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BOOLEAN CONGRUENCE LATTICES OF ORTHODOX SEMIGROUPS

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1. **Introduction.** The problem of characterizing the semigroups with Boolean congruence lattices has been solved for several classes of semigroups. Hamilton [9] and the author of this paper [1] studied the question for semilattices. Hamilton and Nordahl [10] considered commutative semigroups, Fountain and Lockley [7,8] solved the problem for Clifford semigroups and idempotent semigroups, in [1] the author generalized their results to completely regular semigroups. Finally, Zhitomirskiy [19] studied the question for inverse semigroups. For a collection of various results concerning the relations between the structure of a semigroup and the structure of its congruence lattice consult Mitsch [13]. In [2] the author proved that in general the problem can be treated in a similar way for the following lattice properties: sectionally complemented, relatively complemented, complemented modular, Boolean, respectively, and it can be shown that the same is true for the property dually sectionally complemented. The aim of this paper is to give a structural characterization of those orthodox semigroups S whose congruence lattice $\mathcal{C}(S)$ has one of these five properties, which we shortly denote by P. In Section 2 we shall give some basic definitions and then prepare a few results about subdirect products of rectangular groups and fundamental inverse semigroups. In Section 3 we shall prove that for an orthodox semigroup S its congruence lattice $\mathcal{C}(S)$ has the property P if and only if

(i) S is a subdirect product of a rectangular group $B = I \times G \times \Lambda$ and a fundamental inverse semigroup A such that $\mathcal{C}(B)$ and $\mathcal{C}(A)$ have property P and

(ii) each congruence ρ on S is the direct product of the congruences ρ_A and ρ_B, respectively, denoting the congruences on A and B which are induced by ρ (Theorem 1).

This description raises the problem to characterize all subdirect products S of B and A such that each congruence on S can be written as the direct product of a congruence on B and a congruence on A. Such a characterization is given in Section 4 in terms of kernels of idempotent separating congruences on S and certain rectangular sub-bands of S (Theorem 2). In Section 5 we study the problem for fundamental inverse semigroups and prove that for this class only the case *Boolean* of property P is possible. We conclude that for an orthodox semigroup S whose congruence lattice $\mathcal{C}(S)$ is P, $\mathcal{C}(S)$ is at least relatively complemented. Finally, in Section 6 we use a characterization of general semigroups

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whose congruence lattice is P, given by the author in [2], to obtain a more detailed structural description of the semigroups under study. In the main Theorem of the paper this characterization then is given in form of a list of seven classes of orthodox semigroups which do have the desired property (Theorem 4). Furthermore, for such semigroups the congruence lattice can be decomposed in a direct product of certain lattices in a natural way.

2. **Preliminaries.** Throughout the paper, the congruence lattice of a semigroup S is denoted by $(\mathcal{C}(S), \vee, \cap)$. The universal and identity relations on S are denoted by $\omega = \omega_S$ and $\varepsilon = \varepsilon_S$, respectively. For an arbitrary set X, $\mathcal{E}(X)$ is the lattice of all equivalence relations on X and $\mathcal{P}(X)$ is the power set lattice of X. A lattice L with a least element ε is *sectionally complemented* if each interval $[\varepsilon, \xi]$ in L is a complemented lattice. A lattice L with a greatest element ω is *dually sectionally complemented* if each interval of the form $[\xi, \omega]$ in L is a complemented lattice. A lattice L is *relatively complemented* if each interval $[\xi, \eta]$ in L is a complemented lattice. It is well known that the following implications hold: Boolean $\Rightarrow$ complemented modular $\Rightarrow$ relatively complemented $\Rightarrow$ (dually) sectionally complemented.

For an arbitrary semigroup S, $E(S)$ denotes the set of idempotents of S and $\leq$ is the natural partial order on $E(S)$, that is, $e \leq f$ if and only if $e = ef = fe$ for $e,f \in E(S)$. In order to avoid confusion the symbol "$\leq$" is used for this order on $E(S)$ whereas "$\subseteq$" stands for the inclusion relation on $\mathcal{C}(S)$. Also by "$\leq$" we shall denote the natural partial order on an inverse semigroup A which is defined by $a \leq b$ if and only if $a = eb$ for some $e \in E(A)$ and $a, b \in A$ (see [17]). For $x \in S$, $V(x)$ is the set of all inverses of x in S, that is, $V(x) = \{y \in S \mid x = xyx \text{ and } y = yxy\}$. In what follows let S be a regular semigroup. We shall need the following notions and results due to Pastijn and Petrich (see [15,16]): For $\rho, \theta \in \mathcal{C}(S)$ let

$$\rho\ U\ \theta \Leftrightarrow \rho \cap \leq = \theta \cap \leq,$$
$$\rho\ T\ \theta \Leftrightarrow \rho|E(S) = \theta|E(S),$$
$$\rho\ V\ \theta \Leftrightarrow \rho\ U\ \theta \text{ and } \ker\rho = \ker\theta$$

where $\ker\rho = \bigcup\{e\rho : e \in E(S)\}$. Then U, T, V are complete congruences on the lattice $\mathcal{C}(S)$. The congruence classes $\rho U(\rho T, \rho V)$ have a greatest and a least element to be denoted by $\rho^U(\rho^T, \rho^V)$ and $\rho_U(\rho_T, \rho_V)$, respectively. Let $\mathcal{K}$ be a class of semigroups and $\rho \in \mathcal{C}(S)$. Then ρ *is over* $\mathcal{K}$ if $e\rho \in \mathcal{K}$ for all $e \in E(S)$. Using this concept the relations U, T, V can be characterized in the following way:

$$\rho\ U\ \theta \Leftrightarrow \rho/\rho \cap \theta \text{ and } \theta/\rho \cap \theta$$

are over completely simple semigroups,

$$\rho\ T\ \theta \Leftrightarrow \rho/\rho \cap \theta \text{ and } \theta/\rho \cap \theta \text{ are over groups},$$
$$\rho\ V\ \theta \Leftrightarrow \rho/\rho \cap \theta \text{ and } \theta/\rho \cap \theta \text{ are over rectangular bands}.$$

A regular semigroup S is *fundamental* if $\varepsilon^T = \varepsilon$, that is, ε is the only idempotent separating congruence on S. For a fundamental inverse semigroup we shall use the term *antigroup*. The following results have been proved by the author [3]:

RESULT 1. An orthodox semigroups S has a complemented congruence lattice $\mathcal{C}(S)$ if and only if

(i) S is isomorphic to a subdirect product of a rectangular group $I \times G \times \Lambda$ and an antigroup A,

(ii) $\mathcal{C}(G)$ and $\mathcal{C}(A)$ are complemented.

RESULT 2. If ε^X and ω_X both have complements in $\mathcal{C}(S)$ then they are mutual complements for $X = U, T, V$.

For the remainder of this section we collect a few technical lemmas concerning subdirect products of rectangular groups and antigroups. Let S be a regular subdirect product of the rectangular group $B = I \times G \times \Lambda$ and the fundamental inverse semigroup A. For $e \in E(A)$ let

$$RG(e) = \{(i, g, \lambda) \mid (i, g, \lambda, e) \in S\} = I \times G \times \Lambda \times \{e\} \cap S \text{ and}$$
$$RB(e) = \{(i, 1, \lambda) \mid (i, 1, \lambda, e) \in S\} = I \times \{1\} \times \Lambda \times \{e\} \cap S.$$

Each inverse of (i, g, λ, e) in S is of the form (j, g^{-1}, μ, e). So $RG(e)$ is a regular subsemigroup of $I \times G \times \Lambda$. Hence $RG(e)$ is a rectangular group $I(e) \times G(e) \times \Lambda(e)$ for some subsets $I(e) \subseteq I$, $\Lambda(e) \subseteq \Lambda$ and a subgroup $G(e) \subseteq G$. Also, $RB(e)$ is the rectangular band $I(e) \times \{1\} \times \Lambda(e)$.

LEMMA 1. *If $f \leq e$ for some $e, f \in E(A)$ then $RG(e) \subseteq RG(f)$ and $RB(e) \subseteq RB(f)$.*

PROOF. Let $e, f \in E(A)$, $f \leq e$ and $(i, g, \lambda) \in RG(e)$. Now $x = (j, h, \mu, f) \in S$ for some $(j, h, \mu) \in I \times G \times \Lambda$. Then $(j, 1, \mu, f) = xx'x'x$ for some $x' \in V(x)$ and so $(j, 1, \mu, f) \in S$. Since $RG(e)$ is a rectangular group, $(i, g, \lambda, e) \in S$ implies that $(i, 1, \lambda, e) \in S$. Therefore, $(i, g, \lambda, f) = (i, g, \lambda, e)(j, 1, \mu, f)(i, 1, \lambda, e) \in S$.

LEMMA 2. *If $(i, g, \lambda, a) \in S$ and $e \in E(A)$ with $e \leq a$ then $(i, g, \lambda, e) \in S$.*

PROOF. An inverse of (i, g, λ, a) is of the form (j, g^{-1}, μ, a^{-1}). Then $(i, 1, \lambda, aa^{-1}a^{-1}a) \in S$. If $e \leq a$ then also $e \leq aa^{-1}a^{-1}a$. By Lemma 1, $(i, 1, \lambda, e) \in S$. Hence $(i, g, \lambda, e) = (i, g, \lambda, a)(i, 1, \lambda, e) \in S$.

