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TENSOR PRODUCTS OF FUNDAMENTAL REPRESENTATIONS

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Let G be a reductive group over a field of characteristic zero. Fix a Borel subgroup B of G which contains a maximal torus T. For each dominant weight X we have an irreducible representation $V(X)$ of G with highest weight X. For two dominant representation X_1 and X_2 we have a decomposition

$$V(X_1) \otimes V(X_2) = \oplus\, m_\psi V(\psi).$$

This decomposition is determined by the element

$$r(X_1, X_2) \equiv \Sigma\, m_\psi \cdot \psi$$

of the group ring of the group of characters of T.

The objective of this paper is to compute $r(X_1, X_2)$ for all pairs X_1 and X_2 of fundamental weights. This will be used to compute the equations for cones over homogeneous spaces. This problem immediately reduces to the case when G has simple type; A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2. We will give complete details for the classical types. For the case A_n we will work with GL_n. Here the result may be easily deduced by the well known Hardy-Littlewood formula but the casual reader may get the general idea of our proof from this simplest case. We do not know when a generalization of Littlewood-Richardson to the other classical groups will be available, but my results show that the situation is much more complicated than the general linear case.

1. Statement of the results for classical groups. Let $(t_1, \ldots, t_n)$ be the diagonal $n \times n$ matrix with given coefficients. These elements form a maximal torus T of $GL(n)$. Let

$$d_i = \prod_{1 \leqq j \leqq i} t_j$$

be characters of T. Then $d_1, \ldots, d_{n-1}$ are fundamental weights for GL_n. Let $r_n(d_i, d_j)$ be $r(d_i, d_j)$ for $GL(n)$. Then

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THEOREM ($GL(n)$). *If* $0 \leqq i \leqq j \leqq n$, *then*

$$r_n(d_i, d_j) = \sum_{0 \leqq r \leqq i} d_r d_{j+i-r}$$

where $d_* \equiv 0$, *if* $* > n$.

Let $(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_1^{-1})$ be the diagonal $2n \times 2n$ matrix with given coefficients. These elements form a maximal torus T of $Sp(2n)$. Let

$$d_i = \prod_{1 \leqq j \leqq i} t_j$$

be characters of T. Then $d_1, \ldots, d_n$ are fundamental weights for $Sp(2n)$. Let $r_n(d_i, d_j)$ be $r(d_i, d_j)$ for $Sp(2n)$. Then

THEOREM ($Sp(2n)$). *If* $0 \leqq i \leqq j \leqq n$, *then*

$$r_n(d_i, d_j) = \sum_{0 \leqq s \leqq i} \sum_{0 \leqq r \leqq i-s} d_r d_{j+i-r-2s}$$

where $d_* \equiv 0$ *if* $* > n$.

Let $(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1})$ be the diagonal $(2n + 1) \times (2n + 1)$ matrix with given coefficients. These elements form a maximal torus $\widetilde{T}$ of $\mathcal{O}(2n + 1)$. Let

$$d_i = \prod_{1 \leqq j \leqq i} t_j$$

be character of $\widetilde{T}$. Let T be the inverse image of $\widetilde{T}$ in the universal covering group G of $\mathcal{O}(2n + 1)$. Let spin_n be the character $d_n^{1/2}$ of T. Then $d_1, \ldots, d_{n-1}$ and spin_n are the fundamental weights of G. Let

$$r_n(d_i, d_j)(r_n(d_i, \text{spin}_n), r_n(\text{spin}_n, \text{spin}_n))$$

be

$$r(d_i, d_j)(r(d_i, \text{spin}_n), r(\text{spin}_n, \text{spin}_n))$$

for G.

THEOREM ($0(2n + 1)$). (a) *If* $0 \leqq i \leqq j \leqq n$, *then*

$$r_n(d_i, d_j) = \sum_{0 \leqq s \leqq i} \sum_{0 \leqq r \leqq i-s} d_r d_{\{j+i-r-2s\}_n}$$

where

$$\{x\}_n = x \text{ if } x \leqq n$$

and

$$\{x\}_n = 2n + 1 - x \text{ if } x > n.$$

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(b) $r_n(d_i, \text{spin}_n) = \sum_{0 \leqq s \leqq i} d_s \, \text{spin}_n$

(c) $r_n(\text{spin}_n, \text{spin}_n) = \sum_{0 \leqq s \leqq n} d_s.$

Let $(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$ be the diagonal $2n \times 2n$ matrix with given coefficients. These elements form a maximal torus $\tilde{T}$ of $\mathcal{O}(2n)$. Let

$$d_i = \prod_{1 \leqq j \leqq i} t_j \quad \text{if } 0 \leqq i \leqq n - 1$$

and let $d_n^\epsilon = d_{n-1} t_n^{\epsilon 1}$ where $\epsilon = +$ or $-$. Let T be the inverse image of $\tilde{T}$ in the universal covering group G of $\mathcal{O}(2n)$. Let

$$\text{spin}_n^\epsilon = (d_n^\epsilon)^{1/2}$$

be a character of T. Then $d_1, \ldots, d_{n-2}, \text{spin}_n^+, \text{spin}_n^-$ are the fundamental weights of G and

$$\text{spin}_n^+ \, \text{spin}_n^- = d_{n-1}.$$

Let

$$r_n(d_i, d_j)(r_n(d_i, \text{spin}_n^\epsilon), r_n(\text{spin}_n^{\epsilon_1}, \text{spin}_n^{\epsilon_2}))$$

be $r(d_i, d_j)$ (etc.) for G.

