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Tightness of manifolds with H-spherical ends

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Abstract. In this article we prove a Chern–Lashof inequality for immersions of manifolds with H-spherical ends. Related to this inequality we discuss different types of tightness. In particular we shall prove that an immersion of a manifold with at least two H-spherical ends is tightly immersed only if the ends are of a certain geometric type (Wintgen immersion).

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1. Introduction

The starting point for the theory of tightness was the so-called *Chern–Lashof inequality* [4], [5]. This inequality gives a lower estimate (the *Morse number)* for the *total absolute curvature* of an immersion $F\colon Y \to \mathbb{R}^m$, where Y denotes a compact manifold. The Morse number itself is bounded below by the *total Betti number,* as elementary Morse Theory shows.

One calls such an immersion *tight* if the Chern–Lashof inequality is in fact an equality. This geometric condition can also be expressed in terms of homology ([3]). For two-dimensional closed Y tightness is equivalent to the so-called *two-piece-property* (TPP), i.e. every hyperspace cuts the manifold into at most two pieces.

The concept of tightness had been mainly restricted to immersions of compact manifolds for a long time. In [9] this was extended to certain immersions of noncompact manifolds, to *Wintgen immersions* of manifolds with finitely many ends. For such immersions there is a hierarchy of tightness conditions: *strong tightness, tightness* and *weak tightness.*

In this paper we consider almost arbitrary immersions (we shall call them *semi-Wintgen immersions*) of noncompact manifolds. We restrict our consideration to *manifolds with H-spherical ends,* i.e. each end is an homology sphere. The most popular example of this type of manifolds are obtained by removing finitely many points from a compact manifold.

We shall extend the Chern–Lashof inequality stated in [9] to immersions of such manifolds. We shall prove in Section 3 the following inequality:

$$\text{tac}(X, F) \geqslant \beta(X) - 1, \quad \text{if } X \text{ has one end,}$$

and

$$\text{tac}(X, F) \geqslant \beta(X) - 2, \quad \text{if } X \text{ has at least two ends,}$$

where $\text{tac}(X, F)$ denotes the total absolute curvature and $\beta(X)$ the total Betti number. Thus we have to distinguish the discussion by the number of ends of the manifold.

This inequality yields to another tightness condition: *total tightness*, i.e. the inequality above is an equality.

We examine these different conditions of tightness (mentioned above) in detail in Section 4. The main result is there that a submanifold with at least two H-spherical ends is total tight only if the ends are of a certain geometric type: *Wintgen immersion*, in other words there are only finitely many direction along which the immersion moves to infinity. These directions are called *limit directions*. In detail, we shall prove the following relations for an immersion $F: X \to \mathbb{R}^m$ of a manifold with H-spherical ends:

In the case of exactly one end we have the following implications:

$$\text{strong tight} \iff \text{tight} \iff \text{weak tight} \wedge \beta_{n-1}(X) = 0$$

and

$$\text{total tight} \implies F \text{ is not a Wintgen immersion.}$$

In the case of at least two ends the following holds:

$$\text{strong tight} \implies \text{total tight} \iff \text{tight} \iff \text{weak tight} \wedge \text{vol}\,(C_\infty) = c_{m-1}$$

and in particular

$$\text{total tight} \implies F \text{ is a Wintgen immersion.}$$

($\text{vol}(C_\infty = \text{Vol}(S^{m-1})$ means that the convex hull of limit directions fills in the sphere.) In the sequel every manifold and every function is supposed to be smooth. A *manifold* is an unbounded manifold with finitely many ends in the sense of Freudenthal (see e.g. [9]). For basic results in topology we refer to [6] and [7]. For basic results in differential topology and Morse Theory we refer to [14] resp. [13].

The related topic of tautness in the noncompact case is considered in [2].

We shall restrict our considerations to a certain type of homology at infinity of X. For this we recall the definition of the homology at the ends [9]:

PROPOSITION and DEFINITION 1.1. (i) *Let* (U_ν) *be a countable base of punctured neighbourhoods of the end* ∞_κ. *Let* H_* *be a homology theory. Then the definition of the* homology of the end ∞_κ

$$H_*(\infty_\kappa) := \text{ invlim }_{\nu\in\mathbb{N}} H_*(U_\nu)$$

does not depend on the choice of (U_ν).

(ii) *We call* X *a* manifold with H-spherical ends *if* $H_\nu(\infty) := \oplus_\kappa H_\nu(\infty_\kappa)$ *is trivial for every* $\nu \neq 0, n-1$.

Proof. See [9]. □

EXAMPLE 1.2. If one removes finitely many points from a compact manifold one obtains a manifold with H-spherical ends.

