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THE DISCRETENESS OF THE NORMALIZERS OF HIGHER DIMENSIONAL KLEINIAN GROUPS AND THE ISOMORPHISMS BETWEEN KLEINIAN GROUPS INDUCED BY QUASICONFORMAL MAPPINGS*

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Abstract. In this paper, we get a necessary and sufficient condition for the normalizers of higher dimensional Kleinian groups to be discrete. Also we obtain a necessary and sufficient condition for the isomorphisms between two higher dimensional Kleinian groups induced by quasiconformal mappings to be the same.

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1. Introduction and Main Results. In this paper, we will adopt the same notations as in [**9**] such as the n-dimensional sense-preserving Möbius group $M(n)$ acting on $S^n(=\partial B^{n+1})$, where $B^{n+1} = \{x \in R^{n+1} : |x| < 1\}$, etc. It is known that the Poincaré extension of every element of $M(n)$ acts on B^{n+1} as an isometry about the hyperbolic metric of B^{n+1} (we will use the same sign to denote the element of $M(n)$ and its Poincaré extension). A subgroup G of $M(n)$ is called *elementary* if it has a finite G-orbit in $\bar{B}^{n+1}$. Otherwise we call G *non-elementary*, (cf. [**1**, p. 83]).

A subgroup $G \subset M(n)$ is called *Kleinian* if it is non-elementary and discrete.

In [**4**, p. 98], Maskit proved the following result.

THEOREM M. *Let $G \subset M(2)$ be a Kleinian group and let $N(G)(= \{g \in M(2) : gGg^{-1} = G\})$ be the normalizer of G. Then $N(G)$ is also a Kleinian group.*

See [**2**, Theorem 2.3.8] for the Fuchsian case.

In [**5**, **6**], Ratcliffe proved the following result.

THEOREM R_1. *Let $G \subset M(n)$ be a Kleinian group and let $M = B^{n+1}/G$ be a hyperbolic space-form. Then the isometry group $I(M)$ of M is isomorphic to $N(G)/G$.*

Hence it is interesting to generalize Theorem M to the case of $M(n)$ $(n \geq 3)$. In [**6**] (see also [**7**, Theorem 12.1.17]), Ratcliffe proved the following result.

THEOREM R_2. *Let G be a finitely generated Kleinian group of $M(n)$ which leaves no m-plane of B^{n+1} invariant for $m < n$. Then the normalizer $N(G)$ of G in $M(n)$ is discrete.*

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The condition "G being finitely generated" plays a key role in the proof of Theorem R_2, see [**6**] or [**7**, Theorem 12.1.17]. Obviously, when $n = 2$, Theorem R_2 doesn't coincide with Theorem M. In this paper, we will study this problem further and prove the following result.

THEOREM 1.1. *Let G be a non-elementary group of $M(n)$. Then $N(G)$ is a Kleinian group if and only if*

(1) *G is a Kleinian group;*
(2) *$WY(N(G))$ are discrete.*

Here $WY(N(G)) = \{g \in N(G) : \mathrm{fix}(f) \subset \mathrm{fix}(g)$ for any loxodromic element $f \in N(G)\}$ (cf. [**10**] or [**11**]) and $\mathrm{fix}(f) = \{x \in S^n : f(x) = x\}$. Obviously, when G is non-elementary, $WY(N(G))$ is purely elliptic or trivial. Hence $WY(N(G))$ is discrete if and only if it is finite.

REMARK 1.1. The example in [**9**] shows that condition (2) is necessary since in that example the group G is Kleinian but its normalizer $N(G)$ in $M(n)$ is not, where $n \geq 3$.

The following Lemma and [**10**] show that Theorem 1.1 is the complete generalization of Theorems M and R_2.

LEMMA 1.1. *Let G be a finitely generated Kleinian group of $M(n)$ which leaves no m-plane of B^{n+1} invariant for $m < n$. Then $WY(N(G))$ is trivial.*

Let $G \subset M(n)$ be non-elementary. A point $x_0 \in S^n$ is called a limit point of G if $g_n(z) \to x_0$ for some sequence $\{g_n\}$ of distinct elements of G and some fixed point $z \in B^{n+1}$. The set of all limit points of G is denoted by $L(G)$, i.e.,

$$L(G) = S^n \cap \mathrm{cl}(G(z)),$$

where cl denotes closure, $z \in B^{n+1}$. Since every element of G preserves the hyperbolic metric of B^{n+1}, this definition is independent of the choice of z. Then, by [**8**], we have the following.