THEOREM ($\mathcal{O}(2n)$). (a) *If* $0 \leqq i \leqq j \leqq n - 1$, *then*

$$r_n(d_i, d_j) = \sum_{0 \leqq s \leqq i} \sum_{0 \leqq r \leqq i-s}{}' d_r d_{[i+j-r-2s]_n}$$

where $[x]_n = x$ *if* $x \leqq n$ *and* $[x]_n = 2n - x$ *if* $x > n$ *and the prime means replace the term* $d_r d_n$ *by* $(d_r d_n^+ + d_r d_n^-)$.

(b) *If* $0 \leqq i \leqq n - 1$, *then*

$$r_n(d_i, d_n^\epsilon) = \sum_{0 \leqq s \leqq i}{}^{\epsilon} \sum_{0 \leqq r \leqq i-s} (\delta_{i+n-r-2s}^n) d_r d_{i+n-r-2s}$$

where the ϵ *means* $d_n \equiv d_n^\epsilon$ *and* $\delta_*^n = 1$ *if* $* \leqq n$ *and* 0 *if* $* > n$.

(c) $r_n(d_n^\epsilon, d_n^{-\epsilon}) = \sum_{\substack{0 \leqq p \leqq q \leqq n-1 \\ p \equiv q \equiv n-1((2))}} d_p d_q$

(d) $r_n(d_n^\epsilon, d_n^\epsilon) = \sum_{\substack{0 \leqq p \leqq q \leqq n \\ p \equiv q \equiv n((2))}} d_p d_q$

where $d_n = d_n^\epsilon$.

(e) *If* $0 \leqq i \leqq n - 1$ *then*

$$r_n(d_i, \mathrm{spin}_n^\epsilon) = \sum_{0 \leqq s \leqq i} d_{i-s} \, \mathrm{spin}_n^{\delta_s(\epsilon)}$$

where $\delta_s(\epsilon) = \epsilon$ *if* s *is even and* $= -\epsilon$ *if* s *is odd.*

(f) $r_n(\mathrm{spin}_n^\epsilon, \mathrm{spin}_n^{-\epsilon}) = d_{n-1} + \sum_{1 \leqq x \leqq ((n-1)/2)} d_{n-1-2x}.$

(g) $r_n(\mathrm{spin}_n^\epsilon, \mathrm{spin}_n^\epsilon) = d_n^\epsilon + \sum_{1 \leqq x \leqq (n/2)} d_{n-2x}.$

2. The proof for GL_n. We will proceed by induction on n. We embed GL_{n-1} in the upper left-hand part of GL_n. We will use the notation of [**1**] except that $\chi(\psi) = \epsilon \cdot \psi'$ where the previous notation was $\chi(\psi) = \epsilon \cdot V(\psi')$. By the main result of [**1**] we have

$$r_n(\psi_1, \psi_2) = \chi \circ M \circ L(\psi_1(t) \cdot \psi_2(s))$$

where L is a definite linear operator from the group ring of two copies of the weights to itself and M is the linear operator such that

$$M(\psi_1(t) \cdot \psi_2(s)) = \psi_1 \cdot \psi_2.$$

Now let $'$ denote the analogous operations for GL_{n-1}. Then $\chi \cdot \chi' = \chi$ and $L = L' \circ H$ where H is the product

$$L_{(n,n-1)} \circ \ldots \circ L_{(n,1)}.$$

Thus

$$r_n(\psi_1, \psi_2) = \chi \circ \chi' \circ M \circ L' \circ H(\psi_1(t)\psi_2(s)).$$

Hence

(A) $r_n(\psi_1, \psi_2) = \sum c_k \chi(r_{n-1}(\epsilon_k, \eta_k))$

where

$$H(\psi_1(t)\psi_2(s)) = \sum c_k \epsilon_k(t)\psi_k(s).$$

This formula may be derived directly from the fibering argument mentioned in [**2**]. We will use formula (A) for our induction on n.

The representations with weight d_0 or $d_n = \det$ are one dimensional. Therefore

$$r_n(d_*, \psi) = \psi \cdot d_* \quad \text{if } * = 0 \text{ or } n.$$

As $d_i = 0$ if $i > n$, our formulas are true when $i = 0$ or n or $j = 0$ or n. In particular the formulas are true for GL_1. Thus by induction we may assume $n > 1$ and the formulas are true when $n = n - 1$. For any $1 \leqq i \leqq n - 1$ by the formula for H we have

$$H(d_i(t)\psi(s)) = d_i(t)\psi(s) + d_{i-1}(t)\psi(s)s_n.$$

Now let $i \leqq j \leqq n - 1$. Then by induction

$$r_{n-1}(d_i, d_j) = \sum_{\substack{0 \leqq r \leqq i \\ i+j-r \leqq n-1}} d_r d_{i+j-r} \quad \text{and}$$

$$r_{n-1}(d_{i-1}, d_j t_n) = \sum_{\substack{0 \leqq r \leqq i-1 \\ i-1+j-r \leqq n-1}} d_r d_{i-1+j-r} t_n.$$

