Abstract dependence, recursion theory, and the lattice of recursively enumerable filters

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This thesis is divided into two sections. Both sections are devoted to the study of effectiveness in algebra, realized as analyses of sub-structures of recursive structures. Section One deals with closed subsets of a Steinitz closure system with recursive dependence as introduced by Metakides and Nerode. Initially we generalize many results proved by Metakides and Nerode and results concerning $L(V_\infty)$, the lattice of recursively enumerable subspaces, to considerably more general settings. To do this we introduce the notions of *semiregularity* and the *closure intersection property*, and show how they account for most of the observed phenomena in $L(V_\infty)$ and $L(\omega)$, the lattice of recursively enumerable sets. For example, we show that if (U, cl) has the closure intersection property and is semiregular then $\mathrm{Th}(L(U))$ is undecidable. Similarly, we show that as (F_∞, cl) is regular, the Karp-Myhill theorem fails for F_∞ .

Shore defined *nowhere simplicity* in $L(\omega)$. We examine analogues of this notion in $L(U)$, and in particular $L(V_\infty)$. If (U, cl) has the closure intersection property then any recursively enumerable nondecidable closed subset can be decomposed into a pair of recursively enumerable non-decidable nowhere simple closed subsets. We use this for results concerning automorphisms of $L(V_\infty)$. We examine effective nowhere simplicity via the notion of a *maximal pair*. A recursively enumerable set A can be split into a maximal pair if and only if A is simple.

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We introduce a method of associating co-recursively enumerable subsets of recursive bases naturally with recursively enumerable closed sets, and thus answer a question of A. Retzlaff. If A and B are recursively enumerable subsets of a recursive basis R and A is a major subset of B in $L(R)$ then $cl(A)$ is major in $cl(B)$. This also holds more generally in *effective closure systems* (a question of J.B. Remmel).

This method also allows us to deduce many of the results in the literature; and to later examine complementation in $L(U)$ and $L(V_\infty)$.

THEOREM. *Every fully co-recursively enumerable (that is, generated by a co-recursively enumerable subset of a recursive basis) subspace has*

(i) *a decidable complement and*

(ii) *a nondecidable nowhere simple complement.*

THEOREM. *Every immune fully co-recursively enumerable subspace has*

(i) *a supermaximal and*

(ii) *a k-thin complement.*

Finally we examine automorphisms of supermaximal subspaces (a question of A. Nerode) via the notions of *superfluous* and *essential* bases. These bases are reflections of whether or not a supermaximal subspace has supermaximal subspaces.

The second part is an analysis of the lattice $L(Q)$ of recursively enumerable (proper) filters of the atomless free Boolean algebra Q. This study is intimately connected with the study of recursively bounded π_1^0 classes. Initially, we show that many of the results of Jockusch-Soare can be very easily deduced by studying $L(Q)$. This section basically reviews much of the existing folklore in this area. We then examine the degrees of *superthick* and *thick* filters. We say $F \in L(Q)$ is *super-thick* if for all $G \in L(Q)$ with $F \subseteq G$ there exists $\theta \in Q$ such that $\langle F, \theta \rangle = G$ (where $\langle A \rangle$ is the filter generated by A). Similarly F is *thick* if when $F \subseteq G$ there exists $\theta \in Q$ such that $\langle F, \theta \rangle = \langle G, \theta \rangle \neq Q$. A filter F is called *absolutely perfect* if $F = \langle P_i, \overline{P}_j \mid i \in A, j \in B \rangle$ where $A \cap B = \emptyset$ and $\{P_i\}_{i \in \omega}$ denotes a recursive set of free generators for Q.

THEOREMS.   *(i)   There exists a recursively enumerable absolutely perfect thick filter in each recursively enumerable nonzero degree.   There exist   recursively enumerable absolutely perfect superthick filters   F such that*

  (ii) $\left(F \equiv_T\right)$ A ∪ B   is incomplete,

  (iii)   A ∪ B   is a (complete) maximal set,

  (iv)   A ∪ B   is an incomplete maximal set,

  (v)   A ∪ B   has low degree (so it is not even hyperhypersimple or dense simple).

  (vi)   If   F   is (super)thick then   A ∪ B   is hypersimple.

We produce recursively enumerable (super)thick filters with complete decidable extensions (that is, contained in k recursive ultra-filters), and contained in exactly $2^k - 1$ decidable filters but contained in $2^{\aleph_0}$ or $\aleph_0$ nonrecursive ultrafilters. We produce recursive superthick filters contained in $\aleph_0$ recursive ultrafilters and exactly $k \geq 1$ or $\geq \aleph_0$ nonrecursive ones. These results give many "topological" generalizations of results by Jockusch and Soare on π_1^0 classes.

We also show how to embed the lattice of recursively enumerable sets as a filter in $L(Q)$. We give an analysis of properties of recursively enumerable axiomatizations (identifying filters with theories) of filters. Thus, for example, we produce a recursively enumerable theory T contained in a complete decidable theory T' such that T has a recursively enumerable set of axioms which can be extended to an axiomatization of T' , but however no recursively enumerable set of axioms of T is contained in any recursively enumerable set of axioms of T' . We also generalize the concept of maximality to subfilters of a given recursively enumerable non-principal filter. Thus we show that any recursively enumerable non-principal filter has a "maximal" subfilter, but no recursively enumerable non-principal filter has a "supermaximal" subfilter. Finally, we show how we can definably embed $L(\omega)$ into the lattice of

recursively enumerable filters.

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