COROLLARY 1. *If for each $a \in A$ there exists $e \in E(A)$ so that $e \leq a$ then $I \times G \times \Lambda = \bigcup\{RG(e) \mid e \in E(A)\}$.*

For the idempotents of S we find the following structure: $E(S) = I \times \{1\} \times \Lambda \times E(A) \cap S$. In fact, $E(S)$ forms a strong semilattice of rectangular bands, namely $E(S) = [E(A), RB(e) \times \{e\}, \iota_{ef}]$ where ι_{ef} denotes the inclusion mapping

$$\iota_{ef}\colon (i, 1, \lambda, e) \longrightarrow (i, 1, \lambda, f).$$

The proof of the following statement can be found in [18]:

LEMMA 3. *Let $E(S) = [E(A), RB(e) \times \{e\}, \iota_{ef}]$ and ρ be a congruence on $E(S)$. Then the relation*

$$e\ \rho Y\ f \Leftrightarrow (i,1,\lambda,e)\ \rho\ (i,1,\lambda,ef) \text{ and } (j,1,\mu,f)\ \rho\ (j,1,\mu,ef)$$

for all $(i,1,\lambda) \in RB(e)$, $(j,1,\mu) \in RB(f)$ is a congruence on $E(A)$. Furthermore, an equivalent definition of ρY is given by

$$e\ \rho Y\ f \Leftrightarrow (i,1,\lambda,e)\ \rho\ (i,1,\lambda,ef) \text{ and } (j,1,\mu,f)\ \rho\ (j,1,\mu,ef)$$

for some $(i,1,\lambda) \in RB(e)$ and $(j,1,\mu) \in RB(f)$.

In the following, to each congruence ρ on S we associate relations on the semigroups $I \times G \times \Lambda$ and A, respectively, which—as we shall see later—in our case just are the congruences induced by the respective projections.

DEFINITION. Let S be a subdirect product of the rectangular group $I \times G \times \Lambda$ and the antigroup A and $\rho \in \mathcal{C}(S)$. Let $\rho_{IG\Lambda}$ and ρ_A be the relations defined by

$$(i,g,\lambda)\ \rho_{IG\Lambda}\ (j,h,\mu) \Leftrightarrow (i,g,\lambda,e)\ \rho\ (j,h,\mu,e) \text{ for some } e \in E(A),$$

$$a\ \rho_A\ b \Leftrightarrow a^{-1}ea\ \rho Y\ b^{-1}eb \text{ for all } e \in E(A).$$

LEMMA 4. *The relation ρ_A is a congruence on A. If, in addition, for each $a \in A$ there exists $e \in E(A)$ such that $e \leq a$ then $\rho_{IG\Lambda}$ is a congruence on $I \times G \times \Lambda$.*

PROOF. Since ρY is a congruence it is clear that ρ_A is an equivalence. Let $a\ \rho_A\ b$ and $c \in A$. Then $a^{-1}ea\ \rho Y\ b^{-1}eb$ for all $e \in E(A)$ implies that $a^{-1}c^{-1}eca\ \rho Y\ b^{-1}c^{-1}ecb$ for all $e \in E(A)$ and thus $ca\ \rho_A\ cb$. On the other hand, $a^{-1}ea\ \rho Y\ b^{-1}eb$ implies $(i,1,\lambda,a^{-1}ea)\ \rho\ (i,1,\lambda,a^{-1}eab^{-1}eb)$ for all $(i,1,\lambda) \in RB(a^{-1}ea)$. Let (j,g,μ,c), $(k,g^{-1},\nu,c^{-1}) \in S$; then $(k,1,\nu,c^{-1}a^{-1}eac)\ \rho\ (k,1,\nu,c^{-1}a^{-1}eab^{-1}ebc) = (k,1,\nu,c^{-1}a^{-1}eab^{-1}ebcc^{-1}c) = (k,1,\nu,c^{-1}a^{-1}eacc^{-1}b^{-1}ebc)$ so that $c^{-1}a^{-1}eac\ \rho Y\ c^{-1}a^{-1}eacc^{-1}b^{-1}ebc$. Dually we get $c^{-1}b^{-1}ebc\ \rho\ c^{-1}a^{-1}eacc^{-1}b^{-1}ebc$. This holds for all $e \in E(A)$ so that $ac\ \rho_A\ bc$. Under the given assumption on the idempotents of A, $\rho_{IG\Lambda}$ is reflexive and trivially is symmetric. Transitivity is a consequence of Lemma 1 since $E(A)$ is a semilattice. Now let $(i,g,\lambda)\ \rho_{IG\Lambda}\ (j,h,\mu)$ and $(k,l,\nu) \in I \times G \times \Lambda$. By definition, $(i,g,\lambda,e)\ \rho\ (j,h,\mu,e)$ for some $e \in E(A)$. Under the given assumption on the idempotents of A, by Corollary 1, $(k,l,\nu,f) \in S$ for some $f \in E(A)$. An immediate consequence then is the compatibility of $\rho_{IG\Lambda}$.

LEMMA 5. *Let $\rho \in \mathcal{C}(S)$. If $(i,g,\lambda,a)\ \rho\ (j,h,\mu,b)$ then $(i,g,\lambda,e)\ \rho\ (j,h,\mu,e)$ for all $e \in E(A)$ such that $e \leq a, b$.*

PROOF. By Lemma 2, $(i,g,\lambda,e), (j,h,\mu,e) \in S$. Since $RG(e)$ is a rectangular group $(i,1,\mu,e) \in S$. So we obtain $(i,g,\lambda,e) = (i,1,\mu,e)(i,g,\lambda,a)\ \rho\ (i,1,\mu,e)(j,h,\mu,b) = (i,h,\mu,e) = (i,1,\mu,e)(j,h,\mu,b)(i,1,\mu,e)\ \rho\ (i,1,\mu,e)(i,g,\lambda,a)(i,1,\mu,e) = (i,g,\mu,e) = (i,g,\lambda,a)(i,1,\mu,e)\ \rho\ (j,h,\mu,b)(i,1,\mu,e) = (j,h,\mu,e)$.

If $\mathcal{C}(S)$, and thus by Result 1 also $\mathcal{C}(A)$ is complemented then by Result 2, $(\omega_A)_T = \omega_A$ since A is fundamental. In particular, A has no nontrivial group homomorphic image which by [17] implies that in this case the assumption on the idempotents of A, needed in Corollary 1 and and Lemma 4, holds.

3. **A necessary and sufficient condition.** In what follows let S be an orthodox semigroup. By [3] we may assume that if $\mathcal{C}(S)$ is P then S is a subdirect product of a rectangular group $I \times G \times \Lambda$ and an antigroup A and $\mathcal{C}(G)$ and $\mathcal{C}(A)$ are complemented. In the next statements we formulate some important properties of such semigroups.

LEMMA 6.
(i) $(i,g,\lambda,a)\ \omega_U\ (j,h,\mu,b) \Leftrightarrow i=j, g=h, \lambda=\mu$ *and*
(ii) $(i,g,\lambda,a)\ \varepsilon^U\ (j,h,\mu,b) \Leftrightarrow a=b$.

PROOF. (i) Let the congruence ρ on S be defined by

$$(i,g,\lambda,a)\ \rho\ (j,h,\mu,b) \Leftrightarrow i=j,\ g=h,\lambda=\mu.$$

By [16] $\omega_U = \leq^*$, that is, the congruence generated by the order on $E(S)$. It can be seen easily that $(i,1,\lambda,e) \leq (j,1,\mu,f)$ if and only if $i=j$, $\lambda=\mu$ and $e \leq f$. Therefore $\leq \subseteq \rho$ and thus $\omega_U \subseteq \rho$. Conversely let (i,g,λ,a), $(i,g,\lambda,b) \in S$. Now $(k,1,\lambda,a^{-1}a), (l,1,\lambda,b^{-1}b) \in S$ for some $k,l \in I$. Since $C(A)$ is complemented we have $\sigma_A = (\omega_A)_T = \omega_A$ (Result 2) and therefore there exists $e \in E(A)$ such that $e \leq a,b$. Then $(k,1,\lambda,e) \in S$ and $(k,1,\lambda,a^{-1}a)\ \omega_U\ (k,1,\lambda,e)$. Multiplying (i,g,λ,a) on the left yields $(i,g,\lambda,a)\ \omega_U\ (i,g,\lambda,e)$. By analogy we obtain that $(i,g,\lambda,b)\ \omega_U\ (i,g,\lambda,e)$ implying that $\rho \subseteq \omega_U$.

(ii) Let $\mathcal{Y}$ denote the least inverse congruence on S. Then by [16], Prop. 8.5, $\mathcal{Y} = \varepsilon^V$. So $\mathcal{Y} \subseteq \varepsilon^U$ and thus $\mathcal{Y}\ U\ \varepsilon^U$. Equivalently, $\varepsilon^U/\mathcal{Y}$ is over rectangular groups. Since $S/\mathcal{Y}$ is inverse, $\varepsilon^U/\mathcal{Y}$ in fact is over groups, hence $\varepsilon^U\ T\ \mathcal{Y}$. In particular, $(\varepsilon^U)^T = \mathcal{Y}^T$. On the other hand, $\varepsilon^U \subseteq (\varepsilon^U)^T \subseteq (\varepsilon^U)^U = \varepsilon^U$. Hence $\varepsilon^U = \mathcal{Y}^T$. Using [15], Lemma 5.6 we therefore have $(i,g,\lambda,a)\ \varepsilon^U\ (j,h,\mu,b)$ if and only if $(i,g,\lambda,a)\mathcal{Y}\ \varepsilon^T_{S/\mathcal{Y}}\ (j,h,\mu,b)\mathcal{Y}$. It can be seen easily that $(i,g,\lambda,a)\ \mathcal{Y}\ (j,h,\mu,b)$ if and only if $g=h$ and $a=b$. Thus $S/\mathcal{Y}$ is a subdirect product of G and A, namely the projection of S onto $G \times A$. For (g,a), $(h,b) \in S/\mathcal{Y}$ now it can be seen easily that $(g,a)\ \varepsilon^T_{S/\mathcal{Y}}\ (h,b)$ if and only if $a=b$ since A is fundamental.