Here

$$\chi(r_{n-1}(d_i, d_j)) = r_{n-1}(d_i, d_j) \quad \text{and}$$

$$\chi(r_{n-1}(d_i, d_j t_n)) = d_{i+j-n} d_n$$

if $i + j \geqq n$ and is zero otherwise as the terms of $r_{n-1}(d_{i-1}, d_j t_n)$ are singular except when $i - 1 + j - r = n - 1$. Therefore by (A)

$$r_n(d_i, d_j) = \sum_{\substack{0 \leqq r \leqq i \\ i+j-r \leqq n-1}} d_r d_{i+j-r} + \epsilon d_r d_{i+j-r}$$

where $\epsilon = 0$ if $i + j < n$ and 1 if $i + j - r = n$. Hence

$$r_n(d_i, d_j) = \sum_{\substack{0 \leqq r \leqq i \\ i+j-r \leqq n}} d_r d_{i+j-r}$$

and our theorem is true for GL_n.

3. The proof for $Sp(2n)$. Here we do the same type of calculation but the notations are from [**2**]. L, H and χ have different meanings. In fact

$$H = L_{(1,2n-1)} \circ \ldots \circ L_{(1,1)}$$

and we regard $Sp(2(n - 1))$ are contained in the center square of $Sp(2n)$. Then we still have the inductive formula A with the new H and χ. We assume that $0 \leqq i \leqq j \leqq n$. Again the case $i = 0$ is trivial. So assume $i > 0$. If $n = 1$, then $Sp(2) = SL(2)$ and the formula follows from the $GL(2)$ case. Assume that $n > 1$. If $i = 1$, $H(d_i, \psi)$ is

$$d_i(t)\psi(s) + d_{i-1}(t)\psi(s)s_1^{-1} + d_{i+1}(t)\psi(s)s_1^{-1}.$$

If $i > 1$ it is

$$d_i(t)\psi(s) + d_{i-1}(t)\psi(s)s_1^{-1} + d_{i+1}(t)\psi(s)s_1^{-1} + d_i(t)\psi(s)s_1^{-2}.$$

Now when we do the induction we will have to remember that d_* gives the character d_{*-1} on $Sp(2(n - 1))$. We will assume that $n > 1$ and the formulas are true for $n = n - 1$. Then formula A becomes

(A′) if $i = 1$ then

$$r_n(d_i, d_j) = \chi(r_{n-1}(d_i, d_j)) + \chi(r_{n-1}(d_{i-1}, d_jt_1^{-1})) + \chi(r_{n-1}(d_{i+1}, d_jt_1^{-1}))$$

and if $i > 1$ we need to add $\chi(r_{n-1}(d_i, d_jt_1^{-2}))$.

First we will need to compute some Euler characteristics. If $1 \leqq i \leqq j \leqq n$ then

$$\chi(d_id_j) = d_id_j.$$

Also $\chi(d_id_jt_1^{-1}) = 0$ unless $i = 1$ in which case it equals d_0d_j. Next

$$\chi(d_id_jt_1^{-2}) = 0$$

unless $i = 2$ in which case it equals $-d_0d_j$ or $i = j = 1$ where it equals d_0d_0. Further

$$\chi(d_0d_jt_1^{-1}) = 0$$

unless $j = 1$ in which case it equals d_0d_0.

The main term in the formulas is $\chi(r_{n-1}(d_i, d_j))$. It equals

$$\chi\left(\sum_{0 \leqq s \leqq i-1} \sum_{0 \leqq r \leqq i-1-s} d_{r+1}d_{(i-1)+(j-1)+1-r-2s}\right)$$

by induction. Therefore

$$(1) \qquad \chi(r_{n-1}(d_i, d_j)) = \sum_{0 \leqq s \leqq i-1} \sum_{1 \leqq r \leqq i-s} d_rd_{i+j-r-2s}.$$

By the same reasoning and the χ formula

$$(2) \qquad \chi(r_{n-1}(d_i, d_jt_1^{-2})) = -\sum_{0 \leqq s \leqq i-2} d_0d_{i+j-2-2s} + \epsilon d_0d_0$$

where $\epsilon = 0$ if $i < j$ and equals 1 otherwise.

If $i > 1$ then

$$(3) \qquad \chi(r_{n-1}(d_{i-1}, d_jt_1^{-1}))$$

$$= \chi\left(\sum_{0 \leqq s \leqq i-2} \sum_{0 \leqq r \leqq i-2-s} d_{r+1}d_{(i-2)+(j-1)+1-r-2s}t_1^{-1}\right)$$

$$= \sum_{0 \leqq s \leqq i-2} d_0d_{i+j-2-2s}.$$

If $i = 1$ then

$$\chi(r_{n-1}(d_{i-1}, d_jt_1^{-1})) = \chi(d_0d_jt_1^{-1}) = \epsilon d_0d_0.$$

If $i + 1 \leqq j$ then

$$\chi(r_{n-1}(d_{i+1}, d_jt_1^{-1}))$$
$$= \chi\left(\sum_{0\leqq s\leqq i}\sum_{0\leqq r\leqq i-s} d_{r+1}d_{i+(j-1)+1-r-2s}t_1^{-1}\right)$$
$$= \sum_{0\leqq s\leqq i} d_0d_{i+j-2s}.$$

