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WEYL'S THEOREM FOR OPERATOR MATRICES AND THE SINGLE VALUED EXTENSION PROPERTY

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Abstract. If bounded linear operators A and B are each reguloid, and have the single valued extension property, then Weyl's theorem holds for all holomorphic functions of all operator matrices $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

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1. Introduction. Weyl's theorem is said to hold for a bounded linear operator T on a Banach space E if there is equality

$$\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T), \tag{1}$$

where σ and σ_w are the spectrum and the Weyl spectrum, and π_{00} the isolated eigenvalues of finite multiplicity. The weaker condition, that

$$\sigma(T) = \sigma_w(T) \cup \pi_{00}(T), \tag{2}$$

has been described as "Browder's theorem holds". See [**7**]. Curto and Han [**4**] have listed a number of equivalent conditions, among them the implication

$$\lambda \in \pi_{00}(T) \Longrightarrow T \text{ has SVEP at } \lambda. \tag{3}$$

Here $T \in B(E)$ is said to have the *single valued extension property* (SVEP) at λ provided that for arbitrary functions $f : U \to E$ holomorphic on neighborhoods U of λ, we have

$$(T - zI)f(z) \equiv 0 \Longrightarrow f(z) \equiv 0. \tag{4}$$

Let $H(K)$ denote the space of functions holomorphic on an open neighborhood of $K \subset \mathbb{C}$.

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In this note we ask to what extent Weyl's theorem (1) is transmitted from $T = A$ on X and $T = B$ on Y to

$$T = M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{5}$$

on $E = X \times Y$. Our main result is as follows.

THEOREM 1.1. *If $A \in B(X)$ and $B \in B(Y)$ are each reguloid, and have the single valued extension property, then Weyl's theorem holds for $f(M_C)$ for arbitrary $f \in H(\sigma(M_C))$ and for arbitrary bounded operators $C : Y \to X$.*

Here $T \in B(E)$ is said to be *reguloid* if

$$\lambda \in \text{iso}\,\sigma(T) \Longrightarrow T - \lambda I \in (T - \lambda I)B(E)(T - \lambda I) : \tag{6}$$

All isolated points of spectrum give rise to operators that have generalized inverses. Such operators are necessarily *isoloid* in the sense that all isolated points of the spectrum are eigenvalues, and also *closoid*, in the sense that

$$\lambda \in \text{iso}\,\sigma(T) \Longrightarrow (T - \lambda I)(E) = \text{cl}(T - \lambda I)(E), \tag{7}$$

giving rise to operators with closed range. Curto and Han [**4**, Theorem 2.2] showed that if $T \in B(E)$ has SVEP then the condition (7) is necessary and sufficient for Weyl's theorem.

A condition stronger than reguloid is that the operator T be *transaloid*, in the sense that the norm and the spectral radius coincide for all $T - \lambda I$. Curto and Han [**4**, Theorem 2.5] have shown that if $T \in B(E)$ has SVEP and is transaloid then Weyl's theorem holds for all holomorphic images $f(T)$. Our Theorem 1.1 improves this in two ways, by relaxing the transaloid condition to reguloid, and by extending to operator matrices.

2. Proof of Theorem 1.1.

To prove Theorem 1.1, we need several lemmas.

LEMMA 2.1. *Let $A \in B(X)$ and $B \in B(Y)$ have SVEP. If A is isoloid and B is reguloid, then M_C is isoloid.*

Proof. Suppose that $\lambda \in \text{iso}\sigma(M_C)$. If B has SVEP then $\sigma(B)$ coincides with the defect spectrum of B. It follows from [**5**, Theorem 2.3](or [**8**]) that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$. Therefore $\lambda \in \text{iso}\ (\sigma(A) \cup \sigma(B))$. Now we consider two cases. First, suppose that $\lambda \in \sigma(A)$. Then $\lambda \in \text{iso}\sigma(A)$. Since A is isoloid, $(A - \lambda I)^{-1}(0) \neq \{0\}$. Observe that $(A - \lambda I)^{-1}(0) \oplus \{0\} \subseteq (M_C - \lambda I)^{-1}(0)$, and hence $(M_C - \lambda I)^{-1}(0) \neq \{0\}$.