LEMMA 7. *If $\mathcal{C}(S)$ is sectionally complemented then $\rho = \rho^U$ for all $\rho \supseteq \varepsilon^U$.*

PROOF. Let $\rho \supseteq \varepsilon^U$ and τ be a complement of ρ in $[\varepsilon, \rho^U]$. Then $\varepsilon = \rho \cap \tau \cap \leq\ = (\rho \cap \leq) \cap (\tau \cap \leq)$. Since $\tau \subseteq \rho^U$ we also have $\tau^U \subseteq \rho^U$ and therefore $\tau \cap \leq\ = \tau^U \cap \leq\ \subseteq \rho^U \cap \leq\ = \rho \cap \leq$. We conclude that $\tau \cap \leq\ = \varepsilon = \varepsilon \cap \leq$ implying $\tau \subseteq \varepsilon^U \subseteq \rho$. Therefore $\rho = \rho \vee \tau = \rho^U$.

LEMMA 8. *If $\mathcal{C}(S)$ is dually sectionally complemented then $\rho = \rho^U$ for all $\rho \supseteq \varepsilon^U$.*

PROOF. Let $\rho \supseteq \varepsilon^U$ and τ be a complement of ρ^U in $[\rho, \omega]$. Now $\rho = \tau \cap \rho^U$ and hence $\rho \cap \leq\ = \tau \cap \leq \cap\ \rho^U \cap \leq\ = \tau \cap \leq \cap\ \rho \cap \leq$, that is, $\rho \cap \leq\ \subseteq \tau \cap \leq$. By [16] we obtain that $\rho_U \subseteq \tau_U$ and hence $\rho^U \subseteq \tau^U$. Now $\omega = \rho^U \vee \tau = \rho^U \vee \tau^U$ so that $\tau^U = \omega$. Then $\tau \supseteq \omega_U$. Also $\varepsilon^U \subseteq \rho \subseteq \tau$. Since $\mathcal{C}(S)$ is complemented, $\omega = \omega_U \vee \varepsilon^U \subseteq \tau$ and thus $\rho = \rho^U$.

LEMMA 9. *Let $\phi: \mathcal{C}(S) \longrightarrow \mathcal{C}(I \times G \times \Lambda) \times \mathcal{C}(A)$ be defined by $\rho\phi = (\rho_{IG\Lambda}, \rho_A)$. If either $\mathcal{C}(S)$ or $\mathcal{C}(A)$ has the property P then ϕ is surjective.*

PROOF. Let $(\xi, \eta) \in \mathcal{C}(I \times G \times \Lambda) \times \mathcal{C}(A)$ and let a relation $\rho(\xi, \eta)$ be defined by

$$(i, g, \lambda, a)\ \rho(\xi, \eta)\ (j, h, \mu, b) \Leftrightarrow (i, g, \lambda)\ \xi\ (j, h, \mu) \text{ and } a\ \eta\ b.$$

Let $\rho(\xi, \eta)\phi = (\rho_{IG\Lambda}, \rho_A)$. We shall prove that $\xi = \rho_{IG\Lambda}$ and $\eta = \rho_A$. Let $(i, g, \lambda)\ \xi\ (j, h, \mu)$. There exists $e \in E(A)$ such that (i, g, λ, e), $(j, h, \mu, e) \in S$. Since $e\ \eta\ e$ we have $(i, g, \lambda, e)\ \rho(\xi, \eta)\ (j, h, \mu, e)$ and therefore $(i, g, \lambda)\ \rho_{IG\Lambda}\ (j, h, \mu)$. If $(i, g, \lambda)\ \rho_{IG\Lambda}\ (j, h, \mu)$ then $(i, g, \lambda, e)\ \rho\ (j, h, \mu, e)$ for some $e \in E(A)$ and thus $(i, g, \lambda)\ \xi\ (j, h, \mu)$. We have thus obtained $\rho_{IG\Lambda} = \xi$. Now let $a\ \eta\ b$. We have $a^{-1}ea\ \eta\ b^{-1}eb$ for all $e \in E(A)$. Let $e \in E(A)$ be fixed; now $(i, 1, \lambda, a^{-1}ea)\ \rho(\xi, \eta)\ (i, 1, \lambda, a^{-1}eab^{-1}eb)$ for all $(i, 1, \lambda) \in RB(a^{-1}ea)$ and $(j, 1, \mu, b^{-1}eb)\ \rho(\xi, \eta)\ (j, 1, \mu, a^{-1}eab^{-1}eb)$ for all $(j, 1, \mu) \in RB(b^{-1}eb)$. This can be done for each $e \in E(A)$. We obtain $a^{-1}ea\ \rho Y\ b^{-1}eb$ for all $e \in E(A)$ and so $a\ \rho_A\ b$. Conversely, let $a\ \rho_A\ b$, that is, $a^{-1}ea\ \rho Y\ b^{-1}eb$ for all $e \in E(A)$. Then $(i, 1, \lambda, a^{-1}ea)\ \rho(\xi, \eta)\ (i, 1, \lambda, a^{-1}eab^{-1}eb)$ for all $e \in E(A)$ and all $(i, 1, \lambda) \in RB(a^{-1}ea)$, $(j, 1, \mu, b^{-1}eb)\ \rho(\xi, \eta)\ (j, 1, \mu, a^{-1}eab^{-1}eb)$ for all $e \in E(A)$ and all $(j, 1, \mu) \in RB(b^{-1}eb)$. We obtain $a^{-1}ea\ \eta\ b^{-1}eb$ for all $e \in E(A)$ and thus $a\ \eta^T\ b$. If $\mathcal{C}(A)$ has the property P then $\eta = \eta^T$ since $T = U$ (and using Lemmas 7 and 8) and so $a\ \eta\ b$. Now suppose that $\mathcal{C}(S)$ has the property P. We define the relation $\bar{\eta}$ on S by

$$(k, x, \nu, c)\ \bar{\eta}\ (l, y, \pi, d) \Leftrightarrow c\ \eta\ d.$$

Then $\bar{\eta}$ is a congruence on S and $\varepsilon^U \subseteq \bar{\eta}$. By Lemmas 7 and 8 we have $\bar{\eta}^U = \bar{\eta}$ and therefore $\bar{\eta}^T = \bar{\eta}$ since $\bar{\eta} \subseteq \bar{\eta}^T \subseteq \bar{\eta}^U$. We thus have $(k, 1, \nu, e)\ \bar{\eta}\ (l, 1, \pi, f) \Leftrightarrow e\ \eta\ f \Leftrightarrow e\ \eta^T f \Leftrightarrow (k, 1, \nu, e)\ (\eta^T)^-\ (l, 1, \pi, f)$. Therefore $\operatorname{tr}(\eta^T)^- = \operatorname{tr}\bar{\eta}$ and hence $(\eta^T)^- \subseteq \bar{\eta}^T$ implying $\bar{\eta} = (\eta^T)^-$. We conclude that $\eta = \eta^T$. Therefore $a\ \eta\ b$ in this case, too.

We have thus shown that if either $\mathcal{C}(S)$ or $\mathcal{C}(A)$ has the property P then the mapping $(\xi, \eta) \longrightarrow \rho(\xi, \eta) \longrightarrow \rho(\xi, \eta)\phi$ is the identical mapping on $\mathcal{C}(I \times G \times \Lambda) \times \mathcal{C}(A)$ and thus in both cases ϕ is surjective (we do not yet know that "$\mathcal{C}(S)$ is P" implies "$\mathcal{C}(A)$ is P" which is not immediate for the case *sectionally complemented*).

LEMMA 10. *If either $\mathcal{C}(S)$ or $\mathcal{C}(A)$ is P then the mapping ϕ is a lattice homomorphism.*

PROOF. (i) $(\rho \cap \theta)_{IG\Lambda} = \rho_{IG\Lambda} \cap \theta_{IG\Lambda}$. Since ϕ is monotone we have to prove that $\rho_{IG\Lambda} \cap \theta_{IG\Lambda} \subseteq (\rho \cap \theta)_{IG\Lambda}$. Let $x, y \in I \times G \times \Lambda$ and $x(\rho_{IG\Lambda} \cap \theta_{IG\Lambda})y$; we have $(x, e)\ \rho\ (y, e)$ and $(x, f)\ \theta\ (y, f)$ for some $e, f \in E(A)$. Then $(x, ef)\ \rho \cap \theta\ (x, ef)$ and so $x\ (\rho \cap \theta)_{IG\Lambda}\ y$.

(ii) $(\rho \cap \theta)_A = \rho_A \cap \theta_A$. Again we only have to prove that $\rho_A \cap \theta_A \subseteq (\rho \cap \theta)_A$. Let $e < f \in E(A)$ and $e\ \rho_A \cap \theta_A\ f$. Then $(i, 1, \lambda, e)\ \rho \cap \theta\ (i, 1, \lambda, f)$ for all $(i, 1, \lambda) \in RB(f)$. Hence $\operatorname{tr}(\rho_A \cap \theta_A) = \operatorname{tr}(\rho \cap \theta)_A$. By Lemmas 7 and 8 and the proof of Lemma 9 each congruence on A is uniquely determined by its trace. Therefore $\rho_A \cap \theta_A = (\rho \cap \theta)_A$.

