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THE CLASS GROUPS OF THE IMAGINARY ABELIAN NUMBER FIELDS WITH GALOIS GROUP $(\mathbb{Z}/2\mathbb{Z})^n$

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Assuming the Generalised Riemann Hypothesis we determine all imaginary Abelian number fields N whose Galois group $G(N/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ for some integers $n \geqslant 1$ and the square of every ideal of N is principal.

For a finite group G the smallest positive integer m such that g^m is the identity for all $g \in G$ is called the exponent of G and denoted by $\exp(G)$. Chowla([**6**]) proved that there are finitely many imaginary quadratic number fields with class groups of exponents $\leqslant 2$. In [**24, 16**] it is shown that there are at most 66 such fields and that there are exactly 65 such fields under the assumption of the Generalised Riemann Hypothesis. These 65 fields are exactly the ones having one ideal class in each genus and are given in many sources in the literature, for example, [**2, 8, 20, 5**]. K. Horie and M. Horie proved that there are only finitely many imaginary Abelian number fields of 2-power degrees with class groups of exponents $\leqslant 2$. (See [**13, 14**].) Many authors have studied the exponents of class groups of various number fields; for example, [**3, 10, 11, 12, 15, 17, 18, 19, 22**]. In particular, Louboutin determined all nonquadratic imaginary cyclic number fields of 2-power degrees with class groups of exponents $\leqslant 2$. In this paper we show the following.

THEOREM 1. *Assume the Generalised Riemann Hypothesis. There are exactly* $650(= 65+483+99+3)$ *imaginary Abelian number fields N with Galois group isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^n$ *for some integer n and class group of exponent at most* 2. *These fields are all of degree at most* 16 *with conductor at most* 233905 *and class number at most* 32. *All of these fields of degrees* $\geqslant 4$ *over* $\mathbb{Q}$ *are compiled at the end of tnis paper.*

For a number field F let $\mathrm{Cl}(F)$, h_F, O_F, d_F denote the class group, the class number, the ring of algebraic integers, and the absolute value of the discriminant of F, respectively. If F is a CM-field, then we let F^+ and h_F^- denote its maximal totally real subfield and the relative class number of F, respectively. We let f_F be the conductor of F when F is an Abelian number field.

Before proceeding to the proof of Theorem 1 we need the following result.

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PROPOSITION 2. (a) *Let L/F be a quadratic extension of number fields. Assume that* $\exp\big(\mathrm{Cl}(L)\big) \leqslant 2$. *Then* $\exp\big(\mathrm{Cl}(F)\big)$ *divides 4. In addition, the 4-rank of* $\mathrm{Cl}(F)$ *is at most one, and* $\exp\big(\mathrm{Cl}(F)\big) \leqslant 2$ *if L/F is not unramified.*

(b) *Let $n \geqslant 2$ and let $m_1, m_2, \cdots, m_n$ be square free positive integers such that $N_{2^n} = \mathbb{Q}(\sqrt{-m_1}, \sqrt{-m_2}, \cdots, \sqrt{-m_n})$ is an imaginary Abelian number fields of degree 2^n over $\mathbb{Q}$. Then N_{2^n} has at least one imaginary subfield M with $[N_{2^n} : M] = 2$ such that N_{2^n}/M is not unramified. Moreover, if $n \geqslant 3$, then N_{2^n} has at least two distinct imaginary subfields M and M' with $[N_{2^n} : M] = [N_{2^n} : M'] = 2$ such that neither the extension N_{2^n}/M nor the extension N_{2^n}/M' is unramified.*

