ is the F-rational points of a F-group inside the F-group $(\mathbb{G}/\mathbb{N})$. The topology on $G = \mathbb{G}(F)$ coming from the local field F is locally compact and totally disconnected. In particular, every neighborhood of the identity element of G contains an open compact subgroup, and there is a countable fundamental system of open compact subgroups. Denote by $C_c^\infty(G)$ the complex vector space of locally constant compactly supported functions on G.

Let $d_\ell y$ be a left Haar measure on G. Recall the modular function Δ_G is defined as

$$\int_G f(y)d_\ell y = \Delta_G(g)\int_G f(yg)d_\ell y,$$

and $d_r y = \Delta_G(y)^{-1}d_\ell y$ is a right Haar measure. There is natural convolution structure on $C_c^\infty(G)$ that can be written symmetrically in $d_\ell y$ and $d_r y$ as follows. For $\phi,\ \psi \in C_c^\infty(G)$,

$$\begin{aligned}\phi\star\psi := x\mapsto \int_G \phi(y)\psi(y^{-1}x)d_\ell y &= \int_G \phi(y)\psi(y^{-1}x)\Delta_G(y)d_r y\\ = x\mapsto \int_G \phi(xh^{-1})\psi(h)d_r h &= \int_G \phi(xh^{-1})\psi(h)\Delta_G(h)^{-1}\,d_\ell h.\end{aligned}\tag{2.1}$$

The convolution product makes $C_c^\infty(G)$ into a ring. If G is not discrete, then $C_c^\infty(G)$ does not have an identity. Set $C(f) := x \mapsto f(x^{-1})$. For $\phi \in C_c^\infty(G)$ and $f \in C_c^\infty(G)$, the distributions

$$D_{\{\ell,\phi\}}(f) := \int_G \phi(y)f(y)d_\ell y \quad\text{and}\quad D_{\{r,\phi\}}(f) := \int_G \phi(y)f(y)d_r y$$

can be expressed in terms of convolution products as:

$$D_{\{\ell,\phi\}}(f) = \big(\phi\star C(f)\big)(1) \quad\text{and}\quad D_{\{r,\phi\}}(f) = \big(C(f)\star\phi\big)(1).\tag{2.2}$$

In the reverse direction, if f is a function on G and $x \in G$, define $\lambda_x(f) := y \mapsto f(x^{-1}y)$ (resp. $\rho_x(f) := f(yx)$) to be the left (resp. right) translation of f by x. Then the convolutions $\phi\star f$ and $f\star\phi$ can be recovered from the distributions $D_{\{\ell,\phi\}}$ and $D_{\{r,\phi\}}$ by the formulae

$$\phi\star f = x\mapsto D_{\{\ell,\phi\}}\big(\lambda_x(C(f))\big) \quad\text{and}\quad f\star\phi = x\mapsto D_{\{r,\phi\}}\big(\rho_{x^{-1}}(C(f))\big).\tag{2.3}$$

For an arbitrary distribution D on G and $f \in C_c^\infty(G)$, we use the obvious extrapolation of formulae (2.3) to define the convolutions $D\star f$ and $f\star D$.

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Definition 2.1 (see [BD, §1.4]) A distribution D on G is respectively left or right essentially compact if for all $f \in C_c^\infty(G)$, the convolutions $D \star f$ or $f \star D$ belong to $C_c^\infty(G)$. A distribution D on G is essentially compact if it is both left and right essentially compact.

Define complex vector spaces:

$$\begin{aligned}\mathcal{H}\hat{}(G) :&= \{\text{distribution } D \mid \text{for all } f \in C_c^\infty(G),\\ &\qquad \text{the convolution } (D \star f) \text{ belongs to } C_c^\infty(G)\},\\ \hat{}\mathcal{H}(G) :&= \{\text{distribution } D \mid \text{for all } f \in C_c^\infty(G),\\ &\qquad \text{the convolution } (f \star D) \text{ belongs to } C_c^\infty(G)\},\\ \mathcal{U}(G) :&= \{\text{distribution } D \mid \text{for all } f \in C_c^\infty(G),\\ &\qquad \text{both } (D \star f) \text{ and } (f \star D) \text{ belong to } C_c^\infty(G)\}.\end{aligned}$$

Suppose that $D_1, D_2 \in \mathcal{H}\hat{}(G)$. Then for any $f \in C_c^\infty(G)$, the convolution $D_1 \star (D_2 \star f)$ is a well-defined function in $C_c^\infty(G)$. The distribution $D_1 \star D_2$ is defined as the distribution on G, such that for all $f \in C_c^\infty(G)$,

$$(D_1 \star D_2) \star f = D_1 \star (D_2 \star f), \text{so } (D_1 \star D_2)(f) := \big(D_1 \star (D_2 \star C(f))\big)(1).$$

Similarly, if $D_1, D_2 \in \mathcal{H}\hat{}(G)$, the distribution $D_1 \star D_2 \in \mathcal{H}\hat{}(G)$ is defined as

$$f \star (D_1 \star D_2) = (f \star D_1) \star D_2, \text{so}(D_1 \star D_2)(f) := \big((C(f) \star D_1) \star D_2\big)(1).$$

We remark that when G is unimodular, for all $D_1, D_2 \in \mathcal{U}(G)$, the convolution of D_1 and D_2 as elements in $\mathcal{H}\hat{}(G)$ and as elements in $\hat{}\mathcal{H}(G)$ are equal.

In order to provide a better feel for the algebras $\mathcal{H}\hat{}(G)$ and $\hat{}\mathcal{H}(G)$, we review [BD, Lemma 1.2.2]. For $D \in \mathcal{H}\hat{}(G)$ consider the linear map $L_D \colon C_c^\infty(G) \to C_c^\infty(G)$ defined as $L_D(f) := D \star f$. Clearly, $L_D(f \star h) = L_D(f) \star h$ for all $f, h \in C_c^\infty(G)$. Consider the endomorphism algebra

$$\begin{aligned}\mathrm{End}_{(-,C_c^\infty(G))}(C_c^\infty(G)) := \{&\text{linear } T \colon C_c^\infty(G) \to C_c^\infty(G) \mid\\ &T(f \star h) = T(f) \star h \,\forall\, f, h \in C_c^\infty(G)\}.\end{aligned}$$

Obviously, $L_D \in \mathrm{End}_{(-,C_c^\infty(G))}(C_c^\infty(G))$, and furthermore, the map

$$\begin{aligned} L\colon\ \mathcal{H}\hat{}(G) &\to \mathrm{End}_{(-,C_c^\infty(G))}(C_c^\infty(G))\\ D &\mapsto L_D \end{aligned} \tag{2.4}$$

is an algebra homomorphism.

Proposition 2.2 ([BD, Lemma 1.2.2]) *If $T \in \mathrm{End}_{(-,C_c^\infty(G))}(C_c^\infty(G))$, the distribution D_T defined as $D_T(f) := T(C(f))(1)$ is left essentially compact and*

$$T \mapsto D_T \tag{2.5}$$

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is the inverse of (2.4). *In particular,* (2.5) *provides an identification of the algebras* $\mathcal{H}^{\wedge}(G)$ *and* $\operatorname{End}_{(-,C_c^\infty(G))}(C_c^\infty(G))$.

Proof For $T \in \operatorname{End}_{(-,C_c^\infty(G))}(C_c^\infty(G))$ and $f \in C_c^\infty(G)$, we have

$$D_T \star f := x \mapsto D_T\big(\lambda_x(C(f))\big) = T\big(C(\lambda_x(C(f)))\big)(1)$$

The function $C(\lambda_x(C(f)))$ is

$$\begin{aligned} y \mapsto\ & \lambda_x(C(f))(y^{-1}) = C(f)(x^{-1}y^{-1}) = f(yx) \quad \forall f \in C_c^\infty(G) \\ \mapsto\ & f \star \delta_{x^{-1}}. \end{aligned}$$

So, $C(\lambda_x(C(f))) = f \star \delta_{x^{-1}}$. Thus,

$$D_T \star f = x \mapsto T(f \star \delta_{x^{-1}})(1) = x \mapsto T(f)(x).$$

In particular, since $D_T \star f$ equals $T(f)$, it follows that

(i) D_T is essentially compact, and

(ii) $L_{D_T}(f) = D_T \star f = T(f)$,

so the maps (2.4) and (2.5) are inverses of each other. ■

Obviously, we also have

$$\begin{aligned} {}^{\wedge}\mathcal{H}(G) &= \operatorname{End}_{(C_c^\infty(G),-)}(C_c^\infty(G)) \\ &= \big\{ \text{linear } T\colon C_c^\infty(G) \to C_c^\infty(G) \mid \\ &\qquad\qquad T(h \star f) = h \star T(f)\ \forall\, f, h \in C_c^\infty(G) \big\}. \end{aligned}$$

For later use, we recall an equivalent formulation of an element $D \in \mathcal{H}^{\wedge}(G)$. If J is an open compact subgroup of G, let 1_J denote the characteristic function of J, and set $e_J := \frac{1}{\operatorname{meas}(J)} 1_J$. Take $\mathcal{J}$ to be a countable fundamental system of open compact subgroups of G. For $J \in \mathcal{J}$, set $D_J := D \star e_J$, so D_J belongs to $C_c^\infty(G) \star e_J$. The set of functions $\{D_J \mid J \in \mathcal{J}\}$, have the following compatibility property. If $L, J \in \mathcal{J}$ and $L \subset J$, then $e_L \star e_J = e_J$, and consequently $D_L \star e_J = D_J$. Conversely, given a fundamental system of open compact subgroups $\mathcal{J}$, suppose that we have a system of functions $\{D_J \in C_c^\infty(G) \star e_J \mid J \in \mathcal{J}\}$ with the property that

$$D_L \star e_J = D_J \text{ for all } L, J \in \mathcal{J} \text{ satisfying } L \subset J. \tag{2.6}$$

For any $f \in C_c^\infty(G)$, take $J \in \mathcal{J}$ so that $e_J \star C(f) = C(f)$ and consider the function $D_J \star C(f)$. If $L \in \mathcal{J}$ satisfies $L \subset J$, then

$$D_L \star C(f) = D_L \star (e_J \star C(f)) = (D_L \star e_J) \star C(f) = D_J \star C(f). \tag{2.7}$$

It follows that if $J \in \mathcal{J}$ satisfies $e_J \star C(f) = C(f)$, then (2.7) is independent of J. Set $D(f) := (D_J \star C(f))(1)$. Then D is a well-defined element of $\mathcal{H}^{\wedge}(G)$, and furthermore $D \star e_J = D_J$.