We shall examine the *total absolute curvature* of immersions $F: X \to \mathbb{R}^m$ of manifolds X with H-spherical ends.

DEFINITION 1.3. Let $F: X \to \mathbb{R}^m$ be a proper immersion.

(i) The determinant of the shape operator $L: BX \to \mathbb{R}$ is called the *Lipschitz–Killing curvature* defined as a function on the *unit normal bundle* BX. L is the Gaussian curvature in the case of hypersurfaces.

In the case of orientable X the normalized Lebesgue integral

$$\mathrm{tac}(X,F) := \frac{1}{c_{m-1}}\int_{BX} |L|\mathrm{d}A, \qquad c_{m-1} := \mathrm{Vol}(S^{m-1}),$$

where $\mathrm{d}A$ is the induced volume element of BX is called the *total absolute curvature of* F. S^{m-1} denotes the sphere of unit vectors at the origin in $\mathbb{R}^m$. For non-orientable X one defines the total absolute curvature by using the orientable double covering. For more details of these definitions see [3], [8].

(ii) For $e \in S^{m-1}$ we call the mapping

$$h_e: X \to \mathbb{R}, \qquad x \mapsto \langle F(x), e\rangle$$

the e-*height function* h_e *with respect to* e *and* F.

(iii) For $f: X \to \mathbb{R}, \nu \in \mathbb{N}$ the νth *Morse number of* $f, \mu_\nu(f)$, is defined by

$$\mu_\nu(f) := \{x \in X | x \text{ is a non} - \text{degenerate critical point for } f \text{ of index } \nu\}.$$

The *(total) Morse number of* f is defined by:

$$\mu(f) := \sum_{\nu=0}^{n} \mu_\nu(f).$$

$\beta_\nu(\ ) := \dim(H_\nu(\ ))$ is called the νth *Betti number* and $\beta(\ ) := \sum_\nu \beta_\nu(\ )$ the *total Betti number*.

2. Morse Theory and generalized Morse inequality

In this section we generalize the Morse inequality given in [9] to a wider class of functions. We shall consider *Morse–Palais–Smale functions*, i.e. functions that obey the following definition:

DEFINITION 2.1. We call a differentiable function $f: X \to \mathbb{R}$ *Morse–Palais–Smale–function* if f satisfies the *Morse condition* and the *Palais–Smale condition*.

The Morse condition requires that f has only finitely many critical points, in the sense of Braess [1]. The Palais–Smale condition (or condition C, [19]) is satisfied, if there is no sequence $(x_\nu) \in X^{\mathbb{N}}$ such that

$$(x_\nu) \to \infty, \qquad df(x_\nu) \to 0, \qquad |f(x_\nu)| \text{is bounded.}$$

E.g. the Palais–Smale condition is satisfied for proper f.

If the Morse–Palais–Smale function f has critical points we denote the smallest critical value by $r_f \in \mathbb{R}$, otherwise $r_f := 0$. The set $E_f := f^{-1}((-\infty, r_f))$ is called the *lower end of* f.

The Palais–Smale condition requires that there is no critical point at 'infinity'. Note, we do not claim f to be bounded, in contrast to [17], [18] and [19]. This will change essentially the Morse inequalities.

Let us recall the Main Theorem for Morse–Palais–Smale functions:

THEOREM 2.2. *Let $f: X \to \mathbb{R}$ be a Morse–Palais–Smale function. Then the homotopy type of the* sublevelsets $f^{-1}((-\infty, t])$, $t \in \mathbb{R}$ *changes exactly at the critical levels. Indeed, at a critical value the homotopy type changes by adding a cell of the same dimension as the index of the corresponding critical point.*

Proof. See [10]. □

Now we can state a very first version of the desired inequality:

COROLLARY 2.3. *Let $f: X \to \mathbb{R}$ be a Morse–Palais–Smale function. Then*

$$\mu_\nu(f) \geqslant \beta_\nu(X, E_f) \quad \textit{for } \nu \in \mathbb{N}.$$

Proof. We get by Theorem 2.2:

$$\mu_\nu(f) \geqslant \beta_\nu(X \backslash E_{-f}, E_f),$$

as in ([19] or [13]). Furthermore

$$H_*(X \backslash E_{-f}) \cong H_*(X).$$

Therefore we get by considering the exact sequence of the pair $X \backslash E_{-f}, X$:

$$H_*(X, X \backslash E_{-f}) \cong \{0\}.$$

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We conclude with the exact sequence of the triple E_f, $X\backslash E_{-f}, X$:

$$H_*(X \setminus E_{-f}, E_f) \cong H_*(X, E_f). \qquad \square$$

This inequality together with the following homology inequality enables us to state the desired lower bound for $\mathrm{tac}(X, F)$.