PROOF: (a) Let $\mathfrak{a}$ be an ideal of O_F. We have $\mathfrak{a}^2 O_L = (\alpha)$ for some $\alpha \in O_L$ and $N_{L/F}(\mathfrak{a}^2 O_L) = N_{L/F}(\alpha) O_F = \mathfrak{a}^4$, so $\exp\big(\mathrm{Cl}(F)\big)$ divides 4. For the second statement we use some well known results of class field theory. According to Takagi–Artin theorem if L/F is unramified, then $\left|\mathrm{Cl}(F)/N_{L/F}\big(\mathrm{Cl}(L)\big)\right| = [L : F] = 2$ and so the 4-rank of $\mathrm{Cl}(F)$ can not exceed one, where $N_{L/F}$ denotes the norm map from $\mathrm{Cl}(L)$ to $\mathrm{Cl}(F)$. (See [**4**, Theorem 5.1, Chapter VII].) If L/F is not unramified, then the norm map $N_{L/F}$ is surjective and so $\exp\big(\mathrm{Cl}(F)\big) \leqslant 2$. (See [**23**, Theorem 10.1 and Theorem 5 in Appendix].)

(b) Let S be the set of all imaginary subfields M of N_{2^n} with $[N_{2^n} : M] = 2$. Let p be a prime dividing $d_{N_{2^n}^+}$ and let $\mathfrak{p}$ be a prime divisor of $pO_{N_{2^n}}$. Suppose that N_{2^n}/M is unramified for all $M \in S$. Then the inertia field of $\mathfrak{p}$ would be equal to $N_{2^n}^+$. This contradicts the fact that $p \mid d_{N_{2^n}^+}$. For the second statement we take a field $M_1 \in S$ such that N_{2^n}/M_1 is not unramified. Let q be a prime dividing $d_{M_1^+}$ (Since $n \geqslant 3$, we have $[M_1^+ : \mathbb{Q}] \geqslant 2$ and $d_{M_1^+} > 1$.) and $\mathfrak{q}$ a prime divisor of $qO_{N_{2^n}}$. Suppose that N_{2^n}/M is unramified for all $M \in S \backslash \{M_1\}$. Then the inertia field of $\mathfrak{q}$ would be equal to either M_1 or $N_{2^n}^+$. Thus q would be unramified in either $M_1/\mathbb{Q}$ or $N_{2^n}^+/\mathbb{Q}$. This contradicts the fact that $q \mid d_{M_1}$ and $q \mid d_{M_1^+} \mid d_{N_{2^n}^+}$. ∎

COROLLARY 3. *Let N_{2^n} be as above.*

(a) *Assume that* $\exp\big(\mathrm{Cl}(N_4)\big) \leqslant 2$. *Then $N_4 = k_1 k_2$ is a compositum of two distinct imaginary quadratic fields k_1 and k_2 such that* $\exp\big(\mathrm{Cl}(k_1)\big) \leqslant 2$, $\exp\big(\mathrm{Cl}(k_2)\big) \mid 4$ *and at the same time the 4-rank of $\mathrm{Cl}(k_2)$ is at most one. Let t be the number of distinct prime divisors of d_{k_2}. Assuming the Generalised Riemann Hypothesis we have either*

(i) $\mathrm{Cl}(k_2) \cong (\mathbb{Z}/2\mathbb{Z})^{t-1}$ *with $t \leqslant 7$ and $d_{k_2} \leqslant 1.4 \times 10^6$; or*

(ii) $\mathrm{Cl}(k_2) \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{t-2}$ *with $t \leqslant 8$ and $d_{k_2} \leqslant 3.4 \times 10^7$.*

(b) *If* $\exp\big(\mathrm{Cl}(N_{2^n})\big) \leqslant 2$ *with $n \geqslant 3$, then $N_{2^n} = MM'$ is a compositum of two distinct imaginary subfields M and M' such that $[M : \mathbb{Q}] = [M' : \mathbb{Q}] = 2^{n-1}$,* $\exp\big(\mathrm{Cl}(M)\big) \leqslant 2$, *and* $\exp\big(\mathrm{Cl}(M')\big) \leqslant 2$.

PROOF: The first statement of (a), and (b) follow from Proposition 2. We now prove the second statement of (a). By [7, (18.3) Corollary] the 2-rank of $\mathrm{Cl}(k_2)$ is $t - 1$. Let $k_2 = \mathbb{Q}(\sqrt{-m})$ with the square free positive integer m. Assume $\mathrm{Cl}(k_2) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-1}$.