3 Module Action and Functor

A group homomorphism $\tau\colon G \to H$ yields natural left and right module actions of $C_c^\infty(G)$ on $C_c^\infty(H)$. The formula for the action is found in Cartier's article [C, p. 124, display (29)]. For $f_G \in C_c^\infty(G)$ and $f_H \in C_c^\infty(H)$,

$$
\begin{aligned}
f_G \underset{\{\tau,\ell\}}{\star} f_H &:= x \mapsto \int_G f_G(y) f_H(\tau(y)^{-1}x)\Delta_H(\tau(y)) d_r y \\
f_H \underset{\{\tau,r\}}{\star} f_G &:= x \mapsto \int_G f_H(x\tau(y)^{-1}) f_G(y)\Delta_H(\tau(y))^{-1} d_\ell y.
\end{aligned}
\tag{3.1}
$$

Note that if $H = G$ and τ is the identity map, then the formulae in (3.1) reduce to those of (2.1).

These formulae can be reinterpreted in terms of distributions. For $f_G \in C_c^\infty(G)$ and $f_H \in C_c^\infty(H)$, let $D_{\{\ell,f_G,\tau\}}$ and $D_{\{r,f_G,\tau\}}$ be the essentially compact distributions on H defined by

$$
\begin{aligned}
D_{\{\ell,f_G,\tau\}}(f_H) := f_G \underset{\{\tau,\ell\}}{\star} C(f_H)(1) &= \int_G f_G(y)\, f_H(\tau(y))\Delta_H(\tau(y)) d_r y \\
&= \int_G f_G(y) f_H(\tau(y))\Delta_H(\tau(y))\Delta_G(y)^{-1} d_\ell y \\
D_{\{r,f_G,\tau\}}(f_H) := C(f_H) \underset{\{\tau,r\}}{\star} f_G(1) &= \int_G f_H(\tau(y)) f_G(y)\, \Delta_H(\tau(y))^{-1} d_\ell y \\
&= \int_G f_H(\tau(y)) f_G(y)\Delta_H(\tau(y))^{-1}\Delta_G(y) d_r y.
\end{aligned}
$$

Then we have the the following analogues of (2.2):

$$
f_G \underset{\{\tau,\ell\}}{\star} f_H = x \mapsto D_{\{\ell,f_G,\tau\}}(\lambda_x(C(f_H)) \text{ and } f_H \underset{\{\tau,r\}}{\star} f_G = x \mapsto D_{\{r,f_G,\tau\}}(\rho_{x^{-1}}(C(f_H)).
$$

Now consider the distribution algebra $\mathcal{H}\hat{}(G)$. Our goal is to obtain a left module action of the distribution algebra $\mathcal{H}\hat{}(G)$ on $C_c^\infty(H)$ that then yields a ring homomorphism of $\mathcal{H}\hat{}(G)$ into $\mathcal{H}\hat{}(H)$. This is in fact rather easy because the action of $C_c^\infty(G)$ on $C_c^\infty(H)$ is smooth.

Theorem 3.1 (i) *Suppose that $D_G \in \mathcal{H}\hat{}(G)$ and $f_H \in C_c^\infty(H)$. Take J to be an open compact subgroup of G such that*

$$
e_J \underset{\{\tau,\ell\}}{\star} C(f_H) = C(f_H).
$$

Then the rule $(F(\tau)(D_G))$ defined as

$$
(F(\tau)(D_G))(f_H) := \big((D_G \star e_J) \underset{\{\tau,\ell\}}{\star} C(f_H)\big)(1) \tag{3.2}
$$

is independent of J and so defines a distribution on H.

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(ii) $F(\tau)(D_G) \in \mathcal{H}^\wedge(H)$.
(iii) *The map* $D_G \mapsto F(\tau)(D_G)$ *is an algebra homomorphism of* $\mathcal{H}^\wedge(G)$ *to* $\mathcal{H}^\wedge(H)$.
(iv) *The process* $G \mapsto \mathcal{H}^\wedge(G)$ *is a functor from the category* $AlgG_F$ *of* F*-groups and* F*-homomorphisms to the category of rings.*

Proof (i) Suppose that $L \subset J$ is a smaller open compact subgroup, then $e_L \star e_J = e_J = e_J \star e_L$, so that

$$(D_G \star e_J) \underset{\{\tau,\ell\}}{\star} C(f_H) = \big(D_G \star (e_L \star e_J)\big) \underset{\{\tau,\ell\}}{\star} C(f_H) = (D_G \star e_L) \underset{\{\tau,\ell\}}{\star} \big(e_J \underset{\{\tau,\ell\}}{\star} C(f_H)\big)$$
$$= (D_G \star e_L) \underset{\{\tau,\ell\}}{\star} C(f_H)$$

and so

$$(D_G \star e_J) \underset{\{\tau,\ell\}}{\star} C(f_H)(1) = (D_G \star e_L) \underset{\{\tau,\ell\}}{\star} C(f_H)(1).$$

It is now easy to deduce that $F(\tau)(D_G)$ is a well-defined distribution on H.

(ii) By definition, $(F(\tau)(D_G)) \star f_H = x \mapsto (F(\tau)(D_G))(\lambda_x(C(f_H)))$, and

$$(F(\tau)(D_G))(\lambda_x(C(f_H))) = (D_G \star e_J) \underset{\{\tau,\ell\}}{\star} C(\lambda_x(C(f_H)))(1),$$

where $J \subset G$ is any open compact subgroup so that

$$e_J \underset{\{\tau,\ell\}}{\star} C\big(\lambda_x(C(f_H))\big) = C\big(\lambda_x(C(f_H))\big).$$

The function $C(\lambda_x(C(f_H)))$, is $\rho_{x^{-1}}(f_H)$, so the requirement on J is that f_H be left $\tau(J)$-invariant. So,

$$(F(\tau)(D_G)) \star f_H = x \mapsto (D_G \star e_J) \underset{\{\tau,\ell\}}{\star} \rho_{x^{-1}}(f_H)(1) = \big((D_G \star e_J) \underset{\{\tau,\ell\}}{\star} f_H\big)(x).$$

The function $((D_G \star e_J) \underset{\{\tau,\ell\}}{\star} f_H)$ is independent of J satisfying the requirement that f_H be left $\tau(J)$-invariant. But this function has compact support.

(iii) In order to distinguish convolutions on G and H, we write the former as $\star_G$ and the later as $\star_H$. Suppose that $D'_G, D_G \in \mathcal{H}^\wedge(G)$ and $f_H \in C_c^\infty(H)$. Select a sufficiently small $J \subset G$ so that f_H is left τ-invariant and so

$$\big(F(\tau)(D_G)\big) \star_H f_H = (D_G \star_G e_J) \underset{\{\tau,\ell\}}{\star} f_H.$$

Also, select a sufficiently small L so that $e_L \star_G (D_G \star_G e_J) = (D_G \star_G e_J)$. Then

$e_L \underset{\{\tau,\ell\}}{\star} ((D_G \star_G e_J) \underset{\{\tau,\ell\}}{\star} f_H) = (D_G \star_G e_J) \underset{\{\tau,\ell\}}{\star} f_H$, and

$$\begin{aligned}\big(F(\tau)(D'_G)\big) \star_H \big((F(\tau)(D_G)) \star_H f_H\big) &= \big(F(\tau)(D'_G)\big) \star_H \big((D_G \star_G e_J) \underset{\{\tau,\ell\}}{\star} f_H\big)\\ &= (D'_G \star_G e_L) \underset{\{\tau,\ell\}}{\star} \big((D_G \star_G e_J) \underset{\{\tau,\ell\}}{\star} f_H\big)\\ &= \big(D'_G \star_G (e_L \star_G (D_G \star_G e_J))\big) \underset{\{\tau,\ell\}}{\star} f_H\\ &= \big(D'_G \star_G (D_G \star_G e_J)\big) \underset{\{\tau,\ell\}}{\star} f_H\\ &= \big((D'_G \star_G D_G) \star_G e_J\big) \underset{\{\tau,\ell\}}{\star} f_H\\ &= \big(F(\tau)(D'_G \star_G D_G)\big) \star_H f_H.\end{aligned}$$

We deduce from this that $F(\tau)(D'_G \star_G D_G) = F(\tau)(D'_G) \star_H F(\tau)(D_G)$.