If $i = j$ then

$$\chi(r_{n-1}(d_{i+1}, d_jt_1^{-1}))$$
$$= \chi\left(\sum_{0\leqq s\leqq j-1}\sum_{0\leqq r\leqq j-1-s} d_{r+1}d_{(j-1)+i+1-r-2s}t_1^{-1}\right)$$
$$= \sum_{0\leqq s\leqq i-1} d_0d_{i+j-2s}.$$

Thus in any case

$$(4) \qquad \chi(r_{n-1}(d_{i+1}, d_jt_1^{-1})) = \sum_{0\leqq s\leqq i} d_0d_{i+j-2s} - \epsilon d_0d_0.$$

To prove our formula we have three cases; $i = 1 = j = 1$; $i = 1$ and $j > 1$; $i > 1$. In the first case we get the answer

$$d_1d_1 + d_0d_0 + d_0d_0 + d_0d_2 - d_0d_0$$
$$= \sum_{0\leqq s\leqq 1}\sum_{0\leqq r\leqq 1-s} d_rd_{2-r-2s}.$$

In the second case we get the answer

$$d_1 \cdot d_{1+j-1} + d_0d_{1+j} + d_0d_{1+j-2}$$
$$= \sum_{0\leqq s\leqq 1}\sum_{0\leqq r\leqq 1-s} d_rd_{1+j-r-2s}.$$

In the last case we get the answer

$$\sum_{0\leqq s\leqq i-1}\sum_{1\leqq r\leqq i-s} d_rd_{i+j-r-2s} + \epsilon d_0d_0$$
$$- \sum_{0\leqq s\leqq i-2} d_0d_{i+j-2-2s} + \sum_{0\leqq s\leqq i-2} d_0d_{i+j-2-2s}$$
$$+ \sum_{0\leqq s\leqq i} d_0d_{i+j-2s} - \epsilon d_0d_0$$
$$= \sum_{0\leqq s\leqq i}\sum_{0\leqq r\leqq i-s} d_rd_{i+j-r-2s}.$$

Thus the formula is true.

4. The proof for $\mathcal{O}(1 + 2n)$. We proceed by induction on n. If $n = 1$ then $\mathcal{O}(3)$ is the projective group $PGL(1)$ and its double cover is $SL(2)$. Then spin is the fundamental weight of $SL(2)$ and d_1 is its square. The formulas are in this case

$$\begin{aligned} r_1(d_1, d_1) &= d_0 d_{\{2\}_1} + d_1 d_{\{1\}_1} + d_0 d_{\{0\}_1} \\ &= d_0 d_1 + d_1 d_1 + d_0 d_0, \ r_n(d_1, \text{spin}) \\ &= d_0 \text{ spin}_1 + d_1 \text{ spin}_1 \quad \text{and} \end{aligned}$$

$$r_1(\text{spin}_1, \text{spin}_1) = d_0 + d_1.$$

These formulas are easily proved by many methods including ours.

We will assume that $n > 1$ and embed $\mathcal{O}(1 + 2(n - 1))$ in the center square of $\mathcal{O}(1 + 2n)$. When we do this d_* corresponds to d_{*-1} in $\mathcal{O}(1 + 2(n - 1))$ and we use the identity $\{x\}_n = \{x - 1\}_{n-1} + 1$ repeatedly without mention. Also spin_n corresponds to spin_{n-1}. The operator χ differs a little bit from the last example but it has the same form but $L_{(n,1)}$ now has a different meaning [see **2**]. We will prove the formulas by induction on n.

First we will do the formulas involving spin_n. For any character ψ we have

$$H(\text{spin}_n(t)\, \psi(s)) = \text{spin}_n(t) \cdot \psi(s) + \text{spin}_n(t) \cdot \psi(s) s_1^{-1}.$$

Therefore

$$r_n(\text{spin}_n, \psi) = \chi(r_{n-1}(\text{spin}_n, \psi)) + \chi(r_{n-1}(\text{spin}_n, t_1^{-1}).$$

In particular

$$r_n(\text{spin}_n, d_i) = r_{n-1}(\text{spin}_n, d_i) + \chi(r_{n-1}(\text{spin}_n, d_i) t_1^{-1}).$$

By induction

$$r_{n-1}(\text{spin}_n, d_i) = \sum_{0 \leqq s \leqq i-1} d_{s+1} \text{ spin}_n.$$

Hence

$$\chi(r_{n-1}(\text{spin}_n, d_i) t_1^{-1}) = d_0 \text{ spin}_n$$

and hence the formula (b) is true. For (c) we have

$$r_n(\text{spin}_n, \text{spin}_n) = r_{n-1}(\text{spin}_n, \text{spin}_n) + \chi(r_{n-1}(\text{spin}_n, \text{spin}_n) t_1^{-1})$$

where

$$r_{n-1}(\text{spin}_n, \text{spin}_n) = \sum_{0 \leqq s \leqq n-1} d_{s+1}.$$

As

$$\chi(r_{n-1}(\mathrm{spin}_n, \mathrm{spin}_n)t_1^{-1}) = d_0$$

the formula (c) is true.

It remains to prove (a). If $1 \leqq i \leqq j$ and $i < n$ then the same argument as in Section 3 applies. One simply replaces $d_k d_l$ by $d_k d_{\{l\}_n}$.