To determine upper bounds for t and d_{k_2} we divide our determinations into 12 cases according to the values $-m$ *mod* 4 and the decompositions of the ideals (2) and (3) in k_2. Using Theorem 4 below we compute upper bounds for t and d_{k_2} in each case and take the worst upper bounds. We illustrate our method in the case that $-m \equiv 3$ *mod* 4 and (3) is ramified in k_2. Let p_i be the ith prime, that is, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$, et cetera. Set $f(t) = 2\prod_{i=1}^{t} p_i$ and $g(t) = \sqrt{f(t)}/(\ln f(t))$. Then $d_{k_2} = 4m = 2q_1q_2\cdots q_t \geqslant f(t)$, where q_i's are the distinct prime divisors of d_{k_2}. By Theorem 4 point (4) below

$$h_{k_2} = 2^{t-1} \geqslant \frac{2\pi}{3e}\frac{\sqrt{d_{k_2}}}{\ln d_{k_2}} \geqslant \frac{2\pi}{3e}\frac{\sqrt{f(t)}}{\ln f(t)} = \frac{2\pi}{3e}g(t),$$

because $\sqrt{x}/(\ln x)$ is increasing for $x \geqslant e^2$. According to *Bertrand's Postulate* ([**21**, Theorem 8.7]) $p_t < p_{t+1} < 2p_t$. Hence

$$\ln f(t+1) = \ln f(t) + \ln p_{t+1} < \ln f(t) + \ln 2p_t < 2\ln f(t).$$

Therefore

$$\frac{g(t+1)}{g(t)} = \frac{\ln f(t)}{\ln f(t+1)}\sqrt{p_{t+1}} > \frac{\sqrt{p_{t+1}}}{2}.$$

Since $(\sqrt{p_{t+1}})/2 > 2$ for $t \geqslant 6$, $g(t)$ grows more rapidly than 2^{t-1} for $t \geqslant 6$. Note that $f(8) > f(7) > e^2$, $(2\pi/3e)g(8) = 202.2304\cdots > 2^{8-1} = 128$, but $(2\pi/3e)g(7) = 56.2697\cdots < 2^{7-1} = 64$. Therefore, $2^{t-1} \geqslant (2\pi/3e)g(t)$ implies $t \leqslant 7$, and hence $h_{k_2} \leqslant 2^6$. The inequality $2^6 \geqslant (2\pi/3e)(\sqrt{d_{k_2}}/\ln d_{k_2})$ yields $d_{k_2} \leqslant 1.4 \times 10^6$. This proves that if $-m \equiv 3$ *mod* 4 and (3) is ramified in k_2, then $t \leqslant 7$ and $d_{k_2} \leqslant 1.4 \times 10^6$. Similary we compute upper bounds for t and d_{k_2} in the remaining cases and verify that $t \leqslant 7$ and $d_{k_2} \leqslant 1.4 \times 10^6$ are the worst bounds. It follows that if $\mathrm{Cl}(k_2) \simeq (\mathbb{Z}/2\mathbb{Z})^{t-1}$, then $t \leqslant 7$ and $d_{k_2} \leqslant 1.4 \times 10^6$ (see [?, Theorem 2]). In a similar fashion we obtain that if $\mathrm{Cl}(k_2) \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{t-2}$, then $t \leqslant 8$ and $d_{k_2} \leqslant 3.4 \times 10^7$. This completes the proof. ∎

Theorem 4. ([**16**, Theorem 1]) *Let k be an imaginary quadratic field. Assuming the Generalised Riemann Hypothesis we have:*

(1) $h_k \geqslant (\pi/3e)(\sqrt{d_k}/\ln d_k)$.
(2) *When 2 is not inert in k, $h_k \geqslant (\pi/2e)(\sqrt{d_k}/\ln d_k)$.*
(3) *When 3 is not inert in k, $h_k \geqslant (4\pi/9e)(\sqrt{d_k}/\ln d_k)$.*
(4) *When neither 2 nor 3 is inert in k, $h_k \geqslant (2\pi/3e)(\sqrt{d_k}/\ln d_k)$ for k with $d_k \neq 8$.*