(iv) This is immediate from (i), (ii), and (iii). ■

When $\tau\colon G \to H$ is an injection, it is a simple calculation to verify that $F(\tau)\colon \mathcal{H}^\wedge(G) \to \mathcal{H}^\wedge(H)$ is also an injection. In the next two sections we describe the kernel of the algebra map $F(\tau)\colon \mathcal{H}^\wedge(G) \to \mathcal{H}^\wedge(H)$ in terms of the group kernel of $\tau\colon G \to H$. The kernel of $F(\tau)$ is those distributions whose "weighted integration over the cosets of the group kernel" is zero.

4 Kernel Formula

Suppose that $\tau\colon G \to H$ is a (continuous) group homomorphism with kernel K. Let $q\colon G \to (G/K)$ denote the quotient map. When convenient, we shall also use the common notation $\overline{g}$ to denote $q(g)$. Set $\iota\colon (G/K) \to H$ so that

$$\tau = \iota \circ q. \tag{4.1}$$

We will need a formula for left Haar integration over G in terms of the left Haar measures on K and (G/K), so we recall the following lemma, whose proof we leave to the reader.

Lemma 4.1 *For $f \in C_c^\infty(G)$, the functional*

$$f \mapsto \int_{(G/K)} \int_K \qquad \forall f \in C_c^\infty(G) f(yk) d_\ell k d_\ell \overline{y}$$

is a left Haar measure.

The module action of $f_G \in C_c^\infty(G)$ on $f_H \in C_c^\infty(H)$ is

$$\begin{aligned}f_G \underset{\{\tau,\ell\}}{\star} f_H &:= x \mapsto \int_G f_G(y) f_H(\tau(y)^{-1}x) \Delta_H(\tau(y)) d_r y \\ &= x \mapsto \int_G f_G(y) f_H(\tau(y)^{-1}x) \Delta_H(\tau(y)) \Delta_G(y)^{-1} d_\ell y.\end{aligned} \tag{4.2}$$

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To integrate over G, we use Lemma 4.1. The last integral of (4.2) becomes

(4.3)
$$
\begin{aligned}
f_G \underset{\{\tau,\ell\}}{\star} f_H := x \mapsto & \int_{G/K} \Big\{ \int_K f_G(yk) f_H(\tau(yk)^{-1}x)\Delta_H(\tau(yk))\Delta_G(yk)^{-1} d_\ell k \Big\} d_\ell \overline{y} \\
= x \mapsto & \int_{G/K} \Big\{ \int_K f_G(yk)\ f_H(\tau(y)^{-1}x)\Delta_H(\tau(y))\Delta_G(yk)^{-1} d_\ell k \Big\} d_\ell \overline{y} \\
= x \mapsto & \int_{G/K} \Big\{ \int_K f_G(yk) \frac{d_\ell k}{\Delta_G(yk)} \Big\} f_H(\tau(y)^{-1}x)\Delta_H(\tau(y)) d_\ell \overline{y} \\
= x \mapsto & \int_{G/K} \Big\{ \Delta_H(\tau(y)) \int_K f_G(yk) \frac{d_\ell k}{\Delta_G(yk)} \Big\} f_H(\tau(y)^{-1}x) d_\ell \overline{y}.
\end{aligned}
$$

Set

$$
I(f_G) := y \mapsto \Delta_{(G/K)}(\overline{y}) \int_K f_G(yk) \frac{d_\ell k}{\Delta_G(yk)} \tag{4.4}
$$

a function on G.

Lemma 4.2 *The function $I(f_G)$ is K-bi-invariant.*

Proof Right K-invariance is obvious from (4.4). Left K-invariance is then a consequence of K being a normal subgroup of G and right K-invariant. ∎

When F is a bi-K-invariant function of G, let $\overline{F}$ denote F as a function of (G/K). We rewrite (4.3) as:

$$
\begin{aligned}
f_G \underset{\{\tau,\ell\}}{\star} f_H &= x \mapsto \int_{G/K} \big\{ I(f_G)(y) \big\} f_H(\tau(y)^{-1}x)\Delta_H(\tau(y))\Delta_{(G/K)}(\overline{y})^{-1} d_\ell \overline{y} \\
&= \overline{I(f_G)} \underset{\{\iota,\ell\}}{\star} f_H
\end{aligned} \tag{4.5}
$$

So, the left module action of f_G on $C_c^\infty(H)$ is equal to integration of the function f_G along the cosets of K in G in the manner of (4.4) to obtain a bi-K-invariant function $I(f_G)$, which we view as a function $\overline{I(f_G)}$ on (G/K), and then using the left module action of $\overline{I(f_G)}$ on $C_c^\infty(H)$. In terms of distributions, we have $D_{\{\ell, f_G, \tau\}} = D_{\{\ell, \overline{I(f_G)}, \iota\}}$, *i.e.*,

$$
F(\tau)(D_{\{\ell, f_G\}}) = F(\iota)\big(D_{\{\ell, \overline{I(f_G)}\}}\big). \tag{4.6}
$$

We want to extrapolate (4.4) and (4.6) to an arbitrary left essentially compact distribution $D \in \mathcal{H}\hat{}(G)$. In particular, we wish to show that any left essentially compact distribution $D \in \mathcal{H}\hat{}(G)$ can be integrated over the cosets of K to obtain a distribution $\overline{I(D)}$ on the quotient (G/K) such that $\overline{I(D)} \in \mathcal{H}\hat{}((G/K))$ and

$$
F(\tau)(D) = F(\iota)(\overline{I(D)}). \tag{4.7}
$$

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To accomplish this, we take a (countable) fundamental system of open compact subgroup $\mathcal{J}$ of the identity in G. Observe that $\{\overline{J} \mid J \in \mathcal{J}\}$ is a fundamental system of open compact subgroups in (G/K). Recall from Section 2 that an element $D \in \mathcal{H}^{\wedge}(G)$ is equivalent to a system of functions F_J (recall $F_J = D \star e_J$) satisfying the compatibility condition (2.6). Apply the weighted integration operator (4.4) to each F_J to obtain bi-K-invariant functions $I(F_J)$. We shall show that $I(F_J)$ is a compatible system of functions on (G/K) that defines an essentially compact distribution $I(D)$ satisfying (4.7). We summarize the process via the diagram

$$\begin{tikzcd} D \arrow[r] & \{F_J := D \star e_J \mid J \in \mathcal{J}\} \arrow[d] \\ \Phi & \{I(F_J) \mid J \in \mathcal{J}\}. \arrow[l] \end{tikzcd}$$

Here, Φ is the left essentially compact distribution on (G/K) associated with the compatible system of functions $\{I(F_J) \mid J \in \mathcal{J}\}$.

Theorem 4.3 *Suppose that $G = \mathbb{G}(F)$, $H = \mathbb{H}(F)$ are p-adic algebraic groups, and $\tau \colon G \to H$ is a homomorphism with kernel K. Write $\tau = \iota \circ q$ as in* (4.1) *and $F(\tau) \colon \mathcal{H}^{\wedge}(G) \to \mathcal{H}^{\wedge}((G/K))$ as in* (3.2)*. Suppose that $\mathcal{J}$ is a fundamental system of open compact subgroups of G. Given $D \in \mathcal{H}^{\wedge}(G)$, consider the compatible system of functions $\{D \star e_J \mid J \in \mathcal{J}\}$. Then $F(\tau)(D) = F(\iota)(\Phi)$, where $\Phi \in \mathcal{H}^{\wedge}((G/K))$ is the distribution associated with the compatible system of functions $\{I(D \star e_J) \mid J \in \mathcal{J}\}$.*

We prove Theorem (4.3) after we establish two lemmas.

Suppose that $F \in C_c^{\infty}(G)$. Then the function $I(F)$ defined in (4.4) is bi-K-invariant. For any $f \in C_c^{\infty}(G)$, we can form the convolution $I(F) \star f$, in particular, for $f = e_J$, where J is an open compact subgroup of G.