(iii) $(\rho \vee \theta)_{IG\Lambda} = \rho_{IG\Lambda} \vee \theta_{IG\Lambda}$. By monotonity of ϕ we have to prove $(\rho \vee \theta)_{IG\Lambda} \subseteq \rho_{IG\Lambda} \vee \theta_{IG\Lambda}$. Let $x, y \in I \times G \times \Lambda$ and $x\ (\rho \vee \theta)_{IG\Lambda}\ y$, that is, $(x, e)\ \rho \vee \theta\ (y, e)$ for some

$e \in E(A)$. Then $(x,e) = (x_1,a_1)\ \xi_1\ (x_2,a_2)\cdots \xi_n\ (x_n,a_n) = (y,e)$ for some $(x_i,a_i) \in S$ and $\xi_i \in \{\rho,\theta\}$. By the lemmas of Section 2 and since $\mathcal{C}(A)$ is complemented, which implies $\sigma_A = \omega_A$, we obtain that $(x,f) = (x_1,f)\ \xi_1\ (x_2,f)\cdots \xi_n\ (x_n,f) = (y,f)$ for some $f \in E(A)$ with $f \leq a_i$ for all i. Hence $x\ \rho_{IG\Lambda} \vee \theta_{IG\Lambda}\ y$.

(iv) $(\rho \vee \theta)_A = \rho_A \vee \theta_A$. Again we only have to prove $(\rho \vee \theta)_A \subseteq \rho_A \vee \theta_A$. Let $e < f \in E(A)$ and $e\ (\rho \vee \theta)_A\ f$; then $(i,1,\lambda,e)\ \rho \vee \theta\ (i,1,\lambda,f)$ for $(i,1,\lambda) \in RB(f)$. By [15] we have that $\mathrm{tr}(\rho \vee \theta) = \mathrm{tr}(\rho) \vee \mathrm{tr}(\theta)$. So there exist idempotents $(i_k,1,\lambda_k,e_k) \in E(S)$ such that $(i,1,\lambda,e)\ \xi_1\ (i_1,1,\lambda_1,e_1)\ \xi_2 \cdots \xi_n\ (i_n,1,\lambda_n,e_n) = (i,1,\lambda,f)$ where $\xi_i \in \{\rho,\theta\}$. Using this we obtain $e\ \rho_A \vee \theta_A\ f$. We have thus shown that $\mathrm{tr}(\rho \vee \theta)_A = \mathrm{tr}(\rho_A \vee \theta_A)$. By Lemmas 7 and 8 and the proof of Lemma 9 we have that $\xi = \xi^T$ for all $\xi \in \mathcal{C}(A)$ and so each congruence on A is uniquely determined by its trace. Therefore we get $(\rho \vee \theta)_A = \rho_A \vee \theta_A$.

LEMMA 11. *A congruence ρ on S is idempotent separating if and only if $(i,g,\lambda,a)\ \rho\ (j,h,\mu,b)$ implies $i = j$, $\lambda = \mu$, $a = b$.*

PROOF. Let ρ be idempotent separating and $(i,g,\lambda,a)\ \rho\ (j,h,\mu,b)$. Then *a fortiori* ρ is over completely simple semigroups and thus $\rho \subseteq \varepsilon^U$. By Lemma 6 we have $a = b$. Let $(k,g^{-1},\nu,a^{-1}) \in V\big((i,g,\lambda,a)\big)$ then $(k,1,\lambda,a^{-1}a)\ \rho\ (k,g^{-1}h,\mu,a^{-1}a)$. Since ρ is over groups $(k,h^{-1}g,\mu,a^{-1}a) \in S$ and $(k,1,\lambda,a^{-1}a)\ \rho\ (k,h^{-1}g,\mu,a^{-1}a)$. Thus $(k,1,\lambda,a^{-1}a)\ \rho\ (k,1,\mu,a^{-1}a)$ which implies $\lambda = \mu$. An analogous argument shows that $i = j$. Sufficiency is clear since $E(S) = I \times \{1\} \times \Lambda \times E(A) \cap S$.

COROLLARY 2. *The greatest idempotent separating congruence $\mu = \varepsilon^T$ on S is given by*

$$(i,g,\lambda,a)\ \mu\ (j,h,\nu,b) \Leftrightarrow i = j,\ \lambda = \nu,\ a = b.$$

COROLLARY 3. *The kernel of μ is given by*

$$\ker \mu = \bigcup \{RG(e) \times \{e\} : e \in E(A)\} = I \times G \times \Lambda \times E(A) \cap S.$$

The next step is to show that ϕ is injective and thus an isomorphism if $\mathcal{C}(S)$ has the property P.

LEMMA 12. *If $\mathcal{C}(S)$ is sectionally complemented then ϕ is injective.*

PROOF. (i) First we prove that $(\varepsilon,\varepsilon)\phi^{-1} = \{\varepsilon\}$, that is, $\rho\phi = (\varepsilon,\varepsilon)$ if and only if $\rho = \varepsilon$. Let $\rho\phi = (\varepsilon,\varepsilon)$ and $(i,1,\lambda,e),(j,1,\mu,f) \in E(S)$ and $(i,1,\lambda,e)\ \rho\ (j,1,\mu,f)$. Then $(i,1,\lambda,ef)\ \rho\ (j,1,\mu,ef)$ and $\rho_{IG\Lambda} = \varepsilon$ imply that $i = j$, $\lambda = \mu$. On the other hand $(i,1,\lambda,e)\ \rho\ (j,1,\mu,f)$ implies $e\ \rho_A\ f$ and thus $e = f$. Therefore ρ is idempotent separating. Now let $(i,g,\lambda,a)\ \rho\ (j,h,\mu,b)$; by Lemma 11 we have $i = j$, $\lambda = \mu$ and $a = b$. Since $\mathcal{C}(A)$ is complemented there exists $e \in E(A)$ such that $e \leq a$. We obtain $(i,g,\lambda,e)\ \rho\ (i,h,\lambda,e)$ and thus $(i,g,\lambda)\ \rho_{IG\Lambda}\ (i,h,\lambda)$ implying $g = h$. Therefore $\rho = \varepsilon$.

(ii) Let $\rho, \theta \in \mathcal{C}(S)$ such that $\rho\phi = \theta\phi$. We may assume that $\rho \subseteq \theta$ since ϕ is a lattice homomorphism. Let $\xi \in \mathcal{C}(S)$ be a complement of ρ in $[\varepsilon, \theta]$. Then $\rho \cap \xi = \varepsilon$ and $\rho \vee \xi = \theta$. Now $(\varepsilon, \varepsilon) = \varepsilon\phi = \rho\phi \cap \xi\phi = \theta\phi \cap \xi\phi = (\theta \cap \xi)\phi = \xi\phi$ since $\xi \subseteq \theta$. Then $\xi = \varepsilon$ and thus $\rho = \theta$.

LEMMA 13. *If $\mathcal{C}(S)$ is dually sectionally complemented then ϕ is injective.*

PROOF. Let $\rho, \theta \in \mathcal{C}(S)$, $\rho \subseteq \theta$ such that $\rho\phi = \theta\phi$. Suppose that $(x,a)\ \theta\ (y,b)$. Then $(x,e)\ \rho\ (y,e)$ for some $e \leq a, b$. Let $(x', a^{-1}) \in V\big((x,a)\big)$; then $(x'x, a^{-1}e) = (x'x, e) \leq (x'x, a^{-1}a)$. Therefore $(x'x,e)\rho \leq_{S/\rho} (x'x, a^{-1}a)\rho$. By [16] ω_U is the congruence generated by the order on the idempotents. We obtain $(x'x,e)\rho\ \omega_U\ (x'x,a^{-1}a)\rho$ and therefore $(x,e)\rho\ \omega_U\ (x,a)\rho$ where $\omega_U = (\leq_{S/\rho})^*$, the congruence generated by $\leq_{S/\rho}$. The same procedure for (y,b) then yields $(y,b)\rho\ \omega_U\ (y,e)\rho$ and thus $(x,a)\rho\ \omega_U\ (y,b)\rho$ in S/ρ. On the other hand, let $(i,1,\lambda,e), (j,1,\mu,f) \in E(S)$ such that $(i,1,\lambda,e) \leq (j,1,\mu,f)$; then $i = j$, $\lambda = \mu$ and $e \leq f$. If $(i,1,\lambda,e)\ \theta\ (i,1,\lambda,f)$ then $e\ \theta_A\ f$ and thus $e\ \rho_A\ f$. Hence $(i,1,\lambda,e)\ \rho\ (i,1,\lambda,f)$. So $\rho_A = \theta_A$ in fact implies that $\rho\ U\ \theta$. But then θ/ρ is over completely simple semigroups and thus $\theta/\rho \subseteq \varepsilon^U$ in S/ρ. Now $(x,a)\ \theta\ (y,b)$ implies $(x,a)\rho\ \theta/\rho\ (y,b)\rho$ in S/ρ and therefore $(x,a)\rho\ \varepsilon^U\ (y,b)\rho$ in S/ρ. We thus have obtained that $(x,a)\rho\ \omega_U \cap \varepsilon^U\ (y,b)\rho$. Since $\mathcal{C}(S)$ is dually sectionally complemented, $\mathcal{C}(S/\rho)$ is complemented and so by Result 2 $\omega_U \cap \varepsilon^U = \varepsilon$, the identity relation on S/ρ. Hence $(x,a)\ \rho\ (y,b)$.