Proof of Theorem 1: We verify that there are 65 imaginary quadratic fields k_1 with $\exp\left(\mathrm{Cl}(k_1)\right) \leqslant 2$ and 161 imaginary quadratic fields k_2 such that $\mathrm{Cl}(k_2)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{t-2}$, where $t \leqslant 8$ was defined in Corollary 3 point (a). We compute the relative class numbers of $12545\big(= 65 \times \big((64/2) + 161\big)\big)$ quartic composita, and verify that the relative class numbers of all of these 12545 quartic composita are

2-powers. Among those 12545 quartic fields only 9594 ones satisfy exp $\left(\mathrm{Cl}(N_4^+)\right) \leqslant 2$. For these 9594 quartic fields we compute $\mathrm{Cl}(N_4)$ and verify that there are exactly 483 fields N_4 with exp $\left(\mathrm{Cl}(N_4)\right) \leqslant 2$. Our computational results are summerised in Tables 1 and 4.

There are 12922 pairs of $(N_4^{(1)}, N_4^{(2)})$ two distinct imaginary quartic fields such that exp $\left(\mathrm{Cl}(N_4^{(i)})\right) \leqslant 2$ for $i = 1, 2$ and $[N_4^{(1)} \cap N_4^{(2)} : \mathbb{Q}] = 2$. Among those 12922 pairs 8686 ones give distinct composita $N_8 = N_4^{(1)} N_4^{(2)}$. Among those 8686 composita $N_8 = N_4^{(1)} N_4^{(2)}$ there are 172 octic fields N_8 such that every imaginary quartic subfield K of N_8 satisfy that exp $\left(\mathrm{Cl}(K)\right) \mid 4$ and at the same time the 4-rank of $\mathrm{Cl}(K)$ is at most 1. For those 172 octic fields we verify first whether both $h_{N_8}^-$ and $h_{N_8^+}$ are 2-powers, and then verify whether both $\mathrm{Cl}(N_8^+)$ and $\mathrm{Cl}(N_8)$ are elementary 2-groups. Note that using the generalised Bernoulli numbers we can easily compute the relative class numbers of imaginary Abelian number fields. See Chapter 4 in [**23**]. There are 99 octic fields N_8 with exp $\left(\mathrm{Cl}(N_8)\right) \leqslant 2$. Our computational results are given in Tables 2 and 5.

In the same manner we take the composita $N_{16} = N_8^{(1)} N_8^{(2)}$ such that $[N_8^{(1)} \cap N_8^{(2)} : \mathbb{Q}] = 4$, exp $\left(\mathrm{Cl}(N_8^{(i)})\right) \leqslant 2$ for $i = 1, 2$. We verify that there exist exactly three fields N_{16} with exp $\left(\mathrm{Cl}(N_{16})\right) \leqslant 2$. Our computational results are given in Tables 3 and 6.

The three composita of any two of those three fields N_{16} with exp $\left(\mathrm{Cl}(N_{16})\right) \leqslant 2$ are the same field ; $N_{32} = \mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-5}, \sqrt{-7})$. Note that N_{32} has a subfield $M = \mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{-5}, \sqrt{-7})$ such that the primes lying above 3 are ramified in the extension N_{32}/M. However, we verify that $\mathrm{Cl}(M) = \mathbb{Z}/4\mathbb{Z}$. By Proposition 2 point (a), exp $\left(\mathrm{Cl}(N_{32})\right) > 2$. Consequently, according to Proposition 2 point (b) if $n \geqslant 5$, then exp $\left(\mathrm{Cl}(N_{2^n})\right) > 2$. Our proof of Theorem 1 is now complete. $\square$

Our computations were carried out using PARI-GP ([**1**]) and KASH ([**9**]). Our source code is available to anyone interested by request.