Thus we need only compute $r_n(d_n, d_n)$. First of all

$$H(d_n(t)d_n(s)) = d_n(t)d_n(s) + d_n(t)d_n(s)s_1^{-1} + d_n(t)d_n(s)s_1^{-2} + d_{n-1}(t)d_n(s)s_1^{-1}.$$

Therefore

$$r_n(d_n, d_n) = \chi(r_{n-1}(d_n, d_n)) + \chi(r_{n-1}(d_n, d_n)t_1^{-1}) + \chi(r_{n-1}(d_n, d_n)(t_1^{-2}) + \chi(r_{n-1}(d_{n-1}, d_n t_1^{-1})).$$

Now

$$\chi(r_{n-1}(d_n, d_n)) = r_{n-1}(d_n, d_n) = \sum_{0\leqq s\leqq n-1}\ \sum_{0\leqq r\leqq n-1-s} d_{r+1}d_{\{2n-1-r-2s\}_n} = \sum_{0\leqq s\leqq n-1}\ \sum_{1\leqq r\leqq n-s} d_r d_{\{2n-r-2s\}_n}.$$

Next

$$\chi(r_{n-1}(d_n, d_n)t_1^{-1}) = \sum_{0\leqq s\leqq n-1} d_0 d_{\{2n-1-2s\}_n} = \sum_{0\leqq s\leqq n-1} d_0 d_{\{2n-2s\}_n}.$$

The last equations follow from direct calculation of $\{*\}_n$. Also

$$\chi(r_{n-1}(d_n, d_n)t_1^{-2}) = -\sum_{0\leqq s\leqq n-2} d_0 d_{\{2n-2-2s\}_n} + d_0 d_0.$$

Lastly

$$r_{n-1}(d_{n-1}, d_n) = \sum_{0\leqq s\leqq n-2}\ \sum_{0\leqq r\leqq n+2-s} d_{r+1}d_{\{2n-2-r-2s\}_n}.$$

So

$$\chi(r_{n-1}(d_{n-1}, d_n)) = \sum_{0\leqq s\leqq n-2} d_0 d_{\{2n-2-2s\}_n}.$$

Therefore combining all equations

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$$r_n(d_n, d_n) = \sum_{0 \leqq s \leqq n-1} \sum_{1 \leqq r \leqq n-s} d_r d_{\{2n-r-2s\}_n} + \sum_{0 \leqq s \leqq n-1} d_0 d_{\{2n-2s\}_n} + d_0 d_0 - \sum_{0 \leqq s \leqq n-2} d_0 d_{\{2n-2-2s\}_n} + \sum_{0 \leqq s \leqq n-2} d_0 d_{\{2n-2-2s\}_n} = \sum_{0 \leqq s \leqq n} \sum_{0 \leqq r \leqq n-s} d_r d_{\{2n-r-2s\}_n};$$

i.e., (a) is true.

5. The proof for $\mathcal{O}(2n)$. When $n = 2$ the double covering of $\mathcal{O}(2n)^\circ$ is $SL(2) \times SL(2)$. Then spin_2^+ and spin_2^- are the fundamental representation corresponding to the two factors. Also

$$d_2^\epsilon = (\text{spin}_2^\epsilon)^2 \quad \text{and} \quad d_1 = \text{spin}_2^+ \cdot \text{spin}_2^-.$$

For the sake of reality we will give the formulas in this case.

$$r_2(d_1, d_1) = d_0 d_2^+ + d_0 d_2^- + d_1 d_1 + d_0 d_0,$$
$$r_2(d_1, d_2^\epsilon) = d_0 d_1 + d_1 d_2^\epsilon,$$
$$r_2(d_2^+, d_2^-) = d_1 d_1,$$
$$r_2(d_2^\epsilon, d_2^\epsilon) = d_0 d_0 + d_0 d_2^\epsilon + d_2 d_2^\epsilon,$$
$$r_2(d_1, \text{spin}_2^\epsilon) = d_1 \text{spin}_2^\epsilon + d_0 \text{spin}_2^{-\epsilon},$$
$$r_2(\text{spin}_2^+, \text{spin}_2^-) = d_1 \quad \text{and}$$
$$r_2(\text{spin}_2^\epsilon, \text{spin}_2^\epsilon) = d_2^\epsilon + d_0.$$

These formulas are elementary and an irreducible representation of the product $SL(2) \times SL(2)$ is a product of irreducible representations of the factors which are well-known.

We will assume that $n \geqq 3$ and proceed by induction of n. Embed $\mathcal{O}(2(n-1))$ as the central square in $\mathcal{O}(2n)$. Then we have a slightly different operator χ and

$$H = L_{(1, 2n-1)} \circ \ldots \circ L_{(1,n+2)} \circ L_{(1,n-1)} \circ \ldots \circ L_{(1,1)}$$

where $L_{(1,n-1)}$ has a new meaning. We begin with the spin representations. Let ψ be a character. Then

$$H(\text{spin}_n^\epsilon(t)\, \psi(s)) = \text{spin}_n^\epsilon(t)\, \psi(s) + \text{spin}_n^{-\epsilon}(t)\, \psi(s) s_1^{-1}.$$

So

$$r_n(\text{spin}_n^\epsilon, \psi) = \chi(r_{n-1}(\text{spin}_n^{+\epsilon}, \psi)) + \chi(r_{n-1}(\text{spin}_n^{-\epsilon}, t_1^{-1}))$$

as usual spin_n^ϵ corresponds to $\text{spin}_{n-1}^\epsilon$ on $\mathcal{O}(2(n-1))$ and d_* corresponds to d_{*-1}.