Lemma 4.4 *Suppose that J is an open compact subgroup of G and $F \in C_c^{\infty}(G)$. Then $I(F) \star e_J = I(F \star e_J)$.*

Proof We have $I(F) \star e_J := x \mapsto \int_G I(F)(xh^{-1})\, e_J(h)\, d_r h$, and then

$$\begin{aligned} &\int_G I(F)(xh^{-1}) e_J(h) d_r h \ \frac{1}{\operatorname{meas}(J)} \int_J I(F)(xh^{-1}) d_r h \\ &= \frac{1}{\operatorname{meas}(J)} \int_J \Delta_{(G/K)}(\overline{x}) \int_K \frac{F(xh^{-1}k)}{\Delta_G(xk)} d_\ell k d_r h \\ &= \frac{1}{\operatorname{meas}(J)} \int_J \Delta_{(G/K)}(\overline{x}) \int_K \frac{F(x(h^{-1}kh)h^{-1})}{\Delta_G(xk)} d_\ell k \, d_r h. \end{aligned}$$

(changing variables $k' = h^{-1}kh$ and using that h is in the compact group J)

$$= \frac{1}{\operatorname{meas}(J)} \int_J \Delta_{(G/K)}(\overline{x}) \int_K \frac{F(xk'h^{-1})}{\Delta_G(xk')} d_\ell k' d_r h$$

$$= \frac{1}{\operatorname{meas}(J)} \int_K \Delta_{(G/K)}(\overline{x}) \int_J \frac{F(xk'h^{-1})}{\Delta_G(xk')} d_r h d_\ell k'$$

$$= \Delta_{(G/K)}(\overline{x}) \int_K \frac{(F \star e_J)(xk')}{\Delta_G(xk')} dk' = \big(I(F \star e_J)\big)(x). \qquad \blacksquare$$

Lemma 4.5 *Suppose that K is a normal subgroup of G, J is an open compact subgroup of G, and I is a locally constant function on G that is bi-K-invariant. (Recall that $\overline{I}$ means I viewed as a function on the quotient (G/K).)*

(i) *The convolution $I \star e_J$ is bi-K-invariant.*

(ii) $\overline{I \star e_J} = \overline{I} \star_{(G/K)} e_{\overline{J}}$.

Proof (i) We have $I \star e_J = x \mapsto \int_J I(xh^{-1}) \frac{d_r h}{\operatorname{meas}(J)}$. So if $k_1, k_2 \in K$, then

$$(I \star e_J)(k_1 x k_2) = \int_J I(k_1 x k_2 h^{-1}) \frac{d_r h}{\operatorname{meas}(J)} = \int_J I(k_1 x k_2 x^{-1} x h^{-1}) \frac{d_r h}{\operatorname{meas}(J)}.$$

The product $k_1 x k_2 x^{-1}$ belongs to K and I is left K-invariant, so $I \star e_J$ is bi-K-invariant.

(ii) The hypothesis J is an open subgroup, so the group KJ is open. The space of bi-K-invariant, right J-invariant functions has basis the characteristics functions 1_{gKJ} of the cosets gKJ of KJ in G. We verify (ii) on these basis elements. We have $1_{gKJ} \star e_J = 1_{gKJ}$ and $\overline{1_{gKJ}} = 1_{\overline{g}\overline{J}} = 1_{\overline{g}\overline{J}} \star_{(G/K)} e_{\overline{J}}$. $\blacksquare$

Proof of Theorem 4.3 As in Section 2, set $D_J := D \star e_J$. By Lemmas 4.4 and 4.5:

$$\overline{I(D_J)} \star_{(G/K)} e_{\overline{J}} = \overline{I(D_J) \star e_J} = I(D_J) \star e_J = I(D_J \star e_J).$$

Set

$$\Phi_{\overline{J}} := \overline{I(D_J \star e_J)}. \tag{4.8}$$

If $L, J \in \mathfrak{J}$ with $L \subset J$, then

$$\Phi_{\overline{L}} \star_{(G/K)} e_{\overline{J}} = \overline{I(D_L \star e_J)} \star_{(G/K)} e_{\overline{J}} = \overline{I(D_L) \star e_J} = I(D_L \star e_J) = I(D_J) = \Phi_{\overline{J}}. \tag{4.9}$$

We conclude that the system $\{\Phi_{\overline{J}} \mid J \in \mathfrak{J}\}$ satisfies (2.6) and thus defines an element $\Phi \in \mathcal{H}\hat{\ }((G/K))$.

Suppose that $f_H \in C_c^\infty(H)$. Take $J \in \mathfrak{J}$ so that $e_J \underset{\{\tau,\ell\}}{\star} C(f_H) = C(f_H)$. Recall

$$F(\tau)(D)(f_H) := \Big((D \star e_J) \underset{\{\tau,\ell\}}{\star} C(f_H) \Big)(1), \quad \text{and}$$

$$F(\iota)(\Phi)(f_H) := \Big((\Phi \star_{(G/K)} e_{\overline{J}}) \underset{\{\iota,\ell\}}{\star} C(f_H) \Big)(1).$$

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We have set $D_J = D \star e_J$, and we have by (4.8) and (4.9) that $\Phi_{\overline{J}} := \overline{I(D_J \star e_J)} = \Phi_{\overline{J}} \star_{(G/K)} e_{\overline{J}}$. By (4.5), we have

$$D_J \underset{\{\tau,\ell\}}{\star} C(f_H) = \overline{I(D_J)} \underset{\{\iota,\ell\}}{\star} C(f_H).$$

Hence, $F(\tau)(D)(f_H) = F(\iota)(\Phi)(f_H)$. As this is true for all f_H, we conclude $F(\tau)(D) = F(\iota)(\Phi)$. ■

We consider a special case when G and K are taken to be a finite dimensional vector space V over F and a subspace $W \subset V$. The group operation in this situation is the vector space addition. In particular, the commutative addition means all the various modular functions are identically one. Let $\tau\colon V \to (V/W)$ denote the quotient map and consider the algebra map $F(\tau)\colon \mathcal{H}\hat{}(V) \to \mathcal{H}\hat{}((V/W))$. For $f \in C_c^\infty(V)$, the operator (4.4), simplifies to

$$I(f) := y \mapsto \int_W f(y+w)dw$$

i.e., integration along the coset $y + W$.

Remark 4.6 Suppose that F is a p-adic field, V is a finite dimensional F vector space, $W \subset V$ is a subspace, and $\tau\colon V \to V/W$ is the quotient map. Then the algebra map $F(\tau)\colon \mathcal{H}\hat{}(V) \to \mathcal{H}\hat{}((V/W))$ has kernel consisting of those essentially compact distributions D on V with the property that for any sufficiently small open compact subgroup J, the integration of $D \star e_J$ along any coset $y + W$ is zero.

5 Some Essentially Compact Distributions on a p-adic Vector Space

Suppose that F is a p-adic field with ring of integers $\mathcal{R}_F$ and prime ideal $\wp$. Let $\mathrm{val}_F\colon F \to \mathbb{Z}$ be the valuation map. The following proposition provides an interesting essentially compact distribution on a p-adic field.

Proposition 5.1 *Suppose that $\psi\colon F \to \mathbb{C}$ is a non-trivial additive character with conductor $\wp^c$,* i.e., *trivial on $\wp^c$, but non-trivial on $\wp^{c-1}$. Set $B\colon F \to F$ to be $B(x) := x^2$.*

(i) *Suppose that $J = \wp^t$ is an ideal of $\mathcal{R}_F$ with $2t \geq c$. Then*

$$(\psi \circ B) \star e_J(y) = \begin{cases} (\psi \circ B)(y) & \text{if } \mathrm{val}_F(y) \geq c - \mathrm{val}_F(2) - t \\ 0 & \text{if } \mathrm{val}_F(y) < c - \mathrm{val}_F(2) - t. \end{cases}$$

In particular, $(\psi \circ B) \star e_J$ is compactly supported.

(ii) *The distribution $D_{\psi\circ B}(f) := \int_F \psi(x^2) f(x)dx$ is essentially compact.*

Proof (i) For the convolution $(\psi \circ B) \star e_J$, we have

$$
\begin{aligned}
(\psi \circ B) \star e_J &= y \mapsto \int_F (\psi \circ B)(y+x) e_J(-x) dx \\
&= y \mapsto \int_{\wp^t} (\psi \circ B)(y+x) \frac{1}{\operatorname{meas}(\wp^t)} dx \\
&= y \mapsto \frac{1}{\operatorname{meas}(\wp^t)} \int_{\wp^t} \psi(y^2 + 2yx + x^2) dx \\
&= y \mapsto \frac{\psi(y^2)}{\operatorname{meas}(\wp^t)} \int_{\wp^t} \psi(2yx) dx, \text{ since } x^2 \in \wp^c.
\end{aligned}
$$

The function $x \mapsto \psi(2yx)$ is a character on $\wp^t$. The integral over $\wp^t$ equals $\operatorname{meas}(\wp^t)$ or vanishes depending on whether this character is non-trivial or trivial, *i.e.*, according to whether $\operatorname{val}_F(2) + \operatorname{val}_F(y) + t$ is greater than, equal to, or less than c.

(ii) For any $f \in C_c^\infty(F)$, if t is sufficiently large and $J = \wp^t$, then $f = e_J \star f$. Assume t is large enough so that the condition $2t \geq c$ is also satisfied. Then $(\psi \circ B) \star e_J$ is compactly supported, but $(\psi \circ B) \star f = (\psi \circ B) \star (e_J \star f) = ((\psi \circ B) \star e_J) \star f$, the convolution of two compactly supported functions. Hence, $(\psi \circ B) \star e_J$ is compactly supported as required. ■

Corollary 5.2 *Suppose that V is a finite dimensional F vector space, $\psi \colon F \to \mathbb{C}$ is non-trivial additive character and $B \colon V \times V \to F$ is a symmetric non-degenerate bilinear form on V. Then the distribution $D_{\psi \circ B}(f) := \int_V \psi(B(x,x)) f(x) dx$ is essentially compact.*

Proof Let $\{v_1, \dots, v_m\}$ be an orthogonal basis for B, so that for $x = x_1 v_1 + \cdots + x_m v_m$ we have

$$
\psi(B(x,x)) = \psi(a_1 x_1^2 + \cdots + a_m x_m^2) = \psi(a_1 x_1^2) \cdots \psi(a_m x_m^2),
$$

i.e., $(\psi \circ B)$ has a factorization into the direct product of single variable functions $x_i \mapsto \psi(a_i x_i^2)$. The hypothesis B is non-degenerate, so all the a_i's are non-zero, so each single variable function is essentially compact. It follows that $(\psi \circ B)$ is essentially compact. ■

Remark 5.3 For the orthogonal basis $\{v_1, \dots, v_m\}$, it follows from Proposition 5.1(i) that if we take $J = \{x_1 v_1 + \cdots + x_m v_m \mid x_i \in \wp^t\}$ with t sufficiently large, then the support of the convolution $(\psi \circ B) \star e_J(y)$ will vanish outside a cartesian product $\{y_1 v_1 + \cdots + y_m v_m \mid y_i \in \wp^{s_i}\}$, for some s_i's, and inside the cartesian product equals $(\psi \circ B)(y)$. In particular, it follows that

$$
PV\left(\int_V \psi\big(B(x,x)\big)\, dx \right) := \text{the principal value of the integral}
$$

exists.