Table 1

$\mathrm{Cl}(N_4)$	(1)	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^4$	$(\mathbb{Z}/2\mathbb{Z})^5$
number of fields N_4	47	160	112	132	26	6
upper bounds for f_{N_4}	10921	65689	69601	233905	18204	142545

Table 2

$Cl(N_8)$	(1)	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^4$	$(\mathbb{Z}/2\mathbb{Z})^5$
number of fields N_8	17	27	19	32	1	3
upper bounds for f_{N_8}	627	3553	760	5320	3315	6460

Table 3

$Cl(N_{16})$	(1)	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
number of fields N_{16}	0	1	2
upper bounds for $f_{N_{16}}$		120	840

Table 4

The fields $N_4 = \mathbb{Q}(\sqrt{-m_1}, \sqrt{-m_2})$ with $\exp\left(Cl(N_4)\right) \leqslant 2$.

Nr.	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}	Nr.	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}
1	1	2	1	1	2^3	243	11	30	4	8	$2^3 \cdot 3 \cdot 5 \cdot 11$
2	1	3	1	1	$2^2 \cdot 3$	244	2	165	4	16	$2^3 \cdot 3 \cdot 5 \cdot 11$
3	3	15	1	1	$3 \cdot 5$	245	5	67	2	2	$2^2 \cdot 5 \cdot 67$
4	1	5	1	1	$2^2 \cdot 5$	246	11	123	2	2	$3 \cdot 11 \cdot 41$
5	3	7	1	1	$3 \cdot 7$	247	115	345	1	8	$2^2 \cdot 3 \cdot 5 \cdot 23$
6	2	3	1	1	$2^3 \cdot 3$	248	19	1387	1	2	$19 \cdot 73$
7	3	6	1	1	$2^3 \cdot 3$	249	6	177	1	4	$2^3 \cdot 3 \cdot 59$
8	2	6	1	2	$2^3 \cdot 3$	250	2	177	2	8	$2^3 \cdot 3 \cdot 59$
9	1	6	1	2	$2^3 \cdot 3$	251	1	357	2	8	$2^2 \cdot 3 \cdot 7 \cdot 17$
10	1	7	1	1	$2^2 \cdot 7$	252	35	1435	1	4	$5 \cdot 7 \cdot 41$
11	3	11	1	1	$3 \cdot 11$	253	3	1443	2	8	$3 \cdot 13 \cdot 37$
12	7	35	1	1	$5 \cdot 7$	254	2	187	2	4	$2^3 \cdot 11 \cdot 17$
13	3	39	1	2	$3 \cdot 13$	255	35	43	2	2	$5 \cdot 7 \cdot 43$
14	2	10	1	1	$2^3 \cdot 5$	256	11	1507	1	2	$11 \cdot 137$
15	5	10	1	2	$2^3 \cdot 5$	257	35	385	1	8	$2^2 \cdot 5 \cdot 7 \cdot 11$
16	2	5	2	2	$2^3 \cdot 5$	258	7	385	2	8	$2^2 \cdot 5 \cdot 7 \cdot 11$
17	1	10	2	2	$2^3 \cdot 5$	259	11	385	2	8	$2^2 \cdot 5 \cdot 7 \cdot 11$
18	1	11	1	1	$2^2 \cdot 11$	260	3	130	4	8	$2^3 \cdot 3 \cdot 5 \cdot 13$
19	3	51	1	1	$3 \cdot 17$	261	21	133	1	8	$2^2 \cdot 3 \cdot 7 \cdot 19$
20	1	13	1	1	$2^2 \cdot 13$	262	21	57	1	8	$2^2 \cdot 3 \cdot 7 \cdot 19$
21	11	55	1	2	$5 \cdot 11$	263	6	67	2	2	$2^3 \cdot 3 \cdot 67$
22	2	7	1	1	$2^3 \cdot 7$	264	13	403	1	2	$2^2 \cdot 13 \cdot 31$
23	7	14	1	2	$2^3 \cdot 7$	265	1	403	2	2	$2^2 \cdot 13 \cdot 31$
24	3	19	1	1	$3 \cdot 19$	266	7	58	2	4	$2^3 \cdot 7 \cdot 29$
25	5	15	1	2	$2^2 \cdot 3 \cdot 5$	267	11	37	2	4	$2^2 \cdot 11 \cdot 37$
26	1	15	2	2	$2^2 \cdot 3 \cdot 5$	268	15	1635	1	8	$3 \cdot 5 \cdot 109$
27	3	5	2	2	$2^2 \cdot 3 \cdot 5$	269	35	235	1	2	$5 \cdot 7 \cdot 47$
28	1	17	1	2	$2^2 \cdot 17$	270	7	235	2	2	$5 \cdot 7 \cdot 47$
29	1	19	1	1	$2^2 \cdot 19$	271	3	1659	1	4	$3 \cdot 7 \cdot 79$
30	7	11	1	1	$7 \cdot 11$	272	19	22	2	2	$2^3 \cdot 11 \cdot 19$