LEMMA 1.2. *Let G be non-elementary. Then $L(G) = \mathrm{cl}(\{x :$ there exists a loxodromic element $h \in G$ such that $x \in \mathrm{fix}(h)\})$ and for any G-invariant set A, $L(G) \subset A$ if A is nonempty and closed.*

Let m be the least integer such that $L(G)$ is contained in an $m-$sphere of S^n. By conjugating G, we may assume that $L(G) \subset S^m$. As G leaves the convex hull hull(G) of $L(G)$ invariant, G also leaves $\bar{B}^{m+1}$ invariant since $\bar{B}^{m+1}$ is the affine hull of hull(G), where we denote $\bar{B}^{m+1}$ by $\sigma(L(G))$. See [**5**, **9**] for more details.

For any non-elementary subgroup $G \subset M(n)$, as in [**9**], we define the homomorphism ϕ_G concerning G as follows.

$$\begin{aligned} \phi_G : \quad & G \mapsto \phi_G(G) \\ & f \mapsto \phi_G(f) = \widetilde{f_1}, \end{aligned}$$

where $f_1 = f|_{\sigma(L(G))}$ denotes the restriction of f to $\sigma(L(G))$ and $\widetilde{f_1}$ the Poincaré extension of f_1 from $\sigma(L(G))$ to $\bar{B}^{n+1}$ (cf. [**1**, Section 3.3]).

We define

$$(G)_0 = \{f_0 : f_0 = f \circ \widetilde{f_1}^{-1} \text{ for any } f \in G\}.$$

PROPOSITION 1.1. *For any non-elementary subgroup $G \subset M(n)$, $Ker(\phi_G) \subset (G)_0$.*

PROPOSITION 1.2. *For any non-elementary subgroup $G \subset M(n)$, each element of $(G)_0$ fixes $L(G)$ pointwise.*

Let h be a quasiconformal mapping of $\bar{B}^{n+1}$. We say an isomorphism ψ from subgroup $G_1 \subset M(n)$ to subgroup $G_2 \subset M(n)$ is induced by h if for all $f \in G_1$,

$$h \circ f = \psi(f) \circ h.$$

Let $G_i \subset M(n)$ $(i = 1, 2)$ be two non-elementary subgroups and let ψ be an isomorphism from G_1 to G_2. We define the homomorphism ψ' concerning ψ as follows.

$$\begin{aligned} \psi' : (G_1)_0 &\mapsto (G_2)_0 \\ f_0 &\mapsto \psi(f)_0 \end{aligned}$$

for each $f \in G_1$.

Let $G \subset M(n)$ be non-elementary. As in [**5**], we call G a *generic* group if G leaves no $(m+1)-$plane of B^{n+1} invariant for $m < n$. About these subgroups, we will prove the following result.

THEOREM 1.2. *Let $G_i \subset M(n)$ $(i = 1, 2)$ be non-elementary. Suppose that the isomorphism ψ from G_1 to G_2 is induced by a quasiconformal mapping h of $\bar{B}^{n+1}$. Then G_1 is generic if and only if G_2 is generic.*

In the case $n = 2$, Lehto ([**3**, Section 5.1.3]) proved the following result.

THEOREM L. *Let S and S' be two Riemann surfaces with non-elementary covering groups G and G', $\psi_i : S \mapsto S'$ $(i = 1, 2)$ be two quasiconformal mappings, and f_1 a lift of ψ_1. Then ψ_1 and ψ_2 induce the same group isomorphism between G and G' if and only if there is a lift f_2 of ψ_2 which agrees with f_1 on the limit set of G.*

As the second main aim of this paper, we will prove the following.

THEOREM 1.3. *Let G_i $(i = 1, 2)$ be two Kleinian groups of $M(n)$ and ψ_i $(i = 1, 2)$ two isomorphisms from G_1 to G_2, and let h_i $(i = 1, 2)$ be two quasiconformal mappings of $\bar{B}^{n+1}$. If ψ_i $(i = 1, 2)$ are induced by h_i, respectively, then $\psi_1 = \psi_2$ if and only if $h_1|_{L(G_1)} = h_2|_{L(G_1)}$ and $\psi_1' = \psi_2'$.*

The following result follows from Theorem 1.3 and Lemma 1.1.

COROLLARY 1.1. *Under the assumptions of Theorem 1.3, if G_1 is generic, then $\psi_1 = \psi_2$ if and only if $h_1|_{L(G_1)} = h_2|_{L(G_1)}$.*

REMARK 1.2. By [**10**] and Corollary 1.1, we see that Theorem 1.3 is a generalization of Theorem L.

2. The proofs of Theorem 1.1 and Lemma 1.1.

First, we prove a lemma.