Proposition 5.4 *Suppose that V is a finite dimensional F vector space, $\psi\colon F \to \mathbb{C}$ is non-trivial additive character, and $B\colon V \times V \to F$ is a symmetric non-degenerate bilinear form on V. Suppose also that $W \subset V$ is a subspace of V so that the restriction of B to W is non-degenerate. Let $W^\perp$ denote the perpendicular subspace to W with respect to B, let B_W (resp. $B_{W^\perp}$) be the restriction of B to W (resp. $W^\perp$), and let $\tau\colon V \to W$ be orthogonal projection. Then*

$$F(\tau)(D_{(\psi\circ B)}) = D_{(\psi\circ B_W)} \cdot PV\left(\int_{W^\perp} \psi(B_{W^\perp}(x,x))dx\right).$$

Proof Apply Remark 5.3 and Theorem 4.3. ∎

The following theorem is a generalization of Proposition 5.1 to polynomials of degree at least 2 with a less precise formulation of the support of the convolution.

Theorem 5.5 *Suppose that $\psi\colon F \to \mathbb{C}$ is a non-trivial additive character, and $x \mapsto p(x)$ is a polynomial of degree at least 2. Define a distribution $D_{\{\psi,p\}}$ by*

$$D_{\{\psi,p\}}(f) := \int_F \psi(p(x))f(x)dx.$$

Then $D_{\{\psi,p\}}$ is essentially compact.

Proof Suppose that $p(x) = \sum_{i=k}^{0} a_i x^i$ with $a_k \neq 0$. It is convenient for us to replace $x \mapsto \psi(x)$ by $x \mapsto \psi(a_k x)$ and change p suitably to thereby assume p is monic. If p is quadratic monic, we complete the square to reduce to the case when $p(x) = x^2 + b$ with b constant. Then $x \mapsto \psi(x^2 + b) = \psi(b)\psi(x^2)$ is a non-zero scalar multiple of $x \mapsto \psi(x^2)$, an immediate reduction to Proposition (5.1). So, we assume $k \geq 3$. Take $J = \wp^L$ with $L \geq 0$. We have

$$D_{\{\psi,p\}} \star 1_J = y \mapsto \int_F \psi\bigl(p(y+x)\bigr)\, 1_J(-x)dx = \int_{\wp^L} \psi\bigl(p(y+x)\bigr)\, dx.$$

We wish to find a positive integer K so that if $\mathrm{val}_F(y) = -M \leq -K$, then $D_{\{\psi,p\}} \star 1_J(y) = 0$. We shall pick K with the requirement that $K \geq L$, so $\wp^{(k-2)M} \subset \wp^L$. Then the integral over J is a sum of integrals over the cosets $x_0 + \wp^{(k-2)M}$ of $\wp^{(k-2)M}$ in $\wp^L$. The latter can be written as

$$\begin{aligned}(5.1)\qquad &\int_{x_0+\wp^{(k-2)M}} \psi\bigl(p(y+(x_0+v))\bigr)\, dx \\ &= \int_{\wp^{(k-2)M}} \psi\Bigl(\bigl(y+(x_0+v)\bigr)^k + \sum_{i=(k-1)}^{0} a_i(y+(x_0+v))^i\Bigr)dx \\ &= \int_{\wp^{(k-2)M}} \psi\bigl(n_k(y,x_0)v^0 + n_{k-1}(y,x_0)v^1 + \cdots \\ &\qquad \cdots + n_1(y,x_0)v^{k-1} + n_0(y,x_0)v^k\bigr)\, dx\end{aligned}$$

Here, $n_i(y,x_0)$ is a polynomial in y and x_0 that is a sum of homogeneous polynomials of degrees $i, (i-1), \ldots 2, 1, 0$ in the two variables. Furthermore:

(i) The degree k homogeneous part of $n_k(y, x_0)$ equals $y^k + \cdots$

(ii) The degree $(k-1)$ homogeneous part of $n_{k-1}(y, x_0)$ equals $ky^{(k-1)} + \cdots$

In particular, the highest power of y that occurs in $n_i(y, x_0)$ is at most i. Thus, for K (with a dependence on L) sufficiently large, if $\mathrm{val}_F(y) = -M \leq -K$, then for any $x_0 \in \wp^L$ we have

$$\mathrm{val}_F\big(n_{k-j}(y, x_0)\big) \geq -(k-j)M.$$

Since $v \in \wp^{(k-2)M}$, we conclude that

$$\mathrm{val}_F\big(n_{k-j}(y, x_0)v^j\big) \geq -(k-j)M + (k-2)Mj = \big(-j + (j-1)k\big)M.$$

The latter is greater than c, and so $\psi(n_{k-j}(y, x_0)v^j) = 1$ under the assumption $j \geq 2$, $k \geq 3$, and $M \geq c$. The coset integral (5.1) becomes

$$\int_{x_0 + \wp^{(k-2)M}} \psi\big(p(y + (x_0 + v))\big)\,dx = \psi\big(n_k(y, x_0)\big) \int_{\wp^{(k-2)M}} \psi\big(n_{k-1}(y, x_0)v\big)\,dx.$$

Since the degree $(k-1)$ homogeneous part of $n_{k-1}(y, x_0)$ is $ky^{k-1} + \cdots$, it follows that for K sufficiently large we can arrange $\mathrm{val}_F(n_{k-1}(y, x_0))$ equal to $\mathrm{val}_F(k) - (k-1)M$, and so for $v \in \wp^{(k-2)M}$ we have $\mathrm{val}_F(n_{k-1}(y, x_0)v) \geq \mathrm{val}_F(k) - M$. Equality occurs when $\mathrm{val}_F(v) = (k-2)M$. Clearly, we can make the $\mathrm{val}_F(k) - M$ strictly less than c, and therefore the function $v \mapsto \psi(n_{k-1}(y, x_0)v)$, which is a character on $\wp^{(k-2)M}$, is a non-trivial character. In particular, the integral over the coset $x_0 + \wp^{(k-2)M}$ is zero. It follows that $D_{\{\psi,p\}} \star 1_J(y)$ vanishes for y sufficiently large, and so $D_{\{\psi,p\}}$ is essentially compact. ■

Remark 5.6 If p is a linear or constant polynomial, the distribution

$$f \mapsto D_{\{\psi,p\}}(f) := \int_F \psi\big(p(x)\big) f(x)dx$$

is easily seen to be not essentially compact.

6 Invariant Essentially Compact Distributions on a p-adic Lie Algebra

If $\mathbb{G}$ is a connected F-group, then its Lie algebra $\mathrm{Lie}(\mathbb{G})$ is an F-variety. Set $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})(F)$ to be the Lie algebra of F-rational points. The commutative addition of the F vector space $\mathfrak{g}$ means the notions of left and right essentially compact distributions are the same, *i.e.*, $\mathcal{H}^{\wedge}(\mathfrak{g}) = {}^{\wedge}\mathcal{H}(\mathfrak{g})$. For such a distribution D, we have $D \star f = f \star D$ for all $f \in C_c^\infty(\mathfrak{g})$, and if D_1, D_2 are essentially compact distributions on $\mathfrak{g}$, their convolutions products in $\mathcal{H}^{\wedge}\mathfrak{g})$ and ${}^{\wedge}\mathcal{H}(\mathfrak{g})$ are equal and commutative. We use the notation $\mathcal{D}(\mathfrak{g})$ for this distribution algebra. We view it as an analogue of the algebra of constant coefficient differential operators on the Lie algebra of a Lie group.

Let $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ denote the Adjoint map, so the adjoint action of G on $C_c^\infty(\mathfrak{g})$ is $g.f := x \mapsto f(\mathrm{Ad}(g^{-1})(x)$. Define the adjoint action of G on the space of distributions on $\mathfrak{g}$ by $(g.D)(f) := D(g^{-1}.f)$. We have $(g.D \star g.f) = g.(D \star f)$.

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Definition 6.1 A distribution D on $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})(F)$ is G-invariant if $(g.D) = D$ for all $g \in G$, *i.e.*, $D(f) = D(g^{-1}.f)$ for all $f \in C_c^\infty(\mathfrak{g})$ and $g \in G$. Let $\mathcal{D}(\mathfrak{g})^G$ denote the space of G-invariant essentially compact distributions on $\mathfrak{g}$.