Take $\psi = d_i$ with $i \leqq n - 1$. We have

$$r_{n-1}(d_i, \text{spin}_n^\epsilon) = \sum_{0 \leqq s \leqq i-1} d_{i-1-s+1} \, \text{spin}_n^{\delta_s(\epsilon)}$$

$$= \sum_{0 \leqq s \leqq i-1} d_{i-s} \, \text{spin}_n^{\delta_s(\epsilon)}$$

$$= \chi(r_{n-1}(d_i, \text{spin}_n^\epsilon) \quad \text{and}$$

$$\chi(r_{n-1}(d_i, \text{spin}_n^{-\epsilon})t_1^{-1}) = d_0 \, \text{spin}_n^{\delta_{i-1}(-\epsilon)} = d_{i-i} \, \text{spin}_n^{\delta_i(\epsilon)}.$$

Adding the two equations we get formula (e); i.e.,

$$r_n(d_i, \text{spin}_n^\epsilon) = \sum_{0 \leqq s \leqq i} d_{i-s} \, \text{spin}_n^{\delta_s(\epsilon)}.$$

Take $\psi = \text{spin}_n^\epsilon$. We have

$$r_{n-1}(\text{spin}_n^\epsilon, \text{spin}_n^\epsilon) = d_{n-2+1} + \sum_{1 \leqq x \leqq ((n-2)/2)} d_{n-2-2x+1}$$

or

$$\chi(r_{n-1}(\text{spin}_n^\epsilon, \text{spin}_n^\epsilon)) = d_{n-1} + \sum_{1 \leqq x \leqq ((n-2)/2)} d_{n-1-2x}.$$

Also

$$\chi(r_{n-1}(\text{spin}_n^\epsilon, \text{spin}_n^{-\epsilon})t_1^{-1})$$

$$= \chi\left(\left(d_n^\epsilon + \sum_{1 \leqq x \leqq ((n-1)/2)} d_{n-2x}\right)t_1^{-1}\right)$$

$$\begin{cases} = d_0 \text{ if } 1 = n - 2x \text{ for } 1 \leqq x \leqq \dfrac{n-1}{2}; \ n \text{ is odd} \\ = 0 \text{ if } n \text{ is even.} \end{cases}$$

Summing the two χ's we get

$$r_n(\text{spin}_n^\epsilon, \text{spin}_n^\epsilon) = d_{n-1} + \sum_{1 \leqq x \leqq ((n-1)/2)} d_{n-1-2x}$$

which is (f).

Finally take $\psi = \text{spin}_n^{-\epsilon}$. Then

$$\chi(r_{n-1}(\text{spin}_n^\epsilon, \text{spin}_n^{-\epsilon})) = d_n^\epsilon + \sum_{1 \leqq x \leqq ((n-1)/2)} d_{n-2x} \quad \text{and}$$

$$\chi(r_{n-1}(\text{spin}_n^\epsilon, \text{spin}_n^\epsilon)t_1^{-1}) = d_0$$

if n is even and 0 if n is odd. Hence

$$r_n(\mathrm{spin}_n^\epsilon, \mathrm{spin}_n^{-\epsilon}) = d_n^\epsilon + \sum_{1 \leqq x \leqq (n/2)} d_{n-2x}.$$

This settles the spin representations.

Next we will work on formula (a). If $i \leqq n - 2$ then the same argument as in Section 3 applies with the only changes that $d_k d_l$ is replaced by $d_k d_{[l]_n}$ and $d_k d_n$ by $d_k d_n^+ + d_k d_n^-$. Here one uses the identity $[x]_n = [x - 1]_{n-1} + 1$ repeated. So we may assume that $i = j = n - 1$. We find

$$\begin{aligned} H(d_{n-1}(t)d_{n-1}(s)) &= d_{n-1}(t)d_{n-1}(s) + d_n^+(t)d_{n-1}(s)s_1^{-1} \\ &\quad + d_n^-(t)d_{n-1}(s)s_1^{-1} + d_{n-2}(t)d_{n-1}(s)s_1^{-1} \\ &\quad + d_{n-1}(t)d_{n-1}(s)s_1^{-2}. \end{aligned}$$

So

$$\begin{aligned} r_n(d_{n-1}, d_{n-1}) &= \chi(r_{n-1}(d_{n-1}, d_{n-1})) + \chi(r_{n-1}(d_{n-1}, d_{n-1})t_1^{-1}) \\ &\quad + \chi(r_{n-1}(d_{n-2}, d_{n-1})t_1^{-1}) + \chi(r_{n-1}(d_{n-1}, d_n^+)t_1^{-1}) \\ &\quad + \chi(r_{n-1}(d_{n-1}, d_n^-)t_1^{-1}). \end{aligned}$$