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The fields $N_4 = \mathbb{Q}(\sqrt{-m_1}, \sqrt{-m_2})$ with $\exp\left(\mathrm{Cl}(N_4)\right) \leqslant 2$.

$Nr.$	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}	$Nr.$	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}
31	3	21	1	2	$2^2 \cdot 3 \cdot 7$	273	2	418	1	4	$2^3 \cdot 11 \cdot 19$
32	1	21	1	2	$2^2 \cdot 3 \cdot 7$	274	10	43	2	2	$2^3 \cdot 5 \cdot 43$
33	7	21	1	2	$2^2 \cdot 3 \cdot 7$	275	19	91	2	2	$7 \cdot 13 \cdot 19$
34	2	11	1	1	$2^3 \cdot 11$	276	1	435	4	8	$2^2 \cdot 3 \cdot 5 \cdot 29$
35	11	22	1	1	$2^3 \cdot 11$	277	6	438	1	8	$2^3 \cdot 3 \cdot 73$
36	1	22	1	2	$2^3 \cdot 11$	278	7	1771	1	4	$7 \cdot 11 \cdot 23$
37	2	22	1	2	$2^3 \cdot 11$	279	5	445	1	8	$2^2 \cdot 5 \cdot 89$
38	7	91	1	1	$7 \cdot 13$	280	11	163	1	1	$11 \cdot 163$
39	2	13	2	2	$2^3 \cdot 13$	281	7	462	2	8	$2^3 \cdot 3 \cdot 7 \cdot 11$
40	15	35	1	2	$3 \cdot 5 \cdot 7$	282	21	462	1	16	$2^3 \cdot 3 \cdot 7 \cdot 11$
41	7	15	2	2	$3 \cdot 5 \cdot 7$	283	42	462	1	16	$2^3 \cdot 3 \cdot 7 \cdot 11$
42	3	35	2	2	$3 \cdot 5 \cdot 7$	284	2	462	4	16	$2^3 \cdot 3 \cdot 7 \cdot 11$
43	10	15	1	2	$2^3 \cdot 3 \cdot 5$	285	15	93	2	8	$2^2 \cdot 3 \cdot 5 \cdot 31$
44	3	10	2	2	$2^3 \cdot 3 \cdot 5$	286	7	267	2	2	$3 \cdot 7 \cdot 89$
45	2	15	2	2	$2^3 \cdot 3 \cdot 5$	287	10	235	1	2	$2^3 \cdot 5 \cdot 47$
46	5	30	1	4	$2^3 \cdot 3 \cdot 5$	288	2	235	2	2	$2^3 \cdot 5 \cdot 47$
47	10	30	1	4	$2^3 \cdot 3 \cdot 5$	289	13	37	2	4	$2^2 \cdot 13 \cdot 37$
48	1	30	2	4	$2^3 \cdot 3 \cdot 5$	290	21	483	1	8	$2^2 \cdot 3 \cdot 7 \cdot 23$
49	2	30	2	4	$2^3 \cdot 3 \cdot 5$	291	11	1947	1	4	$3 \cdot 11 \cdot 59$
50	3	123	1	1	$3 \cdot 41$	292	2	498	1	4	$2^3 \cdot 3 \cdot 83$
51	3	43	1	1	$3 \cdot 43$	293	15	1995	1	8	$3 \cdot 5 \cdot 7 \cdot 19$
52	1	33	1	2	$2^2 \cdot 3 \cdot 11$	294	3	1995	2	8	$3 \cdot 5 \cdot 7 \cdot 19$
53	11	33	1	2	$2^2 \cdot 3 \cdot 11$	295	19	1995	2	8	$3 \cdot 5 \cdot 7 \cdot 19$