Let dy be a Haar measure on $\mathfrak{g}$. We have

$$\forall f \in C_c^\infty(\mathfrak{g}), \qquad \int_{\mathfrak{g}} f(y)dy = \Delta_G(g)^{-1} \int_{\mathfrak{g}} f(\mathrm{Ad}(g)(y))dy.$$

In particular, if G is unimodular, then $\int_{\mathfrak{g}} f(y)dy = \int_{\mathfrak{g}} f(\mathrm{Ad}(g)(y))dy$.

Suppose that $\mathbb{G}$ is a connected reductive F-group and $\mathfrak{g} \times \mathfrak{g} \xrightarrow{B} F$ is a symmetric non-degenerate Ad-invariant (*i.e.,* $B(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y) = B(x, y)$) bilinear form on $\mathfrak{g}$. Suppose that $\mathbb{M}$ is an F-subgroup of $\mathbb{G}$ with the property that the restriction of $B(\cdot\,,\,\cdot)$ to $\mathfrak{m} := \mathrm{Lie}(\mathbb{M})(F)$ is non-degenerate. A typical example is when $\mathbb{M}$ is a Levi subgroup. That B is non-degenerate on $\mathfrak{m}$ means $M = \mathbb{M}(F)$ is unimodular. Set

$$\mathfrak{m}^\perp := \{x \in \mathfrak{g} \mid B(x, \mathfrak{m}) = 0\} \quad \text{so that } \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^\perp,$$

and for the latter direct sum let $\tau\colon \mathfrak{g} \to \mathfrak{m}$ be the projection. The map τ gives an algebra homomorphism $F(\tau)\colon \mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{m})$. Both $\mathfrak{m}$ and $\mathfrak{m}^\perp$ are Ad-invariant under the group

$$N_G(M) := \text{normalizer in } G \text{ of } M,$$

so there are Adjoint actions of $N_G(M)$ on $C_c^\infty(\mathfrak{m})$ and $\mathcal{D}(\mathfrak{m})$.

Theorem 6.2 *Suppose that $\mathbb{G}$ is a connected reductive F-group and B is a symmetric non-degenerate* Ad*-invariant* (i.e., $B(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y) = B(x, y)$) *bilinear form on $\mathfrak{g}$. Suppose that further that $\mathbb{M}$ is a connected F-subgroup of $\mathbb{G}$ so that the restriction of B to* $\mathfrak{m} := \mathrm{Lie}(\mathbb{M})(F)$ *is non-degenerate. Then the map $F(\tau)\colon \mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{m})$ takes $\mathcal{D}(\mathfrak{g})^G$ to $\mathcal{D}(\mathfrak{m})^{N_G(M)}$.*

Proof Suppose that $D \in \mathcal{D}(\mathfrak{g})^G$ and $n \in N_G(M)$. Fix a fundamental system of open compact subgroups $\mathcal{J}$ of $\mathfrak{g}$, and consider the system of compatible functions $\{D \star e_J \mid J \in \mathcal{J}\}$. Obviously, $\{\mathrm{Ad}(n)J \mid J \in \mathcal{J}\}$ is also a fundamental system of open compact subgroups of $\mathfrak{g}$, and the distribution D can be fully recovered from the compatible systems of functions $\{D \star e_{\mathrm{Ad}(n)J} \mid J \in \mathcal{J}\}$. The function $I(D \star e_J)$ equals

$$x \mapsto \int_{\mathfrak{m}^\perp} (D \star e_J)(x + v)dv.$$

Viewed as functions on $\mathfrak{m} = (\mathfrak{g}\mathfrak{m}^\perp)$, the functions $I(D \star e_J)$ are a compatible system of functions for the fundamental system of open compact subgroups $\{\overline{J} \mid J \in \mathcal{J}\}$ of $\mathfrak{m}$. The essentially compact distribution Φ that they determine is by definition $I(D)$. The functions $n.I(D \star e_J)$ become a fundamental system of functions for the distribution $n.I(D)$ with respect to the fundamental system of open compact subgroups $\{n.\overline{J} \mid J \in \mathcal{J}\}$ of $\mathfrak{m}$. Now, for $x \in \mathfrak{m} \subset \mathfrak{g}$, we have

$$\begin{aligned} n.I(D \star e_J)(x) &= I(D \star e_J)(\mathrm{Ad}(n^{-1})x) \\ &= \int_{\mathfrak{m}^\perp} (D \star e_J)(\mathrm{Ad}(n^{-1})x + v)dv \\ &= \int_{\mathfrak{m}^\perp} (D \star e_J)(\mathrm{Ad}(n^{-1})\,(x + v))dv \\ &= I(n.(D \star e_J))(x). \end{aligned}$$

Key in this is that Haar measure on $\mathfrak{m}^\perp$ is $\mathrm{Ad}(N_G(M))$-invariant. Since $I(n.(D\star e_J)) = I(n.D \star n.e_J)$, and by hypothesis, D is $\mathrm{Ad}(G)$-invariant, we have $I(n.(D \star e_J)) = I(D\star n.e_J)$. We interpret the functions $I(D\star n.e_J)$ as a compatible system of functions for $I(D)$ with respect to the the fundamental system $\{n.\overline{J} \mid J \in \mathcal{J}\}$ of open compact subgroups of $\mathfrak{m}$. Whence, $n.I(D) = I(D)$. ■

Theorem 6.3 *Suppose that $\psi\colon F \to \mathbb{C}$ is a non-trivial additive character, $\mathbb{G}$ is a connected reductive F-group, $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})(F)$, and B is a symmetric non-degenerate bilinear form on $\mathfrak{g}$,* Ad*-invariant (*i.e., $B(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y) = B(x, y)$*).*

(i) *The distribution $D_{\psi\circ B}(f) := \int_{\mathfrak{g}} \psi(B(x,x)) f(x)dx$ belongs to $\mathcal{D}(\mathfrak{g})^{\mathrm{Ad}(G)}$.*

(ii) *Suppose that $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}$ such that the restriction $B_{\mathfrak{m}}$ of B to $\mathfrak{m}$ is non-degenerate. Write $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^\perp$, and let $\tau\colon \mathfrak{g} \to \mathfrak{m}$ be the orthogonal projection to $\mathfrak{m}$. Then*

$$F(\tau)(D_{\psi\circ B}) = D_{\psi\circ B_{\mathfrak{m}}} \cdot PV\left(\int_{\mathfrak{m}^\perp} \psi(B(x,x))dx\right).$$

Proof Apply Proposition 5.4. ■

7 The Group $SL(2)$

In this section we consider $G = \mathrm{SL}(2)(F)$ and show under a mild condition that the essentially compact distribution $D_{\psi\circ B}$ on the lie algebra $\mathfrak{g} = \mathfrak{sl}(2)(F)$, when restricted to the set of topologically nilpotent elements, transfers by the Cayley transform to a G-invariant essentially compact distribution on G. We first review some facts. Recall that if $B(\,\cdot\,,\,\cdot\,)$ is the Killing form on $\mathfrak{sl}(2)(F)$, then $B(x,x) = -8\det(x)$. By (6.3), for any non-trivial additive character ψ of F, the G-invariant distribution $D_{\psi\circ\det}(f) := \int_{\mathfrak{g}} f(x)\psi(\det(x))dx$ is essentially compact. Let I denote the identity matrix in G. If $x \in \mathfrak{g}$ and $\mathrm{I} - x$ is invertible, the Cayley transform $C(x) \in G$ is defined as

$$C(x) := \frac{\mathrm{I} + x}{\mathrm{I} - x}. \tag{7.1}$$

The relation between the determinant $d = \det(x)$ of x and the trace $s = \mathrm{trace}(C(x))$ of $C(x)$ is

$$d = \frac{2-s}{2+s}, \quad s = 2\frac{1-d}{1+d}.$$

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The set of topologically nilpotent elements $\mathcal{E}_{\rm nil}$ of $\mathfrak{gl}(2)(F)$ (resp. $\mathcal{T}_{\rm nil}$ of $\mathfrak{g}$) and the set of topologically unipotent elements $\mathcal{E}_{\rm uni}$ of $\mathrm{GL}(2)(F)$ (resp. $\mathcal{T}_{\rm uni}$ of G) is defined as

$$\begin{aligned}\mathcal{E}_{\rm nil} &:= \{x \in \mathfrak{gl}(2)(F) \mid x^{p^r} \to 0 \text{as} r \to \infty\}\\ &= \{x = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right] \in \mathfrak{gl}(2)(F) \mid \mathrm{trace}(x), \text{and} \det(x) \in \wp\}\\ \mathcal{T}_{\rm nil} &:= \mathfrak{g} \cap \mathcal{E}_{\rm nil}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_{\rm uni} &:= \{y \in \mathrm{GL}(2)(F) \mid y^{p^r} \to \mathrm{I} \text{ as } r \to \infty\}\\ &= \{y = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right] \in \mathrm{GL}(2)(F) \mid \mathrm{trace}(y) - 2, \text{and } \det(y) - 1 \in \wp\}\\ \mathcal{T}_{\rm uni} &:= G \cap \mathcal{E}_{\rm uni}.\end{aligned}$$

For $x \in \mathcal{T}_{\rm nil}$, the Cayley transform $C(x)$ has the convergent series:

$$C(x) = \mathrm{I} + 2x + 2x^2 + \cdots .$$

Moreover, C maps $\mathcal{T}_{\rm nil}$ bijectively with the set $\left(G \cap (\mathrm{I} + 2\mathcal{E}_{\rm nil})\right)$. Let C^{-1} denote the inverse function. For $\left(G \cap (\mathrm{I} + 2\mathcal{E}_{\rm nil})\right)$, the inverse $C^{-1}(y)$ is given by the convergent series:

$$C^{-1}(y) = \Big(\frac{y-\mathrm{I}}{2}\Big) - \Big(\frac{y-\mathrm{I}}{2}\Big)^2 + \Big(\frac{y-\mathrm{I}}{2}\Big)^3 - \cdots .$$

We recall that when ψ is a non-trivial additive character of F, the conductoral exponent $f(\psi)$ is defined as the smallest integer so that ψ is trivial on $\wp^{f(\psi)}$.