LEMMA 2.4.*Let X have H-spherical ends and $f: X \to \mathbb{R}$ be a Morse–Palais–Smale function. Then*

$$\tfrac{1}{2}(\beta(X, E_f) + \beta(X, E_{-f})) \geqslant \beta(X) - 1, \quad \text{if } k = 1,$$

$$\tfrac{1}{2}(\beta(X, E_f) + \beta(X, E_{-f})) \geqslant \sum_{\nu=0}^{n} |\beta_\nu(X) - \tfrac{1}{2}\beta_\nu(\infty)| = \beta(X) - 2, \text{if } k \geqslant 2.$$

Moreover, equality in the second case implies that $E_f \cup E_{-f}$ is a punctured neighbourhood of every end. (k denotes the number of ends of X.)

Proof. In the case of a surface X, i.e. $n = 2$, we get by the exact sequence of the pair E_f, X:

$$\begin{aligned}\beta_1(X, E_f) &\geqslant \beta_1(X) - \beta_1(E_f) - \beta_0(X) + \beta_0(E_f)\\ &\geqslant \beta_1(X) - 1 + \beta_0(E_f) - \beta_1(E_f),\end{aligned}$$

if $E_f \neq \emptyset$, otherwise

$$\beta_\nu(X, E_f) = \beta_\nu(X).$$

This shows the assertion.

Let now $n \geqslant 3$. Let $U_1, \ldots, U_k$ be disjoint connected neighbourhoods of the ends $\infty_1, \ldots, \infty_k$. We define:

$$\hat{E}_f := E_f \backslash \cup_{U_\kappa \cap E_{-f} = \emptyset} U_\kappa, \qquad \hat{E}_{-f} := E_{-f} \backslash \cup_{U_\kappa \cap E_f = \emptyset} U_\kappa,$$

where $E_f \cup E_{-f} \subseteq \cup_{\kappa=1}^{k} U_\kappa$, w.l.o.g. If $\hat{E}_f = \emptyset$ or $\hat{E}_{-f} = \emptyset$, i.e. $E_f \cup E_{-f}$ is a punctured neighbourhood of every end, the proof can be finished as in [9].

Suppose $\hat{E}_f \neq \emptyset$ and $\hat{E}_{-f} \neq \emptyset$. (That means there are ends which are cut into at least two pieces. The homology of these end pieces is arbitrary, in general. But the homology differs from the homology of the ends which are cut in exactly one piece by vanishing of the $(n-1)$th homology. Therefore we shall distinguish between these two types.)

Consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & \downarrow & & & & \\
\longrightarrow & H_\nu(\hat{E}_f) & \longrightarrow & H_\nu(E_f) & \longrightarrow H_\nu(E_f,\hat{E}_f) \longrightarrow & H_{\nu-1}(\hat{E}_f) & \longrightarrow & & \\
 & & & & \downarrow & & & & \\
 & & & & H_\nu(X,\hat{E}_f) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_\nu(X,E_f) & & & & \\
 & & & & \downarrow & & & & \\
\longrightarrow & H_{\nu-1}(\hat{E}_f) & \longrightarrow & H_{\nu-1}(E_f) & \longrightarrow H_{\nu-1}(E_f,\hat{E}_f) \longrightarrow & H_{\nu-2}(\hat{E}_f) & \longrightarrow & & \\
 & & & & \downarrow & & & &
\end{array}
$$

For $\nu \neq 1, 2, n-1, n$ we get

$$H_\nu(E_f, \hat{E}_f) = \{0\}, \qquad H_{\nu-1}(E_f, \hat{E}_f) = \{0\},$$

therefore

$$\beta_\nu(X, E_f) = \beta_\nu(X, \hat{E}_f).$$

The diagram gives also:

$$\beta_n(X, \hat{E}_f) + \beta_{n-1}(E_f, \hat{E}_f) + \beta_{n-1}(X, E_f) = \beta_n(X, E_f) + \beta_{n-1}(X, \hat{E}_f),$$

for this we used $\beta_n(E_f, \hat{E}_f) = 0$ (since $\beta_{n-1}(\hat{E}_f) = 0$). By the same argument we get by considering the next exact sequence

$$\to H_n(E_f, \hat{E}_f) \to H_{n-1}(\hat{E}_f) \to H_{n-1}(E_f) \to$$

$$\to H_{n-1}(E_f, \hat{E}_f) \to H_{n-2}(\hat{E}_f) \overset{\cong}{\to} H_{n-2}(E_f) \to,$$

the following equality

$$\beta_{n-1}(E_f) = \beta_{n-1}(E_f, \hat{E}_f).$$

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Thus, since $\beta_n(X, \hat{E}_f) = 0$, we obtain:

$$\beta_{n-1}(E_f) + \beta_{n-1}(X, E_f) = \beta_n(X, E_f) + \beta_{n-1}(X, \hat{E}_f). \qquad (*)$$