54	3	33	1	2	$2^2 \cdot 3 \cdot 11$	296	22	253	1	4	$2^3 \cdot 11 \cdot 23$
55	7	19	1	1	$7 \cdot 19$	297	11	2035	2	8	$5 \cdot 11 \cdot 37$
56	2	34	1	2	$2^3 \cdot 17$	298	2	267	2	4	$2^3 \cdot 3 \cdot 89$
57	5	35	1	2	$2^2 \cdot 5 \cdot 7$	299	11	195	4	8	$3 \cdot 5 \cdot 11 \cdot 13$
58	5	7	2	2	$2^2 \cdot 5 \cdot 7$	300	3	715	4	8	$3 \cdot 5 \cdot 11 \cdot 13$
59	1	35	2	2	$2^2 \cdot 5 \cdot 7$	301	3	2163	1	4	$3 \cdot 7 \cdot 103$
60	1	37	1	1	$2^2 \cdot 37$	302	6	91	4	8	$2^3 \cdot 3 \cdot 7 \cdot 13$
61	2	19	1	1	$2^3 \cdot 19$	303	7	78	4	8	$2^3 \cdot 3 \cdot 7 \cdot 13$
62	3	13	2	4	$2^2 \cdot 3 \cdot 13$	304	19	115	2	4	$5 \cdot 19 \cdot 23$
63	11	15	2	4	$3 \cdot 5 \cdot 11$	305	43	51	2	4	$3 \cdot 17 \cdot 43$
64	3	42	1	2	$2^3 \cdot 3 \cdot 7$	306	37	555	2	8	$2^2 \cdot 3 \cdot 5 \cdot 37$
65	2	42	1	2	$2^3 \cdot 3 \cdot 7$	307	1	555	4	8	$2^2 \cdot 3 \cdot 5 \cdot 37$
66	7	42	1	2	$2^3 \cdot 3 \cdot 7$	308	13	43	2	4	$2^2 \cdot 13 \cdot 43$
67	6	7	2	4	$2^3 \cdot 3 \cdot 7$	309	19	30	4	8	$2^3 \cdot 3 \cdot 5 \cdot 19$
68	6	42	1	4	$2^3 \cdot 3 \cdot 7$	310	19	123	2	2	$3 \cdot 19 \cdot 41$
69	6	21	1	4	$2^3 \cdot 3 \cdot 7$	311	35	67	2	2	$5 \cdot 7 \cdot 67$
70	1	42	2	4	$2^3 \cdot 3 \cdot 7$	312	1	595	4	8	$2^2 \cdot 5 \cdot 7 \cdot 17$
71	2	21	2	4	$2^3 \cdot 3 \cdot 7$	313	15	163	2	2	$3 \cdot 5 \cdot 163$
72	21	42	1	8	$2^3 \cdot 3 \cdot 7$	314	5	123	4	8	$2^2 \cdot 3 \cdot 5 \cdot 41$
73	1	43	1	1	$2^2 \cdot 43$	315	11	58	2	2	$2^3 \cdot 11 \cdot 29$
74	11	187	1	1	$11 \cdot 17$	316	11	235	2	4	$5 \cdot 11 \cdot 47$
75	15	195	1	4	$3 \cdot 5 \cdot 13$	317	21	93	1	8	$2^2 \cdot 3 \cdot 7 \cdot 31$
76	3	195	2	4	$3 \cdot 5 \cdot 13$	318	13	51	4	16	$2^2 \cdot 3 \cdot 13 \cdot 17$
77	3	67	1	1	$3 \cdot 67$	319	3	2667	1	4	$3 \cdot 7 \cdot 127$
78	7	203	1	2	$7 \cdot 29$	320	10	67	2	2	$2^3 \cdot 5 \cdot 67$
79	1	51	2	4	$2^2 \cdot 3 \cdot 17$	321	15	2715	1	8	$3 \cdot 5 \cdot 181$
80	11	19	1	1	$11 \cdot 19$	322	30	345	1	16	$2^3 \cdot 3 \cdot 5 \cdot 23$