Theorem 7.1 *Suppose that the conductoral exponent $f(\psi)$ of $\psi\colon F \to \mathbb{C}$ satisfies $f(\psi) \geq 2$. Then the following hold:*

(i) *The restriction of the G-invariant essentially compact distribution $D_{\psi\circ\det}$ to the topologically nilpotent set $\mathcal{T}_{\rm nil}$ remains a G-invariant essentially compact distribution.*

(ii) *Suppose that the residue characteristic p is odd (so that $2\mathcal{E}_{\rm nil} = \mathcal{E}_{\rm nil}$). Let $D = D_{\psi\circ\det\circ C^{-1}}$ be the G-invariant distribution with support $(G \cap (\mathrm{I} + 2\mathcal{E}_{\rm nil}))$, given as integration against the G-invariant function*

$$y \mapsto \psi(\det(C^{-1}(y))) = \psi\Big(\frac{2 - \mathrm{trace}(y)}{2 + \mathrm{trace}(y)}\Big).$$

Then D is essentially compact.

Proof (i) For a positive integer r, let $\mathcal{K}_r \subset \mathfrak{g}$ denote the open compact congruence lattice consisting of those matrices whose entries belong to $\wp^r$. Then

$$\text{(7.2)}\quad \big(\big(\psi \circ \det\big)\big|_{\mathcal{T}_{\rm nil}}\big) \star e_{\mathcal{K}_r} = x \mapsto \frac{1}{\mathrm{meas}(\mathcal{K}_r)} \int_{\mathcal{K}_r} \psi(\det(x+k)) 1_{\mathcal{T}_{\rm nil}}(x+k)dk$$

$$= \left[\begin{smallmatrix} A & B\\ C & -A\end{smallmatrix}\right] \mapsto \frac{1}{\mathrm{meas}(\mathcal{K}_r)} \int_{(\wp^r\times\wp^r\times\wp^r)'} \psi\big(-(A+a)^2 - (B+b)(C+c)\big)\, dadbdc.$$

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The integral is over the set $(\wp^r \times \wp^r \times \wp^r)'$ of $(a,b,c) \in (\wp^r \times \wp^r \times \wp^r)$ satisfying $((A+a)^2 + (B+b)(C+c)) \in \wp$. We need to show the integral in (7.2) vanishes when A, B, C is sufficiently large. Clearly, we need only show this vanishing under the condition $2r \geq f(\psi)$.

We observe that the function $((\psi \circ \det)|_{\mathcal{T}_{\rm nil}}) \star e_{\mathcal{K}_r}$ is $\mathcal{K}_r$-spherical. This observation and the fact that the support of $((\psi \circ \det)|_{\mathcal{T}_{\rm nil}})$ is in the topologically nilpotent set allows us to deduce the convolution is completely determined by its restriction to the topologically nilpotent set. So, we may assume $A^2 + BC \in \wp$ and at least one of A, B, or C is large. Under this assumption, for $(a,b,c) \in (\wp^r \times \wp^r \times \wp^r)$ the condition $((A+a)^2 + (B+b)(C+c)) \in \wp$ becomes the linear condition $2Aa + Bc + Cb \in \wp$. This linear condition specifies a $\mathcal{R}_F$-submodule $\mathcal{L}$ of $(\wp^r \times \wp^r \times \wp^r)$. Furthermore, $((A+a)^2 + (B+b)(C+c)) - (A^2 + BC) = (2Aa + Bc + Cb) + (a^2 + bc)$, and therefore, since $2r \geq f(\psi)$, we have

$$\psi\big(-(A+a)^2 - (B+b)(C+c)\big) = \psi\big(-(A^2+BC)\big)\,\psi(-(2Aa + Bc + Cb)).$$

Provided one of A, B, or C is sufficiently large, the function

$$(a,b,c) \mapsto \psi(-(2Aa + Bc + Cb))$$

is a non-trivial character on the lattice $\mathcal{L}$. Thus the integral (7.2) is zero.

(ii) For a positive integer r, let $K_r \subset G$ denote the r-th congruence subgroup. The proof is similar to the proof of part (i), *e.g.*, the convolution $D \star e_{K_r}$ is K_r-spherical and determined by its values on the topologically unipotent set. Write the variables $y \in G$ and $k \in K_r$ as $y = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and $k = \mathrm{I} + \left[\begin{smallmatrix} x_a & x_b \\ x_c & x_d \end{smallmatrix}\right]$, where $x_a, x_b, x_c, x_d \in \wp^r$. We have

$$\operatorname{trace}(yk) = (A+D) + (Ax_a + Bx_c + Cx_b + Dx_d).$$

So, if y is topologically unipotent, *i.e.*, $(A+D) - 2 \in \wp$, then yk is topologically unipotent precisely when $(Ax_a + Bx_c + Cx_b + Dx_d) \in \wp$. Set

$$(K_r)' = \{k \in K_r \mid (Ax_a + Bx_c + Cx_b + Dx_d) \in \wp\}.$$

We remark that if we replace x_a by $x_\alpha + (1+x_\alpha)x_b x_c$ and we replace $1 + x_d$ by $\frac{1}{1+x_\alpha}$, *i.e.*, $x_d = -x_\alpha + x_\alpha^2 - x_\alpha^3 + \cdots$, then any $k \in K_r$ is parametrized by a unique triple $x_\alpha, x_b, x_c \in \wp^r$. Denote the inverse of the Cayley transform (7.1) as γ. Then

(7.3)

$$\begin{aligned} &D \star e_{K_r} \\ &= y \mapsto \frac{1}{\operatorname{meas}(K_r)} \int_{K_r} \psi\big(\det(\gamma(yk))\big)\, 1_{\mathcal{T}_{\rm nil}}\big(\gamma(yk)\big)\, dk \\ &= y \mapsto \frac{1}{\operatorname{meas}(K_r)} \int_{K_r} \psi\Big(\frac{2 - \operatorname{trace}(yk)}{2 + \operatorname{trace}(yk)}\Big)\; 1_{\mathcal{T}_{\rm uni}}(yk) dk \\ &= \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \mapsto \frac{1}{\operatorname{meas}(K_r)} \int_{(K_r)'} \psi\Big(\frac{2 - (A+D) + (Ax_a + Bx_c + Cx_b + Dx_d)}{2 + (A+D) + (Ax_a + Bx_c + Cx_b + Dx_d)}\Big)\, dk. \end{aligned}$$

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The assumptions $\mathrm{trace}(y), \mathrm{trace}(yk) \in 2 + \wp$, and $p \neq 2$ mean the denominator in the last line of (7.3) is a unit. The assumption that y is topologically unipotent, *i.e.*, $A + D - 2 \in \wp$ also means that when y is sufficiently large, that either B or C has the maximal absolute value among A, B, C, D. By symmetry we assume B has maximal absolute value. Let $U \subset K_r$ be the root subgroup satisfying the conditions $x_\alpha = x_b = 0$, and $Bx_c \in \wp^{f(\psi)-1}$. Then $(K_r)'$ is right U-invariant. When B is sufficiently large, the restriction of the integrand in the final line of (7.3) to a coset kU inside $(K_r)'$ transforms by a nontrivial character of U. Whence the integral over kK_r is zero and so the integral in (7.3) vanishes. So D is essentially compact. ■

We now compute the action of the distribution $D = D_{\psi \circ \det \circ \gamma}$ on an irreducible cuspidal representation (π, V). We formulate the initial aspects in the more general context of an invariant essentially compact distribution D on a p-adic group H and an irreducible smooth representation (τ, W) of H. By Schur's Lemma, $\tau(D)$ is a scalar operator on (τ, W). For convenience we also let $\tau(D)$ denote the scalar by which $\tau(D)$ acts on W. Suppose that J is an open compact subgroup of H and $w \in W$ generates a J-irreducible representation σ inside W. Let χ_σ denote the character of σ, and let e_σ the idempotent in $C_c^\infty(H)$ equal $\frac{\deg(\sigma)}{\mathrm{meas}(J)}\overline{\chi}_\sigma$ on J and zero off J. The convolution $D \star e_\sigma$ belongs to the center of the algebra $e_\sigma \star C_c^\infty(H) \star e_\sigma$.