Now

$$\begin{aligned} \chi(r_{n-1}(d_{n-1}, d_{n-1})) &= r_{n-1}(d_{n-1}, d_{n-1}) \\ &= \sum_{0 \leqq s \leqq n-2} \ \sum_{0 \leqq r \leqq n-2-s}{}' \ d_{r+1} d_{[2n-3-r-2s]_n} \\ &= \sum_{0 \leqq s \leqq n-2} \ \sum_{1 \leqq r \leqq n-1-s}{}' \ d_r d_{[2n-2-r-2s]_n}. \end{aligned}$$

Hence

$$\chi(r_{n-1}(d_{n-1}, d_{n-1})t_1^{-2}) = -\sum_{0 \leqq s \leqq n-3}{}' \ d_0 d_{[2n-4-2s]_n} + d_0 d_0.$$

Next

$$r_{n-1}(d_{n-2}, d_{n-1}) = \sum_{0 \leqq s \leqq n-3} \ \sum_{0 \leqq r \leqq n-3-s}{}' \ d_{r+1} d_{[2n-4-r-2s]_n}.$$

So

$$\chi(r_{n-1}(d_{n-2}, d_{n-1})t_1^{-1}) = \sum_{0 \leqq s \leqq n-3}{}' \ d_0 d_{[2n-4-2s]_n}.$$

Now

$$r_{n-1}(d_{n-1}, d_n^\epsilon) = \sum_{0 \leqq s \leqq n-2} \ \sum_{0 \leqq r \leqq n-2s} \delta^n_{2n-2-r-2s} d_{r+1} d_{2n-r-2s}.$$

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Hence

$$\chi(r_{n-1}(d_{n-1}, d_n^{\epsilon})t_1^{-1}) = \sum_{0 \leqq s \leqq n-2} \delta^n_{2n-2-2s} d_0 d_{2n-2-2s}.$$

Therefore after cancellation our formula gives

$$r_n(d_{n-1}, d_{n-1}) = \sum_{0 \leqq s \leqq n-2} \sideset{}{'}\sum_{1 \leqq r \leqq n-1-s} d_r d_{[2n-2-r-2s]_n} + d_0 d_0$$
$$+ \sideset{}{^+}\sum_{0 \leqq s \leqq n-2} \delta^n_{2n-2-2s} d_0 d_{2n-2-2s}$$
$$+ \sideset{}{^-}\sum_{0 \leqq s \leqq n-2} \delta^n_{2n-2-2s} d_0 d_{2n-2-2s}.$$

Thus to prove the formula we need to show that the sum of the last three terms is

$$\sideset{}{'}\sum_{0 \leqq s \leqq n-1} d_0 d_{[2n-2-2s]_n}.$$

If n is odd both expressions equal

$$d_0 d_0 + \sum_{1 \leqq i \leqq ((n-1)/2)} d_0 d_{2i} + \sum_{1 \leqq i \leqq ((n-1)/2)} d_0 d_{2i}.$$

If n is even both equal

$$d_0 d_0 + \sum_{1 \leqq i \leqq (n/2)} d_0 d_{2i} + d_0 d_n^+ + \sum_{1 \leqq i \leqq (n/2)} d_0 d_{2i} + d_0 d_n^- .$$

This proves (a).

To prove the rest of the formulas let ψ be a character. Then

$$H(d_n^{\epsilon}(t)\psi(s)) = d_n^{\epsilon}(t)\psi(s) + d_{n-1}(t)\psi(s)s_1^{-1} + d_n^{-\epsilon}(t)\psi(s)s_1^{-2}.$$

Thus

$$r_n(\psi, d_n^{\epsilon}) = \chi(r_{n-1}(\chi, d_n^{\epsilon})) + \chi(r_{n-1}(\chi, d_{n-1})t_1^{-1})$$
$$+ \chi(r_{n-1}(\psi, d_n^{-\epsilon})t_1^{-2}).$$

Let $\psi = d_i$ where $i < n$. Then

$$\chi(r_{n-1}(d_i, d_n^{\epsilon})) = r_{n-1}(d_i, d_n^{\epsilon})$$
$$= \sideset{}{^\epsilon}\sum_{0 \leqq s \leqq i-1} \sum_{0 \leqq r \leqq i-1-s} \delta^{n-1}_{i-1+n-1-r-2s} d_{r+1} d_{i-1+n-1-r-2s+1}$$
$$= \sideset{}{^\epsilon}\sum_{0 \leqq s \leqq i-1} \sum_{1 \leqq r \leqq i-s} \delta^n_{i+n-r-2s} d_r d_{i+n-r-2s}.$$

Next

$$r_{n-1}(d_i, d_{n-1}) = \sum_{0 \leqq s \leqq i-1} \sum_{0 \leqq r \leqq i-s-1}' d_{r+1} d_{[i+n-2-r-2s]_n}.$$

So

$$\chi(r_{n-1}(d_i, d_{n-1})t_1^{-1}) = \sum_{0 \leqq s \leqq i-1}' d_0 d_{[i+n-2-2s]_n}.$$

Now

$$\chi(r_{n-1}(d_i, d_n^{-\epsilon})t_1^{-2}) = -\sum_{0 \leqq s \leqq i-2}^{-\epsilon} \delta_{i+n-2-2s}^n d_0 d_{i+n-2-2s}.$$