Secondly suppose that $\lambda \in \sigma(B) \setminus \sigma(A)$. Then $\lambda \in \text{iso}\sigma(B)$ and $A - \lambda$ is invertible. Since B is reguloid, $B - \lambda I$ is regular. This implies that $M_C - \lambda I$ is also regular because

$$M_C - \lambda I = \begin{pmatrix} I & 0 \\ 0 & B - \lambda I \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & 0 \\ 0 & I \end{pmatrix}. \tag{8}$$

In particular, $M_C - \lambda I$ has an invertible generalized inverse since $\lambda \in \text{iso}\sigma(M_C) \subseteq \partial\sigma(M_C)$ by [**6**, (3.8.6.1)]. Thus by [**6**, Theorem 7.3.4] we have

$$(M_C - \lambda I)^{-1}(0) \cong X/(M_C - \lambda I)(X \oplus Y). \tag{9}$$

Suppose that $(M_C - \lambda I)^{-1}(0) = \{0\}$. Then, by (9), $M_C - \lambda I$ is invertible. This contradicts the fact that $\lambda \in \sigma(M_C)$, and thus $(M_C - \lambda I)^{-1}(0) \neq \{0\}$. As we have now considered both cases, M_C is isoloid. $\square$

In this paper, assuming that $A \in B(X)$ and $B \in B(Y)$ both have SVEP, we deal with Weyl's theorem for operator matrices M_C. From this viewpoint, we have the following result.

LEMMA 2.2. *Let $A \in B(X)$ and $B \in B(Y)$ both have SVEP. If A and B are closoid, then Weyl's theorem holds for M_C.*

Proof. Since A and B both have SVEP, it follows from [**8**, Proposition 3.1] that M_C has SVEP. Thus, to prove Weyl's theorem for M_C, by (7), it suffices to show that

$$M_C - \lambda I \text{ has closed range for every } \lambda \in \pi_{00}(M_C). \tag{10}$$

Assume that $\lambda \in \pi_{00}(M_C)$; i.e.,

$$0 < \dim(M_C - \lambda I)^{-1}(0) < \infty. \tag{11}$$

Since B has SVEP, it follows as in Lemma 2.1 that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$. Hence $\lambda \in \text{iso}\,(\sigma(A) \cup \sigma(B))$. Now, let either $\lambda \in \sigma(A) \setminus \sigma(B)$ or $\lambda \in \sigma(B) \setminus \sigma(A)$. Then, observing that

$$M_C - \lambda I = \begin{pmatrix} I & 0 \\ 0 & B - \lambda I \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & 0 \\ 0 & I \end{pmatrix}, \tag{12}$$

it is obvious that $M_C - \lambda I$ has closed range because A and B are closoid and $\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ is invertible.

On the other hand, let $\lambda \in \sigma(A) \cap \sigma(B)$, so that λ is a common isolated point of $\sigma(A)$ and $\sigma(B)$. Because

$$(A - \lambda I)^{-1}(0) \oplus \{0\} \subseteq (M_C - \lambda I)^{-1}(0),$$

(11) implies that $(A - \lambda I)^{-1}(0)$ is finite dimensional. Since A is closoid, $A - \lambda I$ is semi-Fredholm. By the continuity of index, $A - \lambda I$ is Weyl. We claim that $B - \lambda I$ is Weyl, too. To prove this claim, it suffices to show that

$$(B - \lambda I)^{-1}(0) \text{ is finite dimensional} \tag{13}$$

because, in this case, the fact that B is closoid forces that $B - \lambda I$ is semi-Fredholm, and so Weyl. To prove (13), we borrow an argument in the proof of [**11**, Theorem 2.4]. To the contrary we assume that

$$(B - \lambda I)^{-1}(0) \text{ is infinite dimensional.} \tag{14}$$

We consider two cases.

Case 1. Suppose that $C((B - \lambda I)^{-1}(0))$ is finite dimensional. Then $C^{-1}(0)$ must contain an infinite sequence $\{y_i\}$ of linearly independent vectors in $(B - \lambda I)^{-1}(0)$. However, we can see that $(M_C - \lambda I)(0 \oplus y_i) = 0$, for each $i = 1, 2, \ldots$, and this implies that $(M_C - \lambda I)^{-1}(0)$ is infinite dimensional, a contradiction.

Case 2. Suppose that $C((B - \lambda I)^{-1}(0))$ is infinite dimensional. As mentioned above, $A - \lambda I$ is Weyl, and so $X/(A - \lambda I)(X)$ is finite dimensional. This implies that

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$C((B-\lambda I)^{-1}(0)) \cap (A-\lambda I)(X)$ is infinite dimensional. Thus we can find a sequence $\{y_i\}$ of linearly independent vectors in $(B-\lambda I)^{-1}(0)$ for which there exists a sequence $\{x_i\} \subseteq X$ such that $(A-\lambda I)x_i = Cy_i$ for each $i = 1, 2, \ldots$. Then we can see that $(M_C - \lambda I)(x_i \oplus -y_i) = 0$ for each $i = 1, 2, \ldots$, and this implies that $(M_C - \lambda I)^{-1}(0)$ is infinite dimensional, a contradiction. Hence we proved that if $\lambda \in \sigma(A) \cap \sigma(B)$, then $A - \lambda I$ and $B - \lambda I$ are Weyl, and so $M_C - \lambda I$ has closed range because $M_C - \lambda I$ also is Weyl. Consequently, this completes the proof. □