DEFINITION. Let $S \subseteq A \times B$ be a subdirect product of semigroups A and B. A congruence ρ on S is the direct product of the congruences ξ on A and η on B if for all (a,b), $(c,d) \in S$

$$(a,b)\ \rho\ (c,d) \Leftrightarrow a\ \xi\ c \text{ and } b\ \eta\ d.$$

By Lemmas 12 and 13 we have that if $\mathcal{C}(S)$ has the property P then each congruence on S is the direct product of some congruence on $I \times G \times \Lambda$ and some congruence on A. In this case $\mathcal{C}(S)$ is isomorphic to $\mathcal{C}(I \times G \times \Lambda) \times \mathcal{C}(A)$. Therefore if $\mathcal{C}(S)$ has the property P then $\mathcal{C}(I \times G \times \Lambda)$ and $\mathcal{C}(A)$ both have the property P. Conversely, if S is a subdirect product of the rectangular group $I \times G \times \Lambda$ and the antigroup A and $\mathcal{C}(I \times G \times \Lambda)$ and $\mathcal{C}(A)$ have the property P then the mapping $\phi : \rho \longrightarrow (\rho_{IG\Lambda}, \rho_A)$ is a surjective homomorphism. If, in addition, ϕ is injective, or equivalently: each congruence on S is the direct product of a congruence on $I \times G \times \Lambda$ and A, respectively, then $\mathcal{C}(S)$ is isomorphic to the direct product of $\mathcal{C}(I \times G \times \Lambda)$ and $\mathcal{C}(A)$ and therefore $\mathcal{C}(S)$ itself has the property P. We thus have obtained the following

THEOREM 1. *Let S be an orthodox semigroup. Then $\mathcal{C}(S)$ has the property P if and only if*

(i) S is isomorphic to a subdirect product of a rectangular group $I \times G \times \Lambda$ and an antigroup A,

(ii) $\mathcal{C}(I \times G \times \Lambda)$ and $\mathcal{C}(A)$ both have the property P,

(iii) *each congruence on S is the direct product of a congruence on $I \times G \times \Lambda$ and a congruence on A, respectively.*

REMARK. Since for each $a \in A$ there exists $e \in E(A)$ such that $e \leq a$ it follows that $\rho_{IG\Lambda} = (\rho \vee \pi_{IG\Lambda})/\pi_{IG\Lambda}$ where $\pi_{IG\Lambda}$ is the kernel of the projection of S on $I \times G \times \Lambda$. Similarly it can be seen that $\rho_A = \big((\rho \vee \pi_A)/\pi_A\big)^T = (\rho \vee \pi_A)/\pi_A$ (by the proof of Lemma 9). The reason for the artificial-seeming definitions we gave for these congruences is that in this form they can be expressed by means of a closed formula rather than by the transitive closures of the relations $\rho \cup \pi_{IG\Lambda}$ and $\rho \cup \pi_A$, respectively.

4. **Subdirect products whose congruences are direct products.** The next step is to find necessary and sufficient conditions for a subdirect product S of a rectangular group $I \times G \times \Lambda$ and an antigroup A in order that each congruence on S be the direct product of congruences on the rectangular group and the antigroup, respectively. Suppose that $\mathcal{C}(A)$ has the property P. We are looking for necessary and sufficient conditions for the mapping ϕ to be injective

By [6] each idempotent separating congruence on S is uniquely determined by its kernel which is a full selfconjugate regular subsemigroup K contained in $\ker \mu$. Selfconjugate means that $x'Kx \subseteq K$ for all $x \in S$ and $x' \in V(x)$ and full says that $E \subseteq K$. For each normal subgroup N of G the set $N\Pi_G^{-1} \cap \ker \mu = I \times N \times \Lambda \times E(A) \cap S$ is the kernel of an idempotent separating congruence.

We now will show that necessary and sufficient conditions for ϕ to be injective are:

(i) Π_G, the projection of S onto G induces an order preserving bijection between $\Delta(S)$, the set of all kernels of idempotent separating congruences and the lattice of all normal subgroups of G,

(ii) $|RB(f)| = 1$ if f is not minimal in $E(A)$.

LEMMA 14. *Let S be a subdirect product of the rectangular group $I \times G \times \Lambda$ and the antigroup A such that $\mathcal{C}(A)$ has the property P. Then the mapping $\phi: \rho \longrightarrow (\rho_{IG\Lambda}, \rho_A)$ is injective if and only if*

(i) *if $(i, g, \lambda, e), (i, g, \lambda, f) \in S$ for $e, f \in E(A)$ and $e < f$ then (i, g, λ, f) is contained in the full selfconjugate regular subsemigroup (contained in $\ker \mu$) which is generated by (i, g, λ, e),*

(ii) *if $(i, 1, \lambda, e)$, $(j, 1, \mu, e) \in E(S)$ and $f \in E(A)$ with $f < e$ then $(i, 1, \lambda, e)\ \xi\ (j, 1, \mu, e)$ where ξ denotes the congruence on S which is generated by the pair $\{(i, 1, \lambda, f), (j, 1, \mu, f)\}$.*

PROOF. Necessity. Suppose that ϕ is injective. Let (i, g, λ, e), $(i, g, \lambda, f) \in S$, $e, f \in E(A)$ and $e < f$. Let N denote the full selfconjucate regular subsemigroup which is generated by (i, g, λ, e). Then $N \subseteq \ker \mu$. Let $M = (N\Pi_G\Pi_G^{-1}) \cap \ker \mu$ and let $\rho(N)$ and $\rho(M)$ denote the idempotent separating congruences associated with N and M, respectively. We have $\rho(N)_A = \rho(M)_A = \varepsilon$ since both congruences are idempotent separating. Also $\rho(N) \subseteq \rho(M)$ implies $\rho(N)_{IG\Lambda} \subseteq \rho(M)_{IG\Lambda}$. Let $(l, k, \nu)\ \rho(M)_{IG\Lambda}\ (j, h, \mu)$ then $(l, k, \nu, d)\ \rho(M)\ (j, h, \mu, d)$ for some $d \in E(A)$ and thus $l = j, \nu = \mu$ since

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$\rho(M)$ is idempotent separating. Since $RG(d)$ is a rectangular group $(l,k^{-1},\nu,d)\in S$ and $(l,1,\nu,d)\ \rho(M)\ (l,k^{-1}h,\nu,d)$. Therefore $(l,k^{-1}h,\nu,d)\in M=N\Pi_G\Pi_G^{-1}\cap\ker\mu$ and thus $k^{-1}h\in N\Pi_G$. There exist $p\in I$, $\pi\in\Lambda$, $c\in E(A)$ such that $(p,k^{-1}h,\pi,c)\in N$ and so $(p,k^{-1}h,\pi,c)\ \rho(N)\ (p,1,\pi,c)$. Multiplying (l,k,ν,d) from the left and $(l,1,\nu,d)$ from the right then yields $(l,h,\nu,cd)\ \rho(N)\ (l,k,\nu,cd)$ implying $(l,k,\nu)\ \rho(N)_{IG\Lambda}\ (l,h,\nu)$. We obtain $\rho(N)_{IG\Lambda}=\rho(M)_{IG\Lambda}$. By injectivity of ϕ we get $\rho(N)=\rho(M)$. Since $(i,g,\lambda,f)\in M$ we have shown assertion (i). Now let $(i,1,\lambda,e),(j,1,\mu,e)\in S$ and $f\in E(A)$ satisfying $f<e$. Then $(i,1,\lambda,f),(j,1,\mu,f)\in S$. Let ξ_e and ξ_f denote the congruences generated by $\{(i,1,\lambda,e),(j,1,\mu,e)\}$ and $\{(i,1,\lambda,f),(j,1,\mu,f)\}$, respectively. It can be seen easily that the G- and A-entries, respectively, of two $\xi_f(\xi_e)$-equivalent elements are equal and so both congruences are over rectangular bands (and in particular idempotent pure). *A fortiori*, both congruences are contained in ε^U so we have $(\xi_e)_A=(\xi_f)_A=\varepsilon$. If $(k,g,\nu)\ (\xi_e)_{IG\Lambda}\ (l,h,\pi)$ then $(k,g,\nu,c)\ \xi_e\ (l,h,\pi,c)$ and $g=h$. Then also $(k,1,\nu,c)\ \xi_e\ (l,1,\pi,c)$. Therefore there exist $x_1,y_1,\dots x_n,y_n\in S^1$ such that $(k,1,\nu,c)=x_1u_1y_1, x_iv_iy_i=x_{i+1}u_{i+1}y_{i+1}, x_nv_ny_n=(l,1,\pi,c)$ and $\{u_i,v_i\}=\{(i,1,\lambda,e),(j,1,\mu,e)\}$. Let $a\in E(A)$ such that $a\le x_i\Pi_A$, $y_i\Pi_A$ for all i such that $x_i,y_i\in S$ and $a\le c,f$. Let $z_i^*=z_i\Pi_{IG\Lambda}\times\{a\}$ for $z=x,y$, $z_i^*=1$ if $z_i\notin S$ and $z_i^*=z_i\Pi_{IG\Lambda}\times\{f\}$ for $z=u,v$. Then $(k,1,\nu,a)=x_1^*u_1^*y_1^*$, $x_i^*v_i^*y_i^*=x_{i+1}^*u_{i+1}^*y_{i+1}^*$, $x_n^*v_n^*y_n^*=(l,1,\pi,a)$, that is, $(k,1,\nu,a)\ \xi_f\ (l,1,\pi,a)$. Hence $(k,1,\nu)\ (\xi_f)_{IG\Lambda}\ (l,1,\pi)$ and therefore also $(k,g,\nu)\ (\xi_f)_{IG\Lambda}\ (l,g,\pi)$ which implies $(\xi_f)_{IG\Lambda}=(\xi_e)_{IG\Lambda}$. By injectivity of ϕ we get $\xi_e=\xi_f$.