In the case of three-dimensional X the exact sequence of the pair X, E_f, resp. X, $\hat{E}_f$,

$$\to H_\nu(E_f) \to H_\nu(X) \to H_\nu(X, E_f) \to,$$

$$\to H_\nu(\hat{E}_f) \to H_\nu(X) \to H_\nu(X, \hat{E}_f) \to$$

shows the equalities

$$\chi(X) = \chi(X, E_f) + \chi(E_f)$$

and

$$\chi(X) = \chi(X, \hat{E}_f) + \chi(\hat{E}_f).$$

Now:

$$\begin{aligned}
\sum_{\nu=0}^{3} \beta_\nu(X, E_f) &= 2\beta_2(X, E_f) - \chi(X, E_f) \\
&= 2\beta_2(X, E_f) - \chi(X, \hat{E}_f) - \chi(\hat{E}_f) + \chi(E_f) \\
&= \sum_{\nu=0}^{3} \beta_\nu(X, \hat{E}_f) + 2(\beta_2(X, E_f) - \beta_2(X, \hat{E}_f) \\
&\quad +\beta_0(E_f) - \beta_1(E_f) + \beta_2(E_f) - \beta_0(\hat{E}_f) + \beta_1(\hat{E}_f) \\
&= \sum_{\nu=0}^{3} \beta_\nu(X, \hat{E}_f) + 2(\beta_2(X, E_f) - \beta_2(X, \hat{E}_f) + 2\beta_2(E_f) \\
&\overset{(*)}{=} \sum_{\nu=0}^{3} \beta_\nu(X, \hat{E}_f) + 2\beta_3(X, E_f). \qquad (**)
\end{aligned}$$

Let now $n \geqslant 4$. We get again by the diagram above:

$$\beta_1(E_f, \hat{E}_f) - \beta_0(\hat{E}_f) + \beta_0(E_f) - \beta_0(E_f, \hat{E}_f) = 0,$$

$$\beta_0(E_f, \hat{E}_f) + \beta_1(X, \hat{E}_f) + \beta_2(X, E_f) = \beta_2(X, \hat{E}_f) + \beta_1(E_f, \hat{E}_f) + \beta_1(X, E_f),$$

therefore

$$\beta_1(X, \hat{E}_f) + \beta_2(X, E_f) + \beta_0(E_f) = \beta_2(X, \hat{E}_f) + \beta_1(X, E_f) + \beta_0(\hat{E}_f)$$

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Altogether for $n \geqslant 4$

$$\begin{aligned}\sum_{\nu=0}^{n} \beta_\nu(X, E_f) &= \sum_{\nu=0}^{n} \beta_\nu(X, \hat{E}_f) - \beta_1(X, \hat{E}_f) - \beta_2(X, \hat{E}_f) - \beta_{n-1}(X, \hat{E}_f) \\ &\quad + \beta_1(X, E_f) + \beta_2(X, E_f) + \beta_{n-1}(X, E_f) + \beta_n(X, E_f) \\ &= \sum_{\nu=0}^{n} \beta_\nu(X, \hat{E}_f) + 2\beta_n(X, E_f) + 2(\beta_2(X, E_f) - \beta_2(X, \hat{E}_f)).\end{aligned}$$

The five Lemma applied to the following diagram:

$$\begin{array}{ccccccccccc} \longrightarrow & H_2(\hat{E}_f) & \longrightarrow & H_2(X) & \longrightarrow & H_2(X, \hat{E}_f) & \longrightarrow & H_1(\hat{E}_f) & \longrightarrow & H_1(X) & \longrightarrow \\ & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong & \\ \longrightarrow & H_2(E_f) & \longrightarrow & H_2(X) & \longrightarrow & H_2(X, E_f) & \longrightarrow & H_1(E_f) & \longrightarrow & H_1(X) & \longrightarrow \end{array}$$

gives $\beta_2(X, \hat{E}_f) = \beta_2(X, E_f)$. Thus

$$\sum_{\nu=0}^{n} \beta_\nu(X, E_f) = \sum_{\nu=0}^{n} \beta_\nu(X, \hat{E}_f) + 2\beta_n(X, E_f), \qquad (***)$$

which is valid for $\nu \geqslant 3$ (see $(**)$).