Therefore

$$r_n(d_i, d_n^{-\epsilon}) = \sum_{0 \leqq s \leqq i-1}^{\epsilon} \sum_{1 \leqq r \leqq i-s} \delta_{i+n-r-2s}^n d_r d_{i+n-r-2s}$$
$$+ \sum_{1 \leqq s \leqq i}' d_0 d_{[i+n-2s]_n}$$
$$- \sum_{1 \leqq s \leqq i-1}^{-\epsilon} \sigma_{i+n-2s}^n d_0 d_{i+n-2s}.$$

For the formulas (b) to hold the difference of the last two terms should be

$$\sum_{0 \leqq s \leqq i}^{\epsilon} \delta_{i+n-2s}^n d_0 d_{i+n-2s} d_0 d_{i+n-2s},$$

which equals

$$\sum_{(i/2) \leqq s \leqq i}^{\epsilon} d_0 d_{i+n-2s}.$$

The difference equals

$$d_0 d_{n-i} + \sum_{\substack{1 \leqq s \leqq i \\ i+n-2s \geqq n}}^{\epsilon} d_0 d_{[i+n-2s]_n} = \sum_{1 \leqq s \leqq (i/2)}^{\epsilon} d_0 d_{[i+n-2s]_n} + d_0 d_{n-1}$$
$$= \sum_{0 \leqq s \leqq (i/2)}^{\epsilon} d_0 d_{-i+n+2s}.$$

As the two expressions are equal, formula (b) is proven.

For formula (c) take $\psi = d_n^{-\epsilon}$. Then

$$r_n(d_n^{\epsilon}, d_n^{-\epsilon}) = \chi(r_{n-1}(d_n^{\epsilon}, d_n^{-\epsilon}))$$
$$+ \chi(r_{n-1}(d_{n-1}, d_n^{-\epsilon})t_1^{-1}) + \chi(r_{n-1}(d_n^{-\epsilon}, d_n^{-\epsilon})t_1^{-2}).$$

By induction

$$\chi(r_{n-1}(d_n^{\epsilon}, d_n^{-\epsilon})) = r_{n-1}(d_n^{\epsilon}, d_n^{\epsilon})$$
$$= \sum_{\substack{0 \leqq p \leqq q \leqq n-1 \\ p \equiv q \equiv n-2((2))}} d_{p+1}d_{q+1} = \sum_{\substack{1 \leqq p \leqq q \leqq n \\ p \equiv q \equiv n-1((2))}} d_p d_q.$$

Next

$$\chi(r_{n-1}(d_n^{-\epsilon}, d_n^{-\epsilon})t_1^{-2}) = \chi\left(t_1^{-2}\left(\sum_{\substack{0 \leqq p \leqq q \leqq n-1 \\ p \equiv q \equiv n-1((2))}} d_{p+1}d_{q+1}\right)\right).$$

If n is even it equals

$$-\sum_{\substack{1 \leqq q \leqq n-1 \\ q \equiv n-1((2))}} d_0 d_{q+1} = -\sum_{\substack{0 < q \leqq n \\ q \equiv n((2))}} d_0 d_q$$

where $d_n = d_n^{-\epsilon}$. If n is odd it equals $d_0 d_0$. Now

$$r_{n-1}(d_{n-1}, d_n^{-\epsilon})$$
$$= \sum_{0 \leqq i \leqq n-2}^{-\epsilon} \sum_{0 \leqq r \leqq n-2-i} \delta^n_{2n-2-r-2s} d_{r+1} d_{2n-2-r-2s}.$$

So

$$\chi(r_{n-1}(d_{n-1}, d_n^{-\epsilon})t_1^{-1}) = \sum_{0 \leqq s \leqq n-2}^{-\epsilon} \delta^n_{2n-2-2s} d_0 d_{2n-2-2s}$$
$$= \sum_{\substack{k \equiv 0((2)) \\ 1 \leqq k \leqq (n/2)}} d_0 d_{2k}$$

where $d_n = d_n^{-\epsilon}$.

Lastly we want to use the triple sum to show that

$$r_n(d_n^{\epsilon}, d_n^{-\epsilon}) = \sum_{\substack{0 \leqq p \leqq q \leqq n \\ p \equiv q \equiv n-1((2))}} d_p d_q.$$

If n is even this equals the first summand. Hence we want

$$\chi(r_{n-1}(d_n^{-\epsilon}, d_n^{-\epsilon})t_1^{-2}) = -\chi(r_{n-1}(d_{n-1}, d_n^{-\epsilon})t_1^{-1}),$$

which is true. If n is odd we want

$$d_0 d_0 + \sum_{\substack{0 \leqq q \leqq n \\ q \equiv n-1((2))}} d_0 d_q$$
$$= \chi(r_{n-1}(d_n^{-\epsilon}, d_n^{-\epsilon})t_2^{-2}) + \chi(r_{n-1}(d_{n-1}, d_n^{-\epsilon})t_1^{-1})$$

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which is true. Thus formula (c) is true.

The proof of (d) is analogous so we will skip the details. Just use the formula

$$r_n(d_n^\epsilon, d_n^\epsilon) = \chi(r_{n-1}(d_n^\epsilon, d_n^\epsilon)) + \chi(r_{n-1}(d_{n-1}, d_n^\epsilon)t_1^{-1}) + \chi(r_{n-1}(d_n^\epsilon, d_n^{-\epsilon})t_1^{-2}).$$

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