Sufficiency. Suppose that (i) and (ii) hold and that $\rho\phi=\theta\phi$ for some $\rho,\theta\in\mathcal{C}(S)$. First we show that ρ and θ have the same trace. Let $(i,1,\lambda,e),(j,1,\mu,f)\in E(S)$ and $(i,1,\lambda,e)\ \rho\ (j,1,\mu,f)$. Then $(i,1,\lambda,e)\ \rho\ (i,1,\lambda,ef)\ \rho\ (j,1,\mu,ef)\ \rho\ (j,1,\mu,f)$. We have $e\ \rho_A\ ef\ \rho_A\ f$ and therefore $e\ \theta_A\ f$ implying that $(i,1,\lambda,e)\ \theta\ (i,1,\lambda,ef)$ and $(j,1,\mu,f)\ \theta\ (j,1,\mu,ef)$. Furthermore $(i,1,\lambda,ef)\ \rho\ (j,1,\mu,ef)$ implies $(i,1,\lambda)\ \rho_{IG\Lambda}\ (j,1,\mu)$. By $\rho_{IG\Lambda}=\theta_{IG\Lambda}$ we obtain that $(i,1\lambda,g)\ \theta\ (j,1,\mu,g)$ for some $g\in E(A)$. Then also $(i,1,\lambda,efg)\ \theta\ (j,1,\mu,efg)$. Applying (ii) we obtain that $(i,1,\lambda,ef)\ \theta\ (j,1,\mu,ef)$ and hence $(i,1,\lambda,e)\ \theta\ (j,1,\mu,f)$. We have proved that $\operatorname{tr}\rho\subseteq\operatorname{tr}\theta$ and an analogous argument shows $\operatorname{tr}\theta\subseteq\operatorname{tr}\rho$. Now let $(i,g,\lambda,a)\in\ker\rho$, that is, $(i,g,\lambda,a)\ \rho\ (j,1,\mu,e)$ for some $e\in E(A)$. The kernel is inverse closed (see [12]) so that $(k,g^{-1},\nu,a^{-1})\ \rho\ (l,1,\pi,f)$ for certain $k,l\in I$, $\nu,\pi\in\Lambda$ and $f\in E(A)$. We conclude that $a\ \rho_A\ ef$. Since A has the property P each congruence on A is uniquely determined by its trace and so $a\ (\rho_A)_T\ ef$, that is, $ah=efh$ for some $h\in E(A)$ satisfying $a^{-1}a\ \rho_A\ ef\ \rho_A\ h$. Now $(m,1,\kappa,h)\in E(S)$ and we get $(i,g,\kappa,ah)=(i,g,\kappa,efh)\ \rho\ (j,1,\kappa,eh)$. Multiplying $(m,1,\kappa,efh)$ on the right we get $(i,g,\kappa,efh)\ \rho\ (j,1,\kappa,efh)$. Then $(i,g,\kappa)\ \rho_{IG\Lambda}\ (j,1,\kappa)$ implies $(i,g,\kappa)\ \theta_{IG\Lambda}\ (j,1,\kappa)$ and therefore $(i,g,\kappa,c)\ \theta\ (j,1,\kappa,c)$ for some $c\in E(A)$. So $(i,g,\kappa,cefh)\ \theta\ (j,1,\kappa,cefh)$ and hence $(i,g,\kappa,d)\ \theta\cap\mu\ (i,1,\kappa,d)$ for a $d\le cefh$. By assumption (i) we obtain $(i,g,\kappa,efh)\ \theta\ (i,1,\kappa,efh)$ and therefore $(i,g,\kappa,ah)=(i,g,\kappa,efh)\in\ker\theta$. Multiplying $(k,1,\lambda,a^{-1}a)$ on the right we obtain $(i,g,\lambda,aha^{-1}a)=(i,g,\lambda,ah)\in\ker\theta$. Also $a^{-1}a\ \rho_A\ h$ implies $a^{-1}a\ \theta_A\ h$ and therefore $(k,1,\lambda,a^{-1}a)\ \theta\ (k,1,\lambda,a^{-1}ah)$. Multiplying (i,g,λ,a) on the left we get $(i,g,\lambda,a)\ \theta\ (i,g,\lambda,ah)\in\ker\theta$ and hence $(i,g,\lambda,a)\in\ker\theta$. So $\ker\rho\subseteq\ker\theta$ and an analogous argument shows $\ker\theta\subseteq\ker\rho$.

COROLLARY 4. *If f is not minimal in $E(A)$ then $|RB(f)| = 1$.*

PROOF. Let $e, f \in E(A)$, $e < f$ and $(i,1,\lambda,f), (j,1,\mu,f) \in S$ and suppose $i \neq j$. We have $(i,1,\lambda,e), (j,1,\mu,e) \in S$. By Lemma 14 there exist $x_i, y_i \in S^1$ $(i = 1,\ldots n)$ such that $(i,1,\lambda,f) = x_1u_1y_1$, $x_iv_iy_i = x_{i+1}u_{i+1}y_{i+1}$, $x_nv_ny_n = (j,1,\mu,f)$ and $\{u_i, v_i\} = \{(i,1,\lambda,e), (j,1,\mu,e)\}$. Since $j \neq i$ there exists k such that $x_k \notin S$. Let k be the smallest index such that either $x_k \notin S$ or $y_k \notin S$. Let a_i and b_i denote the entries in the A-component of x_i and y_i, respectively. Then $f = a_1eb_1 = \cdots a_{k-1}eb_{k-1} = a_ke$ (or $= eb_k$). But $f = a_ke$ (or $= eb_k$) implies $f \leq e$, a contradiction to $e < f$.

LEMMA 15. *Let $(i,g,\lambda,e) \in \ker\mu$. The full selfconjugate regular subsemigroup of S which is generated by (i,g,λ,e) is the set of all products of the form $x_1'y_1x_1\cdots x_n'y_nx_n$ where $x_i \in S$, $x_i' \in V(x_i)$, $y_i \in E(S)\cup\{(i,g,\lambda,e), (i,g^{-1},\lambda,e)\}$.*

PROOF. Let K be the set of all these products. Then $E(S) \subseteq K$ and K is a regular subsemigroup. Let $x = (j,h,\mu,a) \in S$ and $x' = (l,h^{-1},\nu,a^{-1})$ be an inverse of x; $x_i = (j_i,h_i,\mu_i,a_i), x_i' = (l_i,h_i^{-1},\nu_i,a_i^{-1})$ and $y_i = (k_i,1,\pi_i,e_i)$ or $y_i = (i,g^{\pm1},\lambda,e)$. Now $z = x_1'y_1x_1x_2'y_2\cdots x_n'y_nx_n = (l_1, h_1^{-1}g_1h_1h_2^{-1}\cdots h_n^{-1}g_nh_n, \mu_n, a_1^{-1}e_1a_1\cdots a_n^{-1}e_na_n)$ where $g_i \in \{1,g,g^{-1}\}$. Multiplying x' on the left and x on the right we get $x'zx = (l, h^{-1}h_1^{-1}g_1h_1hh^{-1}\cdots h^{-1}h_n^{-1}g_nh_nh, \mu, a^{-1}a_1^{-1}e_1a_1\cdots a_n^{-1}e_na_na)$. For the A-component we have $a^{-1}a_1^{-1}e_1a_1\cdots a_n^{-1}e_na_na = a^{-1}aa^{-1}a_1^{-1}e_1a_1\cdots a_n^{-1}e_na_na = a^{-1}a_1^{-1}e_1a_1aa^{-1}a_2^{-1}e_2\cdots aa^{-1}a_n^{-1}e_na_na$ since idempotents in A commute. Then also $x'zx = (x'x_1')y_1(x_1x)(x'x_2')y_2\cdots(x'x_n')y(x_nx)$. Since $x'x_i'$ is an inverse of x_ix we see that $x'zx \in K$.

LEMMA 16. *Let $(i,g,\lambda,e), (i,g,\lambda,f) \in \ker\mu$, $g \neq 1$ and $e < f$. Then (i,g,λ,f) is contained in the full selfconjugate regular subsemigroup which is generated by (i,g,λ,e) if and only if there exist $x_i = (j_i,h_i,\mu_i,a_i) \in S$ $(i = 1,\ldots n)$ such that*

$$g = h_1^{-1}g^{\pm1}h_1h_2^{-1}g^{\pm1}h_2\cdots h_n^{-1}g^{\pm1}h_n \text{ and}$$
$$f \leq a_1^{-1}ea_1a_2^{-1}ea_2\cdots a_n^{-1}ea_n.$$

PROOF. Sufficiency.

Let K denote the full selfconjugate regular subsemigroup generated by (i,g,λ,e). Let $c = a_1^{-1}ea_1a_2^{-1}\cdots a_n^{-1}ea_n$ then $(l_1,g,\mu_n,c) \in K$ for some $l_1 \in I$. Since $RG(f)$ is a rectangular group $(i,1,\lambda,f) \in S$ and so $(i,g,\lambda,f) = (i,1,\lambda,f)(l_1,g,\mu_n,c)(i,1,\lambda,f) \in K$.

Necessity.

By Lemma 15 there exist $x_i = (j_i,h_i,\mu_i,a_i)$ and $x_i' = (l_i,h_i^{-1},\nu_i,a_i^{-1}) \in V(x_i)$ such that $(i,g,\lambda,f) = x_1'y_1x_1x_2'y_2x_2\cdots x_n'y_nx_n$ and $y_i \in E(S)\cup\{(i,g,\lambda,e),(i,g^{-1},\lambda,e)\}$. Let $i_1,\ldots,i_s$ denote those indices such that $y_{i_k} \notin E(S)$. Since $g \neq 1$ this set is not empty. Then we have $g = h_{i_1}^{-1}g^{\pm1}h_{i_1}\cdots h_{i_s}^{-1}g^{\pm1}h_{i_s}$ and $f \leq a_{i_1}^{-1}ea_{i_1}\cdots a_{i_s}^{-1}ea_{i_s}$.