We denote the ends which have a nonvoid intersection with $\hat{E}_f$ or $\hat{E}_{-f}$ by $\hat{\infty}$, i.e.

$$\hat{\infty} = \cup_{U_\kappa \cap E_f \neq \emptyset} U_\kappa \cup_{U_\kappa \cap E_{-f} \neq \emptyset} U_\kappa.$$

Consider the next diagram:

$$\begin{array}{ccccccccc} \longrightarrow & H_{\nu+1}(X, \hat{\infty}) & \longrightarrow & H_\nu(\hat{\infty}) & \xrightarrow{i_1} & H_\nu(X) & \longrightarrow & H_\nu(X, \hat{\infty}) & \longrightarrow \\ & \downarrow\cong & & \downarrow j_1 & & \downarrow j_2 & & \downarrow\cong & \\ \longrightarrow & H_{\nu+1}(X, \hat{\infty}) & \longrightarrow & H_\nu(\hat{\infty}, \hat{E}_f) & \xrightarrow{i_2} & H_\nu(X, \hat{E}_f) & \longrightarrow & H_\nu(X, \hat{\infty}) & \longrightarrow \end{array}$$

Diagram chasing ([7]) shows:

$$\ker(j_2) = i_1(\ker(j_1)).$$

We receive

$$\beta_\nu(X) \leqslant \beta_\nu(X, \hat{E}_f) \quad \text{for } \nu \neq n-1, 0,$$

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because $i_1 = j_1 = 0$ for $\nu \neq n-1, 0$.

We conclude:

$$\begin{aligned}
&\sum_{\nu=0}^{n}\beta_\nu(X,E_f)+\sum_{\nu=0}^{n}\beta_\nu(X,E_{-f})-2\sum_{\nu=0}^{n}\beta_\nu(X)+2\\
&\quad=\sum_{\nu=0}^{n}\beta_\nu(X,\hat{E}_f)+\sum_{\nu=0}^{n}\beta_\nu(X,\hat{E}_{-f})\\
&\qquad+2\beta_n(X,E_f)+2\beta_n(X,E_{-f})-2\sum_{\nu=0}^{n}\beta_\nu(X)+2\\
&\quad\overset{(***)}{\geqslant}\beta_{n-1}(X,\hat{E}_f)+\beta_{n-1}(X,\hat{E}_{-f})-2\beta_{n-1}(X)\\
&\qquad+2\beta_n(X,E_f)+2\beta_n(X,E_{-f})\\
&\quad\overset{(*)}{\geqslant}\beta_{n-1}(E_f)-\beta_{n-1}(X)+\beta_{n-1}(X,E_f)\\
&\qquad+\beta_{n-1}(E_{-f})-\beta_{n-1}(X)+\beta_{n-1}(X,E_{-f})\\
&\quad\geqslant 0,
\end{aligned}$$

at the last step we used the exact sequence of the pair E_f, X resp. E_{-f}, X. This shows the assertion for $k = 1$. For the case $k \geqslant 2$ we finish by showing the following inequality:

$$\beta(X)-1>\sum_{\nu=0}^{n}|\beta_\nu(X)-\tfrac{1}{2}\beta_\nu(\infty)|=\beta(X)-2.$$

(Thus equality in the desired inequality may only appear if every end is cut in exactly one piece.)

The exact sequence of the pair ∞, X:

$$\{0\}\to H_n(X,\infty)\to H_{n-1}(\infty)\to H_{n-1}(X)\to H_{n-1}(X,\infty)\to$$

together with $\beta_n(X,\infty)=1$ proves:

$$1+\beta_{n-1}(X)\geq\beta_{n-1}(\infty).$$

Thus we get for $k\geqslant 2$

$$\beta_{n-1}(X)\geqslant\frac{\beta_{n-1}(\infty)}{2}.$$

This completes the proof. □

COROLLARY 2.5. *Let X have H-spherical ends and $f: X \to \mathbb{R}$ be a Morse–Palais–Smale function. Then the following Morse inequality holds:*

$$\mu(f) \geq \beta(X) - 1, \quad \text{if } k = 1,$$

and

$$\mu(f) \geqslant \sum_{\nu=0}^{n} |\beta_\nu(X) - \tfrac{1}{2}\beta_\nu(\infty)| = \beta(X) - 2, \quad \text{if } k \geqslant 2.$$