LEMMA 2.1. *Suppose G is a non-elementary subgroup of $M(n)$. Then $L(G) = L(N(G))$.*

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Proof. Observe that for any loxodromic element $h \in G$ and any $f \in N(G)$,

$$f(\text{fix}(h)) = \text{fix}(fhf^{-1}).$$

This implies that the set

$$F = \{x \in S^n : x \in \text{fix}(h) \text{ for some loxodromic element } h \in G\}$$

is $N(G)$-invariant. Hence the closure $\bar{F}$ of F is nonempty, closed and $N(G)$-invariant since G is non-elementary. It follows from Lemma 1.2 that

$$L(G) \subset L(N(G)) \subset \bar{F} \subset L(G).$$

Hence

$$L(N(G)) = L(G).$$

□

The proof of Theorem 1.1.

The necessity is obvious. We only need to prove the sufficiency.

Suppose that both G and $WY(N(G))$ are discrete, but $N(G)$ is not discrete. Then there is $\{f_i\} \subset N(G)$ such that $f_i \neq I$ and

$$f_i \to I \text{ as } i \to \infty.$$

Then for any fixed $g \in G$,

$$[f_i, g] \to I \text{ as } i \to \infty.$$

Since $[f_i, g] \in G$, we know that $[f_i, g] = I$ for all sufficiently large i. This means

$$f_i g = g f_i.$$

If $g \in G$ is loxodromic, then for large enough i,

$$\text{fix}(g) \subset \text{fix}(f_i).$$

Since G is non-elementary, by **[10]**, we know that there exists $M > 0$ such that

$$0 \leq \dim\left(\left[\bigcap_{i \geq M} \text{fix}(f_i)\right]\right) \leq n - 2.$$

By similar discussions as in the proof of Theorem 2 in **[10]**, we know that there exists $M_1 (\geq M)$ such that for all $i > M_1$, f_i keeps $L(G)$ invariant pointwise. It follows from Lemma 2.1 that f_i keeps $L(N(G))$ invariant pointwise. This means that $f_i \in WY(N(G))$. This is a contradiction.

The proof of Lemma 1.1.

Let $f \in WY(N(G))$. Then f fixes $L(G) = L(N(G))$ pointwise. Let m be the least integer such that $L(G)$ is contained in an m-sphere of S^n. By the proof of [**7**, Theorem 12.1.17], $m = n$. This implies that $f = I$.

3. The proofs of Theorems 1.2 and 1.3.

The proof of Theorem 1.2.
By the assumptions, we know that

$$h \circ f = \psi(f) \circ h$$

for any $f \in G_1$.
Then for any $x_0 \in L(G_1)$, there exists a sequence $\{f_n\} \subset G_1$ such that

$$f_n(0) \to x_0 \text{ as } n \to \infty,$$

where 0 denotes the origin of B^{n+1}.
Since

$$h(f_n(0)) = \psi(f_n)(h(0)),$$

it follows that

$$\psi(f_n)(h(0)) \to h(x_0).$$

Hence $h(x_0) \in L(G_2)$. This shows $h(L(G_1)) \subset L(G_2)$.
Since h is a quasiconformal mapping of $\bar{B}^{n+1}$, by considering $h^{-1} \circ \psi(f_n) = f_n \circ h^{-1}$, we know that $h^{-1}(L(G_2)) \subset L(G_1)$. Hence $h(L(G_1)) = L(G_2)$. Our result follows.

The proof of Theorem 1.3.
Assume that $\psi_1 = \psi_2$. Then

$$h_1 \circ g \circ h_1^{-1} = h_2 \circ g \circ h_2^{-1}$$

for any $g \in G_1$.
Let $f = h_1^{-1} \circ h_2$. Then

$$g \circ f = f \circ g.$$

Assume x_0 is an attractive fixed point of some loxodromic element $g \in G_1$, i.e.,

$$g^n(0) \to x_0 \text{ as } n \to \infty.$$

It follows that

$$g^n(f(0)) \to x_0 \text{ and } f(g^n(0)) \to f(x_0)$$

as $n \to \infty$. Hence $f(x_0) = x_0$. We know $h_1(x) = h_2(x)$ for all $x \in L(G_1)$ since the set of attractive fixed points of loxodromic elements of G_1 is dense in $L(G_1)$ (cf. **[8]**). The necessity follows.
Assume that $h_1 = h_2$ on $L(G_1)$. Then

$$h_1 \circ g \circ h_1^{-1} = h_2 \circ g \circ h_2^{-1}$$

at every point of $L(G_2)$. This shows that

$$h_1 \circ g \circ h_1^{-1}|_{\sigma(L(G_2))} = h_2 \circ g \circ h_2^{-1}|_{\sigma(L(G_2))}.$$

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It follows from

$$\psi_1' = \psi_2'$$

that

$$h_1 \circ g \circ h_1^{-1} = h_2 \circ g \circ h_2^{-1}.$$

The sufficiency follows.

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