We thus have obtained

THEOREM 2. *Let S be an orthodox semigroup. Then $\mathcal{C}(S)$ has the property P if and only if there exist a rectangular group $I \times G \times \Lambda$ and an antigroup A such that*

(i) *S is a subdirect product of $I \times G \times \Lambda$ and A,*

(ii) *$\mathcal{C}(I \times G \times \Lambda)$ and $\mathcal{C}(A)$ have the property P,*

(iii) *for (i, g, λ, e), $(i, g, \lambda, f) \in \ker \mu, g \neq 1$, $e < f$ there exist $x_j = (l_j, h_j, \mu_j, a_j) \in S$ $(j = 1, \ldots n)$ such that $g = h_1^{-1} g^{\pm 1} h_1 h_2^{-1} \cdots h_n^{-1} g^{\pm 1} h_n$ and $f \leq a_1^{-1} e a_1 \cdots a_n^{-1} e a_n$,*

(iv) *$|RB(f)| = 1$ iff f is not minimal in $E(A)$.*

Furthermore, in this case $\mathcal{C}(S) \cong \mathcal{C}(I \times G \times \Lambda) \times \mathcal{C}(A)$.

Property (i) of Lemma 14 and thus property (iii) of the above theorem simply are technical formulations for the fact that Π_G induces an order preserving bijection between $\mathcal{N}(G)$, the lattice of all normal subgroups of G, and $\Delta(S)$, the lattice of all kernels of idempotent separating congruences on S. Suppose that (i) of Lemma 14 holds. Let $N, M \in \Delta(S)$ such that $N\Pi_G = M\Pi_G$ and let $(i, g, \lambda, e) \in N$. Then $(j, g, \mu, f) \in M$ for some j, μ, f. Since $RG(e)$ is a rectangular group $(i, 1, \lambda, e) \in S$. So $(i, 1, \lambda, e)(j, g, \mu, f)(i, 1, \lambda, e) = (i, g, \lambda, ef) \in M$. In particular, M contains the full selfconjugate regular subsemigroup generated by (i, g, λ, ef). By (i) we obtain that $(i, g, \lambda, e) \in M$ and thus $N \subseteq M$. By symmetricity, $N = M$. The mapping $N \longrightarrow N\Pi_G$ $\big(N \in \Delta(S)\big)$ therefore is injective (and obviously is surjective and monotone in both directions). Conversely suppose that $N \longrightarrow N\Pi_G$ is injective. Let $(i, g, \lambda, e), (i, g, \lambda, f) \in S$ and $N(i, g, \lambda, e)$ and $N(i, g, \lambda, f)$ denote the full selfconjugate regular subsemigroups of S generated by the respective element. By Lemma 15 it follows that $N(i, g, \lambda, e)\Pi_G = N(i, g, \lambda, f)\Pi_G$ and thus $N(i, g, \lambda, e) = N(i, g, \lambda, f)$.

REMARK. For inverse semigroups an equivalent formulation of Theorem 2 was proved by Zhitomirskiy [19] (for the Boolean case). In this case, I and Λ are trivial one element sets and (iv) can be omitted. The methods of his paper can be applied to the weaker variants of property P, too.

5. **Antigroups.** The aim of this section is to prove that for antigroups the only possibility of property P is Boolean. Let A be an antigroup and suppose that $\mathcal{C}(A)$ has the property P. By Lemmas 7 and 8 we observe that $\rho^T = \rho$ for all $\rho \in \mathcal{C}(A)$. Hence each congruence on A is uniquely determined by its trace. Therefore, the congruence lattice $\mathcal{C}(A)$ is isomorphic to $\mathcal{N}_A\big(E(A)\big)$, the lattice of all normal congruences on $E(A)$ which is a sublattice of $\mathcal{C}\big(E(A)\big)$. The following result for congruence lattices of semilattices was proved by Papert [14]:

Let $\tau, \rho \in \mathcal{C}(Y)$ and $\Sigma \subseteq \mathcal{C}(Y)$ for some semilattice Y. If $\sigma \cap \rho = \tau$ for all $\sigma \in \Sigma$ then $(\bigvee\{\sigma \in \Sigma\}) \cap \rho = \tau$. It is clear that this property also holds for each complete sublattice of the congruence lattice of a semilattice and therefore holds in $\mathcal{C}(A)$. In particular $\mathcal{C}(A)$ is pseudocomplemented, that is, for each $\rho \in \mathcal{C}(A)$ there exists a greatest element ρ^* satisfying $\rho \cap \rho^* = \varepsilon$. Furthermore, Papert showed that the elements of the form ρ^* form a complete Boolean sublattice of $\mathcal{C}(A)$ in which * is the operation of unique complementation. Now we are ready to prove the desired results:

LEMMA 17. *If for an antigroup A, $\mathcal{C}(A)$ is sectionally complemented then $\mathcal{C}(A)$ is Boolean.*

PROOF. Let $\rho \in \mathcal{C}(A)$ and ξ, η be complements of ρ. By the above result $\xi \vee \eta$ is also a complement of ρ. Since $\mathcal{C}(A)$ is sectionally complemented there exists $\zeta \in \mathcal{C}(A)$ satisfying $\zeta \cap \xi = \varepsilon$ and $\zeta \vee \xi = \xi \vee \eta$. By $\zeta \subseteq \xi \vee \eta$ we have $\rho \cap \zeta = \varepsilon$ and therefore $\varepsilon = (\xi \vee \rho) \cap \zeta = \omega \cap \zeta = \zeta$. Hence $\xi = \xi \vee \eta$. Therefore, $\mathcal{C}(A)$ is uniquely complemented. Since ρ and ρ^{**} are complements of $\rho^*, \rho = \rho^{**}$. By the cited result of Papert, $\mathcal{C}(A)$ is Boolean.

LEMMA 18. *If for an antigroup A, $\mathcal{C}(A)$ is dually sectionally complemented then $\mathcal{C}(A)$ is Boolean.*

PROOF. Let $\rho \in \mathcal{C}(A)$ and $\varepsilon \neq \rho \neq \omega$. Let $\mathrm{Co}(\rho)$ denote the set of all complements of ρ and $\mathrm{Nu}(\rho) = \{\tau | \tau \cap \rho = \varepsilon\}$. Then $\mathrm{Co}(\rho) \subseteq \mathrm{Nu}(\rho)$ and thus $\bigvee \mathrm{Co}(\rho) \subseteq \bigvee \mathrm{Nu}(\rho)$. We obtain that $\rho^* = \bigvee \mathrm{Nu}(\rho)$ is a complement of ρ, in fact the greatest element of $\mathrm{Co}(\rho)$. Now let $\tau \in \mathrm{Co}(\rho)$. Since $\mathcal{C}(A)$ is dually sectionally complemented there exists $\xi \in \mathcal{C}(A)$ satisfying $\xi \cap \rho^* = \tau$ and $\xi \vee \rho^* = \omega$. First $\xi \supseteq \tau$ implies $\xi \vee \rho = \omega$. Also $\xi \subseteq \xi^{**}$ and $\rho \subseteq \rho^{**}$ imply $\varepsilon^* = \omega = \xi^{**} \vee \rho^{**} = (\xi^* \cap \rho^*)^*$. Hence $\xi^* \cap \rho^* = \varepsilon = \xi^* \cap \xi$. Therefore $\varepsilon = \xi^* \cap (\rho^* \vee \xi) = \xi^* \cap \omega = \xi^*$. By $\xi^* \vee \xi = \omega$ we get $\xi = \omega$. Then $\tau = \rho^* \cap \xi$ implies $\tau = \rho^*$. Again $\mathcal{C}(A)$ is uniquely complemented and thus, by the same argument as above, even is Boolean.

Now let $\{A_j | j \in J\}$ be a set of congruence free antigroups (for an extensive treatment of these semigroups consult [17]). An inverse subdirect product S of the semigroups A_j is a *direct sum of the semigroups A_j* if for all $(a_j), (b_j) \in S$, $a_j = b_j$ for all $j \in J \setminus K$ where K is a finite subset of J. In this case we write $S = \sum\{A_j | j \in J\}$. Furthermore, given a direct sum of congruence free antigroups $S = \sum A_j$ we may assume that the representation is reduced, that is, for each $j \in J$ Π_{j^*}, the projection of S onto $\Pi\{A_i | i \neq j\}$, is not injective. For a direct sum of congruence free antigroups $S = \sum\{A_j | j \in J\}$ and a set $K \subseteq J$ the relation $(a_j)\ \rho_K\ (b_j) \Leftrightarrow a_k = b_k$ for all $k \in K$ is a congruence on S. By a result of Tanaka (see [5]) we get that the congruence lattice of an antigroup A is Boolean if and only if $A = \sum\{A_j | j \in J\}$, a direct sum of congruence free antigroups such that each congruence on A is of the form ρ_K for some $K \subseteq J$. In this case $\mathcal{C}(A) \cong \mathcal{P}(J)$. The *problem* now is to give a characterization of those direct sums of congruence free antigroups such that each congruence on such a semigroup is of the form ρ_K for some subset of the index set. This question is answered only partially (see next section).

A conclusion for orthodox semigroups is the following:

COROLLARY 5. *If the congruence lattice $\mathcal{C}(S)$ of an orthodox semigroup S has the property P then $\mathcal{C}(S)$ is at least relatively complemented.*

PROOF. By [11], $\mathcal{C}(I \times G \times \Lambda) \cong \mathcal{E}(I) \times \mathcal{C}(G) \times \mathcal{E}(\Lambda)$ and so if $\mathcal{C}(S)$ has the property P then $\mathcal{C}(S) \cong \mathcal{E}(I) \times \mathcal{C}(G) \times \mathcal{E}(\Lambda) \times \mathcal{C}(A)$. Now $\mathcal{C}(G)$ is complemented and hence complemented modular, $\mathcal{C}(A)$ is Boolean and the lattice of all equivalence relations of a set is relatively complemented.