Proof. Combine Lemma 2.4 and Corollary 2.3. □

3. Total absolute curvature and tightness

We are interested in a lower bound of the total absolute curvature of an immersion $F: X \to \mathbb{R}^m$. The total absolute curvature is equal to the expectation value of critical points of a random non-degenerate height function. Therefore, we need an lower bound of the number of critical points of such an height function. For this we shall use the inequalities of Section 2. Thus we have to guarantee that almost every height function is a Morse–Palais–Smale function, i.e. we shall consider only those immersions that corresponding height functions are Morse-Palais-Smale functions:

DEFINITION 3.1. We call an immersion $F: X \to \mathbb{R}^m$ semi-Wintgen immersion if almost every height function with respect to F is a Morse–Palais–Smale function.

F is called Wintgen immersion if there are only finitely many limit direction corresponding to F. (A limit direction is a direction $v \in S^{m-1}$ such that there exists a sequence $(x_\nu) \in X^{\mathbb{N}}$ with $\lim_{\nu\to\infty} \frac{F(x_\nu)}{\|F(x_\nu)\|} = v$ [9], [21].)

In a geometric sense semi-Wintgen immersion are immerions whose corresponding normal vectors at infinity have measure zero as a subset of S^{m-1}. Rather odd example show that this does not have to be the case in general, however for our purpose this will be the case,see below.

Every Wintgen immersion is a semi-Wintgen immersion.

PROPOSITION 3.2. *Let $F: X \to \mathbb{R}^m$ be a proper immersion. F is a semi-Wintgen immersion if μ is almost everywhere continuous.*

Proof. For the definition of μ see Definition 1.3(iii). μ is almost everywhere continuous if and only if μ is almost everywhere locally constant, i.e. if for almost every $e \in S^{m-1}$ there is a neighbourhood in S^{m-1} where μ is constant. This is equivalent to the situation that there is a closed zero set $A \subset S^{m-1}$ such that μ is local constant in the complement of A.

The Gauss map is regular outside a closed zero set $A' \subset S^{m-1}$ (Sard's Theorem). Thus for every e in the complement of A' there is a neighbourhood $U \subset S^{m-1}$ such that the preimage of U under the Gauss map consists of $\mu(h_e)$ diffeomorphic

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subsets of the unit normal bundle which are disjoint to U. Now, if μ is almost everywhere constant we may assume μ to be constant in U. That means all the height functions corresponding to U satisfy the Palais–Smale condition. (For this let $e \in U$ such that h_e fails the Palais–Smale condition. Thus there exists to $\varepsilon \in \mathbb{R}_+$ and $R \in \mathbb{R}_+$ another direction $e' \in S^{m-1}$ such that the corresponding height function possesses a singularity outside the ball of radius R and $\|e' - e\| < \varepsilon$, i.e. the singularity lies outside the preimage of U but $e' \in U$, w.l.o.g. Thus $h_{e'}$ possesses more than $\mu(h_e)$ singularities. Contradiction.) □

We are interested in the concept of tightness of noncompact submanifolds. Obviously tightness implies at least that almost every height function has the same number of critical points (see [3]), i.e. μ is almost everywhere constant. Thus the restriction to semi-Wintgen immersions is redundant for our purpose.

THEOREM 3.3. *Let $F\colon X \to \mathbb{R}^m$ be a semi-Wintgen immersion. Then*

$$\operatorname{tac}(X, F) \geqslant \beta(X) - 1, \quad \text{if } k = 1,$$

and

$$\operatorname{tac}(X, F) \geqslant \sum_{\nu=0}^{n} |\beta_\nu(X) - \tfrac{1}{2}\beta_\nu(\infty)| = \beta(X) - 2, \quad \text{if } k \geqslant 2,$$

with equality in the second case only if F is a Wintgen immersion.

Proof. The integral of total absolute curvature is equal to the expectation value of the number of critical points of a random non-degenerate height function [3], [8]. Thus with the Morse inequality in Corollary 2.5 in turn we get the stated inequalities, where in the second case equality holds only if for almost every height function the corresponding (lower and upper) ends are neighbourhoods of every end of X. The last is only the case if almost every direction is not perpendicular to the set of limit directions, i.e. if there is only one limit direction w.r.t. each end. Thus F has to be a Wintgen immersion. □

According to the compact case we define in the next section *total tightness* as the case of equality.

4. Different types of tightness

We discuss different types of tightness: strong tightness, tightness, weak tightness and *total tightness*:

DEFINITION 4.1. Let X have H-spherical ends. We call a semi-Wintgen immersion $F\colon X \to \mathbb{R}^m$ *total tight* if the Chern–Lashof inequality of Theorem 3.3 is in fact an equality for F.