We recall that when H is a semisimple p-adic group, an irreducible cuspidal representation (τ, W) of H is induced from an open compact subgroup J if there is a $w \in W$ that generates a J-irreducible representation σ satisfying $e_\sigma \star C_c^\infty(H) \star e_\sigma = \mathbb{C}e_\sigma$. In particular, since $D \star e_\sigma$ is always in $e_\sigma \star C_c^\infty(H) \star e_\sigma$, the convolution $D \star e_\sigma$ is a scalar multiple of e_σ. The scalar multiple is precisely the scalar $\tau(D)$, *i.e.*, $D \star e_\sigma = \tau(D)e_\sigma$. In other words, the idempotent e_σ, under convolution against elements of the Bernstein center, is an eigenfunction. Suppose now that the invariant essentially compact distribution D is represented as integration against the function ϕ. We have

$$(7.4) \qquad \phi \star e_\sigma(1) = \int_H \phi(1h)\, e_\sigma(h^{-1})dh = \frac{\deg(\sigma)}{\mathrm{meas}(J)} \int_J \phi(k)\overline{\chi}_\sigma(k^{-1})dk.$$

We now specialize to the case when $H = \mathrm{SL}(2)(F)$ and $D = D_{\psi \circ \det \circ \gamma}$. In particular, D is represented on the set of topologically unipotent elements by the function

$$\phi := g \mapsto \psi\Big(\frac{2 - \mathrm{trace}(g)}{2 + \mathrm{trace}(g)}\Big).$$

Depending on the parity of $f(\psi)$, write $f(\psi)$ as $2t$ or $2t + 1$ with $t \geq 1$. Set s to be t (resp. $t + 1$) when $f(\psi)$ is even (resp. odd). Let $\rho(\pi)$ denote the depth of an irreducible cuspidal representation (π, V) of G. We recall $\rho(\pi) \in \frac{1}{2}\mathbb{N}$. We consider the three cases: (i) $\rho(\pi) = 0$, (ii) $\rho(\pi)$ is a positive integer, and (iii) $\rho(\pi)$ has the form $\frac{1}{2} + n$, with $n \in \mathbb{N}$.

(i) Case $\rho(\pi) = 0$.

Let $\mathbb{F}_q$ denote the finite field $\mathcal{R}_F/\wp$. The two conjugacy classes of maximal compact subgroups of G become one orbit when we allow conjugation by elements of $\mathrm{GL}(2)(F)$. This means it suffices to consider the situation when (π, V) is induced from a cuspidal representation σ of $\mathrm{SL}(2)(\mathbb{F}_q)$ inflated to $K = \mathrm{SL}(2)(\mathcal{R}_F)$. Let K_i denote the i-th congruence subgroup of K. The integral (7.4) is then a sum of integrals over cosets $kK_s \subset K$ of K_s so that the element $kK_1 \in K/K_1$ is unipotent. The character χ_σ is constant on the cosets of K_1 and hence on those of K_s. The restriction of ϕ to K_t factors to a function on K_t/K_s. The integral over a coset kK_s is

$$\begin{aligned}\int_{kK_s} \phi(x)\overline{\chi}_\sigma(x^{-1})dx &= \int_{K_s} \phi(kv)\chi_\sigma(kv)dv = \chi_\sigma(k)\int_{K_s}\phi(kv)\,dv \\ &= \begin{cases} 0 & \text{if } k \notin K_t \\ \deg(\sigma)\phi(k)\operatorname{meas}(K_s) & \text{if } k \in K_t. \end{cases}\end{aligned} \tag{7.5}$$

The vanishing when $k \notin K_t$ is because at least one entry of the matrix $k - \mathrm{I}$ is not in $\wp^t$. Let U_s be the intersection of K_s with the opposite root space group to this entry. The function $v \mapsto \phi(kv)$ transforms by a non-trivial character of U_s, from which we immediately deduce vanishing of the integral. In regards to the assertion when $k \in K_t$, we note that in this situation, the function $v \mapsto \phi(kv)$ is constant on K_s, and so the integral is as stated. Set

$$G(\psi) := \begin{cases} 1 & \text{if } f(\psi) = 2t \\ \displaystyle\sum_{kK_s \in K_t/K_s} \phi(k) & \text{if } f(\psi) = 2t+1 \text{ (a Gauss sum).} \end{cases} \tag{7.6}$$

Then

$$\phi \star e_\sigma = \deg(\sigma)G(\psi)\operatorname{meas}(K_s)e_\sigma. \tag{7.7}$$

(ii) Case $r = \rho(\pi)$ positive integral.

As in case (i), we can assume π is induced from an irreducible representation of $\mathrm{SL}(2)(\mathcal{R}_F)$. We choose a non-zero $v \in V_\pi^{K_{\rho(\pi)+1}}$ so that v transforms by a character χ of $K_{\rho(\pi)}/K_{\rho(\pi)+1}$.

(a) *Subcase* $s \geq (r+1)$. As in case (i), the integral (7.4) is a sum over cosets of K_s. We again have (7.5), (7.6), and (7.7).

(b) *Subcase* $s < (r+1)$. We note that $T := \max(0, f(\psi) - (r+1)) \leq r$. We write the integral (7.4) as a sum over cosets of K_{r+1}. As in (7.5), we have

$$\begin{aligned}\int_{kK_{r+1}} \phi(x)\overline{\chi}_\sigma(x^{-1})dx &= \int_{K_{r+1}} \phi(kv)\chi_\sigma(kv)dv = \chi_\sigma(k)\int_{K_{r+1}}\phi(kv)dv \\ &= \begin{cases} 0 & \text{if } k \notin K_T \\ \deg(\sigma)\chi_\sigma(k)\operatorname{meas}(K_{r+1}) & \text{if } k \in K_T. \end{cases}\end{aligned} \tag{7.8}$$

The coset integrals (7.8) need to be summed over $k \in K_T/K_{r+1}$. Since χ_σ is a non-trivial character of K_T/K_{r+1}, we see $\phi \star e_\sigma = 0\, e_\sigma$.

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(iii) Case $r = \rho(\pi) = \frac{1}{2} + m$ half-integral.

Let $B \subset K$ denote the Iwahori subgroup, which modulo K_1 is the Borel subgroup of upper triangular matrices. for $i \in \frac{1}{2}\mathbb{N}$, let B_i denote the standard i-th filtration subgroup of B. We have $\rho(\pi) + \frac{1}{2} = m + 1$. We choose a non-zero $v \in V_\pi^{B_{m+1}}$ so that v transforms by a character χ of $K_{\rho(\pi)}/K_{\rho(\pi)+\frac{1}{2}}$.

(a) *Subcase $t \geq m + 1$.* We decompose the integral (7.4) into a sum of integrals over the cosets of $B_t \subset B_{m+1}$. In particular, σ is trivial on B_t, and so the integral over a coset kB_t becomes

$$\int_{kB_t} \phi(x)\overline{\chi}_\sigma(x^{-1})dx = \int_{B_t} \phi(kv)\chi_\sigma(kv)dv = \chi_\sigma(k) \int_{B_t} \phi(kv)dv$$
$$= \begin{cases} 0 & \text{if } k \notin B_t \\ \deg(\sigma) \int_{B_t} \phi(v)\, dv & \text{if } k \in B_t. \end{cases}$$

The integral $\int_{B_t} \phi(v)\, dv$ can easily be written as $\text{meas}(B_{t+\frac{1}{2}})$ times a Gauss sum over the group $B_t/B_{t+\frac{1}{2}}$ of order q. Let $G(\psi)$ denote this Gauss sum. Then

$$\phi \star e_\sigma = \deg(\sigma)G(\psi)\, \text{meas}(B_{t+\frac{1}{2}})e_\sigma.$$

(b) *Subcase $t < m + 1$.* We decompose the integral (7.4) into a sum of integrals over the cosets of B_{m+1}. In particular, σ is trivial on B_{m+1}, and so the integral over a coset kB_{m+1} becomes

$$\int_{kB_{m+1}} \phi(x)\overline{\chi}_\sigma(x^{-1})dx = \int_{B_{m+1}} \phi(kv)\chi_\sigma(kv)dv$$
$$= \chi_\sigma(k) \int_{B_{m+1}} \phi(kv)dv.$$

We then argue as in case(ii)(b) to find $\phi \star e_\sigma = 0\, e_\sigma$.

We summarize the above in the following theorem.

Theorem 7.2 *Suppose that p is odd and $H = \mathrm{SL}(2)(F)$. Suppose that ψ is an additive character of F with conductoral exponent $f(\psi) \geq 2$. Write $f(\psi)$ as $f(\psi) = t + s$ with $s = t$ or $s = t + 1$. The scalar by which the distribution $D = D_{\psi\circ\det\circ C^{-1}}$ acts on an irreducible cuspidal representation (π, V_π) is given by the following:*

(i) *If $\rho(\pi) = 0$, then the scalar is $\deg(\sigma)G(\psi)\,\text{meas}(K_s)$.*

(ii) *If $\rho(\pi) = r$ is integral, then the scalar is $\deg(\sigma)G(\psi)\,\text{meas}(K_s)$ for $s \geq \rho(\pi) + 1$ and zero if $s < \rho(\pi) + 1$.*

(iii) *If $\rho(\pi) = m + \frac{1}{2}$ is half-integral, then the scalar is $\deg(\sigma)G(\psi)\,\text{meas}(B_{t+\frac{1}{2}})$ for $t \geq \rho(\pi) + \frac{1}{2}$ and zero if $t < \rho(\pi) + \frac{1}{2}$.*

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