6. **A further decomposition of S.** We use a result of the author to give a more detailed description of orthodox semigroups whose congruence lattices are relatively complemented (complemented modular, Boolean). For an arbitrary semigroup S, $S^* = S$ if S has no zero and $S^* = S\setminus\{0\}$ if 0 is the zero of S. A semilattice X is a locally finite tree if each interval $[x, y]$ in X is a finite chain. In a locally finite tree X to each element $\alpha \in X^*$ there exists a unique $\alpha+ \in X$ which is *covered* by α, denoted by $\alpha \succ \alpha+$, that is, $\alpha > \alpha+$ and $\alpha > \beta \geq \alpha+$ implies $\beta = \alpha+$. We need the following

CONSTRUCTION. Let X be a locally finite tree, to each $\alpha \in X$ associate a 0-simple semigroup I_α $(\neq \{0\})$ so that $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta$. For $\alpha \in X^*$ let $f_\alpha : I_\alpha^* \longrightarrow I_{\alpha+}^*$ be a partial homomorphism where $\alpha+$ denotes the unique element of X such that $a \succ \alpha+$. Let $f_{\alpha,\alpha} = \mathrm{id}_{I_\alpha^*}$ and $f_{\alpha,\beta}$ be defined by $f_{\alpha,\beta} = f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_n}$ where the elements α_i are defined by $\alpha = \alpha_1 \succ \alpha_2 \cdots \alpha_n \succ \beta$. We suppose that for arbitrary $a \in I_\alpha^*$ and $b \in I_\beta^*$ the set

$$D(a,b) = \{\gamma \in X : (af_{\alpha,\gamma})(bf_{\beta,\gamma}) \text{ is defined in } I_\gamma^*\}$$

is not empty. Let $\delta(a,b)$ denote the greatest element of $D(a,b)$. Let $S = \bigcup(I_\alpha^* : \alpha \in X)$ and define a multiplication $*$ on S by the rule

$$a * b = (af_{\alpha,\delta(a,b)})(bf_{\beta,\delta(a,b)}) \qquad (a \in I_\alpha^*, b \in I_\beta^*)$$

where the right hand side product is defined in $I_{\delta(a,b)}^*$.

DEFINITION. The groupoid S is a *tree of 0-simple semigroups*, to be denoted by $S = (X; I_\alpha, f_{\alpha,\beta})$. If each I_α, $\alpha \in X$, is congruence free (with zero and not the null semigroup of order two) then S is a *tree of congruence free semigroups*.

If X has a least element μ then by definition I_μ^* is closed under multiplication and thus is a simple semigroup. If, in addition, S is a tree of congruence free semigroups then the congruence freeness of $I_\mu^* \cup \{0\}$ implies that I_μ^* consists of exactly one element. A straightforward verification shows that S is a semigroup.

Let P be one of the following lattice properties: sectionally complemented, relatively complemented, complemented modular, Boolean. In [2] the author has proved the following

THEOREM 3. *Let S be a globally idempotent semigroup. Then $\mathcal{C}(S)$ has the property P if and only if S is isomorphic to one of the following:*

(i) a simple semigroup I such that $\mathcal{C}(I)$ has property P,

(ii) a tree of congruence free semigroups $(X; I_\alpha, f_{\alpha,\beta})$ such that for each $x \in I_\alpha^*$, $y \in I_\beta^*$ there exists $\gamma \leq \alpha, \beta$ satisfying $xf_{\alpha,\gamma} = yf_{\beta,\gamma}$,

(iii) a tree of 0-simple semigroups $(X; I_\alpha, f_{\alpha,\beta})$ such that X has at least element μ, I_μ^* is of type (i) and S/I_μ^* is of type (ii).

Furthermore $\mathcal{C}(S) \cong \mathcal{P}(X^*)$ for type (ii) and $\mathcal{C}(S) \cong \mathcal{C}(I_\mu^*) \times \mathcal{P}(X^*)$ for type (iii).

REMARK. This theorem also is true for the property dually sectionally complemented.

In the light of this theorem we recognize that the contribution of this paper to the characterization of orthodox semigroups whose congruence lattice is P in fact only is of relevance if S is simple. If S is simple then A is simple. Suppose $E(A)$ has a least element e. Since $\mathcal{C}(A)$ is complemented to $a \in A$ there exists $f \in E(A)$ satisfying $f \leq a$. Then also $e \leq a$ for all $a \in A$. In this case e is the zero of A, a contradiction to simplicity if $|A| > 1$. Therefore if A is simple and $|A| > 1$ by condition (iv) of Theorem 2 we obtain $|I| = |\Lambda| = 1$. Thus for the case when S is simple we have two alternatives: S is a rectangular group $I \times G \times \Lambda$ or a subdirect product of a group G and a simple antigroup A. Furthermore if an orthodox semigroup S is congruence free then S is inverse since $\varepsilon^V = \varepsilon$ or $\varepsilon^V = \omega$ in which latter case S is a rectangular band and hence trivial (see also [4]). Now $\mathcal{E}(X)$ is relatively complemented for each set X, it is modular (Boolean) if and only if $|X| \leq 3$ ($|X| \leq 2$). The congruence lattice of a group G is complemented (and clearly modular) if and only if G is a direct sum of simple groups, and it is Boolean if and only if it is a direct sum of simple groups in which no Abelian factor appears twice (see [7]). We are ready to formulate

THEOREM 4. *Let S be an orthodox semigroup. Then $\mathcal{C}(S)$ is relatively complemented (complemented modular; Boolean) if and only if S is isomorphic to one of the following:*

(i) a rectangular band $I \times \Lambda$ ($|I|, |\Lambda| \leq 3$; $|I|, |\Lambda| \leq 2$),

(ii) a direct sum of simple groups (in which no Abelian factor appears twice for the Boolean case),

(iii) a direct product of a semigroup of type (i) and a semigroup of type (ii),

(iv) a simple direct sum of congruence free antigroups $A = \sum\{A_j | j \in J\}$ such that each congruence ρ on A is of the form $(a_i)\ \rho\ (b_i) \Leftrightarrow a_i = b_i$ for all $i \in K$ for some set $K \subseteq J$,

(v) a subdirect product of a semigroup of type (ii) and a semigroup of type (iv) such that for $(g, e), (g, f) \in S$, $g \neq 1$, $e < f$ where $e, f \in E(A)$ there exist $(x_1, a_1), \ldots (x_n, a_n) \in S$ such that $g = x_1^{-1} g^{\pm 1} x_1 \cdots x_n^{-1} g^{\pm 1} x_n$ and $f \leq a_1^{-1} e a_1 \cdots a_n^{-1} e a_n$,

(vi) a tree of congruence free inverse semigroups $(X; I_\alpha, f_{\alpha,\beta})$ such that for each $x \in I_\alpha^*$, $y \in I_\alpha^*$ there exists $\gamma \leq \alpha, \beta$ satisfying $x f_{\alpha,\gamma} = y f_{\beta,\gamma}$,

(vii) a tree of 0-simple semigroups $(X; I_\alpha, f_{\alpha,\beta})$ where X has a least element μ, S / I_μ^* is of type (vi) and I_μ^* is of type (i)–(v).

Furthermore, for the general case (vii) the congruence lattice can be factored in a direct product in the following way:

$$\mathcal{C}(S) \cong \mathcal{E}(I) \times \mathcal{C}(G) \times \mathcal{E}(\Lambda) \times \mathcal{P}(J) \times \mathcal{P}(X^*).$$

(For the cases (i)–(vi) some of these factors may be trivial.)

Semigroups of type (v) do exist, an example is the direct product of simple group and a simple congruence free antigroup.The structural description of the semigroups of type (iv) is still an open problem. For the remainder we use the factorization of $\mathcal{C}(S)$ to give some simple characterizations of the congruences U, T, V and the greatest and least elements of their classes. Each congruence $\rho \in \mathcal{C}(S)$ can be represented uniquely by a quintuple $(\rho_I, \rho_G, \rho_\Lambda, \rho_J, \rho_X) \in \mathcal{E}(I) \times \mathcal{C}(G) \times \mathcal{E}(\Lambda) \times \mathcal{P}(J) \times \mathcal{P}(X^*)$. The congruences U, T, V can be characterized in the following way:

$$\begin{aligned} \rho\ U\ \theta &\Leftrightarrow \rho_J = \theta_J \text{ and } \rho_X = \theta_X, \\ \rho\ T\ \theta &\Leftrightarrow \rho_I = \theta_I, \rho_\Lambda = \theta_\Lambda, \rho_J = \theta_J, \rho_X = \theta_X, \\ \rho\ V\ \theta &\Leftrightarrow \rho_G = \theta_G, \rho_J = \theta_J, \rho_X = \theta_X. \end{aligned}$$

Therefore, for the greatest and least elements of the associated congruence classes we obtain

$$\begin{aligned} \rho_U &= (\varepsilon, \varepsilon, \varepsilon, \rho_J, \rho_X),\ \rho^U = (\omega, \omega, \omega, \rho_J, \rho_X) \\ \rho_T &= (\rho_I, \varepsilon, \rho_\Lambda, \rho_J, \rho_X),\ \rho^T = (\rho_I, \omega, \rho_\Lambda, \rho_J, \rho_X), \\ \rho_V &= (\varepsilon, \rho_G, \varepsilon, \rho_J, \rho_X),\ \rho^V = (\omega, \rho_G, \omega, \rho_J, \rho_X). \end{aligned}$$

Furthermore, for the congruence lattice we obtain

$$\mathcal{C}(S) \cong \rho U \times \mathcal{C}(S)/U \cong \rho T \times \rho V \times \mathcal{C}(S)/U \qquad \big(\rho \in \mathcal{C}(S)\big).$$

Similar characterizations can be obtained for the congruences T_r and T_l (for a definition see [15,16]).

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