In general, the spectral mapping theorem fails for the Weyl spectrum [**2**, Example 3.3] but one way inclusion always holds [**2**, Theorem 3.2]: that is

$$\sigma_w(p(T)) \subseteq p(\sigma_w(T)), \quad \text{for each polynomial } p.$$

For the reverse inclusion, it is well known [**7**, Theorem 5] that

$$\sigma_w(p(T)) \supseteq p(\sigma_w(T)), \quad \text{for each polynomial } p \tag{15}$$

if and only if

$$i(T-\lambda I)\, i(T-\mu I) \geq 0, \quad \text{for each pair } \lambda, \mu \in \mathbb{C} \setminus \sigma_e(T). \tag{16}$$

Next, we have the following result.

Lemma 2.3. *Let $A \in B(X)$ and $B \in B(Y)$ have SVEP. Then*

$$\sigma_w(f(M_C)) = f(\sigma_w(M_C)) \text{ for } f \in H(\sigma(M_C)). \tag{17}$$

Proof. Since A and B have SVEP, M_C also has SVEP. Since $i(M_C - \lambda I) \leq 0$ for every $\lambda \in \mathbb{C} \setminus \sigma_e(M_C)$, from (15) and (16), we have

$$\sigma_w(p(M_C)) = p(\sigma_w(M_C)) \text{ for each polynomial } p. \tag{18}$$

Hence [**13**, Theorem 2] completes the proof. □

Proof of Theorem 1.1. Since M_C is isoloid by Lemma 2.1, [**10**, Lemma] implies that

$$f\big(\sigma(M_C) \setminus \pi_{00}(M_C)\big) = \sigma(f(M_C)) \setminus \pi_{00}(f(M_C)), \quad \text{for every } f \in H(\sigma(M_C)).$$

It follows from Lemma 2.2 and Lemma 2.3 that

$$\sigma(f(M_C)) \setminus \pi_{00}(f(M_C)) = f(\sigma(M_C) \setminus \pi_{00}(M_C)) = f(\sigma_w(M_C)) = \sigma_w(f(M_C)),$$

which implies that Weyl's theorem holds for $f(M_C)$. □

3. Applications.

Now, there are many interesting works ([**5**], [**7**], [**11**], [**12**]) which deal with Weyl's theorem and Browder's theorem for 2×2 operator matrices. In this section, we introduce two interesting earlier results, [**5**, Theorem 2.5] and [**11**, Theorem 2.4], which are strongly related to our main result. As applications of Theorem 1.1, we give an improvement of [**5**, Theorem 2.5] and an analogue of [**11**, Theorem 2.4], respectively.

We first consider the following result due to Djordjević and Han [**5**].

PROPOSITION 3.1 [**5**, Theorem 2.5]. *Let $A \in B(X)$ and $B \in B(Y)$ both have SVEP. Suppose that the following conditions hold:*

(a) *A and B are isoloid,*
(b) *$H_0(B - \lambda I)$ is finite dimensional for each $\lambda \in iso\, \sigma(B)$,*
(c) *Weyl's theorem holds for $A \oplus B$.*

Then Weyl's theorem holds for M_C, for arbitrary bounded operators $C : Y \to X$.

Here recall that the "quasinilpotent part" of $T \in B(E)$ is defined by

$$H_0(T) = \left\{x \in E : ||T^n x||^{\frac{1}{n}} \to 0\right\}.$$

We should like to point out that Theorem 1.1 and Proposition 3.1 suggest different sufficient conditions for M_C to obey Weyl's theorem. Actually, we can notice that the assumptions of Theorem 1.1 do not imply (b) of Proposition 3.1, and that the assumptions of Proposition 3.1 do not imply that A and B are reguloid. Moreover, as is shown in Theorem 3.3 below, it is interesting that, without any additional conditions, we can reach the same conclusion that Weyl's theorem holds for $f(M_C)$.

To show this, we first need the following lemma.