From ([9]) we recall:

DEFINITION 4.2. *Let* $F: X \to \mathbb{R}^m$ *be a Wintgen immersion.*

(i) F is called *weak tight* if for $\nu \in \mathbb{N}$ for almost every $e \in S^{m-1}$

$$H_i(X_e^r, E_{h_e}) \to H_i(X, E_{h_e})$$

is injective for $r \in \mathbb{R}$.

(ii) F is called *tight* if for $\nu \in \mathbb{N}$ for almost every $e \in S^{m-1}$

$$H_i(X_e^r) \oplus H_i(X_{-e}^r) \to H_i(X_e^{r'}) \oplus H_i(X_{-e}^{r'})$$

is injective for every $r, r' \in \mathbb{R}, r < r'$ or is surjective for every $r, r' \in \mathbb{R}, r < r'$.

(iii) F is called *strong tight* if for $\nu \in \mathbb{N}$ for almost every $e \in S^{m-1}$

$$H_i(X_e^r) \to H_i(X)$$

is injective for $r \in \mathbb{R}$.

Note, that in this definition F is required to be a Wintgen immersion in contrast to Definition 4.1.

In accord to Theorem 3.3 we have to distinguish the discussion between the cases of exactly one end and at least two ends.

THEOREM 4.3. *Let X have H-spherical ends and $F: X \to \mathbb{R}^m$ be a semi-Wintgen immersion. Then in the case of only one end:*

- *if F is total tight F cannot be a Wintgen immersion,*
- *F is tight if and only if F is strong tight, this is only the case if the $(n-1)$-th Betti number of X vanishes. If $\beta_{n-1}(X) = 0$ then (strong) tightness and weak tightness are equivalent.*

In the case of at least two ends F is total tight if and only if F is tight. Thus in this case total tightness implies F is a Wintgen immersion.

Proof. The first part follows by comparing the Chern–Lashof inequality for Wintgen immersions in [9] and the Chern–Lashof inequality in Section 3. The second part follows from the second part of Theorem 3.3. □

We give some examples for the first part of the theorem:

EXAMPLE 4.4. (i) $\mathbb{R}^n \subset \mathbb{R}^m$ is total tight but no Wintgen immersion. Despite there exists a tight immersion.

(ii) Consider the projective space $\mathbb{C}P^n$ as the subset of the euclidean space given by the hermitian mappings of $\mathbb{C}^{n+1}$. ($\mathbb{C}P^n$ is the subset given by the orthogonal projection with respect to a direction). The stereographic projection with respect to an element of $\mathbb{C}P^n$ gives a total tight immersion of the punctured projective space, since every height function possesses only singularities of even index. Evidently, this is no Wintgen immersion.

(iii) Consider the two-dimensional Clifford-torus immersed in S^3. Stereographic projection w.r.t. an element of the torus gives a total tight immersion of the punctured torus. But the punctured torus cannot be tightly immersed since $\beta_1 \neq 0$. For a visualization see [11].

(iv) One can generalize the examples above by taking a compact taut manifold. We may assume that such a submanifold is a subset of a Euclidean sphere. Stereographic projection yields to a total tight immersion.

In the case of one end the types tight and total tight are excluding each other on the other end tightness, weak tightness and strong tightness only differ by the topology of X. This situation is completely different if X has at least two ends. Then tightness and total tightness are the same and there is a hierarchy of strong tightness, tightness and weak tightness. The difference between strong tightness and tightness only depends on the topology of X ([9]). The difference between tightness and weak tightness is more subtle. In order to describe this difference we need the convex cone of limit directions, mentioned by [21]:

DEFINITION 4.5. Let $F\colon X \to \mathbb{R}^m$ be a Wintgen immersion. We call the set

$$\begin{aligned} C_\infty(F) \;:=\; \{e \in S^{m-1} | &\text{there is one limit direction } v \text{ such that } \langle e, v\rangle \geqslant 0,\\ &\text{there is another limit direction } w \text{ such that } \langle e, w\rangle \leqslant 0\} \end{aligned}$$

the *convex cone of limit directions*.