LEMMA 3.2. *Let $A \in B(X)$ and $B \in B(Y)$ both have SVEP. If* (a), (b), *and* (c) *in Proposition* 3.1 *are assumed, then M_C is isoloid.*

Proof. To show that M_C is isoloid, suppose that $\lambda \in \mathrm{iso}\sigma(M_C)$. Then $\lambda \in$ iso $(\sigma(A) \cup \sigma(B))$. If $\lambda \in \sigma(A)$, as in the proof of Lemma 2.1, we have that $(M_C - \lambda I)^{-1}(0) \neq \{0\}$. Alternatively, if $\lambda \in \sigma(B) \setminus \sigma(A)$, then $\lambda \in \mathrm{iso}\sigma(B)$ and $A - \lambda$ is invertible. Thus $\lambda \in \pi_{00}(B)$ by (b) of Proposition 3.1, which implies that $\lambda \in \pi_{00}(A \oplus B)$ by (c) of Proposition 3.1. Hence $M_C - \lambda I$ is Weyl, which forces that $(M_C - \lambda I)^{-1}(0) \neq \{0\}$. Consequently, M_C is isoloid. □

We have an improvement of Proposition 3.1.

THEOREM 3.3. *Let $A \in B(X)$ and $B \in B(Y)$ both have SVEP. If* (a), (b), *and* (c) *in Proposition* 3.1 *are assumed, then Weyl's theorem holds for $f(M_C)$, for every $f \in H(\sigma(T))$.*

Proof. The proof is exactly same as the proof of Theorem 1.1; i.e., after combining Lemma 3.2 and [**10**, Lemma], and applying Proposition 3.1 and Lemma 2.3, we can show that Weyl's theorem holds for $f(M_C)$. □

To state the next, recall [**14**] that the *spectral picture* of an operator $T \in BL(X)$, denoted $\mathcal{SP}(T)$, consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes. The following result is well known [**11**, Lemma 2.2].

LEMMA 3.4. *For $A \in B(X)$ and $B \in B(Y)$, if either $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ have no pseudoholes, then we have*

(a) *$\sigma_w(A \oplus B) = \sigma_w(M_C)$,*
(b) *$\sigma(A) \cup \sigma(B) = \sigma(M_C) \cup \mathcal{U}$, where $\mathcal{U}$ is the union of the holes in $\sigma(M_C)$ which happen to be subsets of $(\sigma(A) \cap \sigma(B)) \setminus \sigma_w(A \oplus B)$.*

Using the lemma above, W. Y. Lee ([**11**, Theorem 2.4]) proved the following result.

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PROPOSITION 3.5. *For $A \in B(X)$ and $B \in B(Y)$, let $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ have no pseudoholes. If*

(a) *A is isoloid for which Weyl's theorem holds,*
(b) *Weyl's theorem holds for $A \oplus B$,*

then Weyl's theorem holds for M_C, for arbitrary bounded operators $C : Y \to X$.

We also notice that Theorem 1.1 and Proposition 3.5 suggest different sufficient conditions for M_C to obey Weyl's theorem. Actually, we notice that the assumptions of Proposition 3.5 and Theorem 1.1 have no direct co-relation, but if the condition that B is reguloid is added to the assumption of Proposition 3.5, then we have the following result.

LEMMA 3.6. *For $A \in B(X)$ and $B \in B(Y)$, let $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ have no pseudoholes. If A is isoloid and B are reguloid, then M_C is isoloid.*

Proof. Suppose that $\lambda \in \text{iso}(M_C)$. If $\lambda \in \text{iso}\sigma(M_C) \setminus \text{iso}\sigma(A \oplus B)$, then $\lambda \in (\sigma(A) \cap \sigma(B)) \setminus \sigma_w(A \oplus B)$. Thus Lemma 3.4 (a) implies that $\lambda \notin \sigma_w(M_C)$; i.e., $M_C - \lambda I$ is Weyl, and hence $M_C - \lambda I$ is not injective; i.e., $(M_C - \lambda I)^{-1}(0) \neq \{0\}$. On the other hand, if $\lambda \in \text{iso}\sigma(M_C) \cap \text{iso}\sigma(A \oplus B)$, then $\lambda \in (\sigma(A) \cup \sigma(B))$, and so we can apply the same argument as in the proof of Lemma 2.2. Hence we have that M_C is isoloid. $\square$

We conclude by proving the following result.

THEOREM 3.7. *For $A \in B(X)$ and $B \in B(Y)$, let $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ have no pseudoholes. If*

(a) *A is isoloid for which Weyl's theorem holds,*
(b) *Weyl's theorem holds for $A \oplus B$,*
(c) *B is reguloid,*

then Weyl's theorem holds for $f(M_C)$, for every $f \in H(\sigma(T))$.

Proof. By combining Lemma 3.6 and [**10**, Lemma], applying Proposition 3.5 and Lemma 2.3, we can get that Weyl's theorem holds for $f(M_C)$. $\square$

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