The volume of this convex cone is a useful measure for the grade of tightness of a weak tight immersion:

PROPOSITION 4.6. *Let X have H-spherical ends and $F\colon X \to \mathbb{R}^m$ be a Wintgen immersion.Then F is weak tight if and only if*

$$\mathrm{tac}(X, F) = \beta(X) - \frac{2}{c_{m-1}}\mathrm{vol}(C_\infty).$$

Proof. In the case of weak tightness a bounded height function possesses exactly $\beta(X)$ singularities since in this case the condition of weak tightness becomes the ‘common’ condition for height function on compact manifolds. We close the proof by proving that an unbounded height function possesses exactly $\beta(X) - 2$ singularities. For this let h_e be an unbounded height function. Consider the following homology sequence:

$$\begin{aligned} &\to H_{\nu+1}(E_{h_e}) \to H_{\nu+1}(X) \to H_{\nu+1}(X, E_{h_e}) \to \\ &\qquad \to H_\nu(E_{h_e}) \to H_\nu(X) \to H_\nu(X, E_{h_e}) \to \end{aligned}$$

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Thus

$$\beta_\nu(X) = \beta_\nu(X, E_{h_e}) \quad \text{for } \nu \in \mathbb{N}, 1 < \nu < n-1, \tag{$*$}$$

since $H_\nu(E_{h_e}) = \{0\}$.

X is noncompact, therefore $\beta_n(X) = 0$. This together with $\beta_{n-2}(E_{h_e}) = 0$ shows:

$$\beta_n(X, E_{h_e}) - \beta_{n-1}(E_{h_e}) + \beta_{n-1}(X) = \beta_{n-1}(X, E_{h_e}). \tag{$**$}$$

In addition:

$$\chi(X, E_{h_e}) + \chi(E_{h_e}) = \chi(X). \tag{$***$}$$

$(*)$ and $(***)$ prove:

$$\begin{aligned}
&\beta_0(X) - \beta_1(X) + (-1)^{n-1}\beta_{n-1}(X) + (-1)^n\beta_n(X)\\
&\qquad -\beta_0(E_{h_e}) - (-1)^{n-1}\beta_{n-1}(E_{h_e})\\
&\quad = \beta_0(X, E_{h_e}) - \beta_1(X, E_{h_e})\\
&\qquad +(-1)^{n-1}\beta_{n-1}(X, E_{h_e}) + (-1)^n\beta_n(X, E_{h_e}).
\end{aligned}$$

$E_{h_e} \neq \emptyset$ gives $\beta_0(X, E_{h_e}) = 0$ (X is supposed to be connected).

Now with $(**)$:

$$\beta_1(X, E_{h_e}) = \beta_1(X) + \beta_0(E_{h_e}) - \beta_0(X),$$

i.e.

$$\beta(X, E_{h_e}) = \beta(X) - 2 + \beta_0(E_{h_e}) + 2\beta_n(X, E_{h_e}) - \beta_{n-1}(E_{h_e}).$$

The ends of X are H-spherical. This implies $\beta_0(E_{h_e}) = \beta_{n-1}(E_{h_e})$. Therefore:

$$\beta(X, E_{h_e}) = \beta(X) - 2 + 2\beta_n(X, E_{h_e}).$$

We conclude by proving $\beta_n(X, E_{h_e}) = 1$ if h_e is unbounded below and $\beta_n(X, E_{h_e}) = 0$ if h_e is unbounded above.

We may assume that h_e has exactly one maximum if h_e is bounded above and h_e has no maximum, otherwise. For this consider the double of the bounded manifold

$$Y := h_e^{-1}([r_{h_e} - t, -r_{h_{-e}} + t]) \quad (t \in \mathbb{R}_+)$$

and apply [15].

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Now, we get by the discussion of Linking-Type and Non-Linking-Type singularities [16]

$$\beta_n(X, E_{h_e}) = 0$$

if h_e is unbounded above.

If h_e is bounded above we get $\beta_n(X, E_{h_e}) \leqslant 1$ since there is only one maximum. By Theorem 2.2:

$$H_*(X, E_{h_e}) \cong H_*(Y, \partial Y).$$

And for the double $Y \cup_\partial Y$ of Y

$$H_*(Y, \partial Y) \cong H_*(Y \cup_\partial Y, Y),$$

by the Excision axiom.

The double of Y is a compact n-dimensional manifold. Therefore

$$H_n(Y \cup_\partial Y) \neq \{0\},$$

by the Poincaré duality.

The exact sequence of the pair $Y, Y \cup_\partial Y$

$$\to H_n(Y) \to H_n(Y \cup_\partial Y) \to H_n(Y \cup_\partial Y, Y) \to$$

gives $H_n(Y \cup_\partial Y, Y) \neq \{0\}$. This proves the assertion. □

COROLLARY 4.7. *A weak tight immersion is (total) tight if and only if the convex cone of limit directions fills in the unit sphere. (X is supposed to have at least two H-spherical ends).*

Proof. Follows by the proposition above and the Chern–Lashof inequality in Section 3. □

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