81	3	219	1	2	$3 \cdot 73$	323	19	37	2	2	$2^2 \cdot 19 \cdot 37$
82	5	11	2	4	$2^2 \cdot 5 \cdot 11$	324	91	403	1	2	$7 \cdot 13 \cdot 31$
83	1	57	1	2	$2^2 \cdot 3 \cdot 19$	325	7	403	2	2	$7 \cdot 13 \cdot 31$
84	19	57	1	2	$2^2 \cdot 3 \cdot 19$	326	7	102	4	8	$2^3 \cdot 3 \cdot 7 \cdot 17$
85	3	57	1	2	$2^2 \cdot 3 \cdot 19$	327	2	357	4	16	$2^3 \cdot 3 \cdot 7 \cdot 17$
86	2	58	1	1	$2^3 \cdot 29$	328	5	715	2	8	$2^2 \cdot 5 \cdot 11 \cdot 13$
87	1	58	2	2	$2^3 \cdot 29$	329	1	715	4	8	$2^2 \cdot 5 \cdot 11 \cdot 13$
88	7	259	1	2	$7 \cdot 37$	330	43	67	1	1	$43 \cdot 67$
89	5	13	2	4	$2^2 \cdot 5 \cdot 13$	331	11	267	2	4	$3 \cdot 11 \cdot 89$
90	6	22	1	2	$2^3 \cdot 3 \cdot 11$	332	7	742	2	8	$2^3 \cdot 7 \cdot 53$
91	3	22	2	2	$2^3 \cdot 3 \cdot 11$	333	3	3003	2	8	$3 \cdot 7 \cdot 11 \cdot 13$
92	6	11	2	2	$2^3 \cdot 3 \cdot 11$	334	33	253	1	8	$2^2 \cdot 3 \cdot 11 \cdot 23$
93	2	66	1	4	$2^3 \cdot 3 \cdot 11$	335	35	435	2	8	$3 \cdot 5 \cdot 7 \cdot 29$
94	6	33	1	4	$2^3 \cdot 3 \cdot 11$	336	7	435	4	8	$3 \cdot 5 \cdot 7 \cdot 29$
95	22	33	1	4	$2^3 \cdot 3 \cdot 11$	337	11	70	4	8	$2^3 \cdot 5 \cdot 7 \cdot 11$
96	2	33	2	8	$2^3 \cdot 3 \cdot 11$	338	22	35	4	8	$2^3 \cdot 5 \cdot 7 \cdot 11$
97	3	267	1	1	$3 \cdot 89$	339	70	385	1	16	$2^3 \cdot 5 \cdot 7 \cdot 11$
98	1	67	1	1	$2^2 \cdot 67$	340	19	163	1	1	$19 \cdot 163$
99	3	91	2	4	$3 \cdot 7 \cdot 13$	341	1	795	4	8	$2^2 \cdot 3 \cdot 5 \cdot 53$
100	10	35	1	2	$2^3 \cdot 5 \cdot 7$	342	2	403	2	2	$2^3 \cdot 13 \cdot 31$

The fields $N_4 = \mathbb{Q}(\sqrt{-m_1}, \sqrt{-m_2})$ with $\exp\left(\mathrm{Cl}(N_4)\right) \leqslant 2$.

Nr.	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}	Nr.	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}
101	2	35	2	2	$2^3\cdot5\cdot7$	343	22	37	2	4	$2^3\cdot11\cdot37$
102	7	10	2	2	$2^3\cdot5\cdot7$	344	5	163	2	2	$2^2\cdot5\cdot163$
103	35	70	1	4	$2^3\cdot5\cdot7$	345	2	427	2	2	$2^3\cdot7\cdot61$
104	7	70	2	4	$2^3\cdot5\cdot7$	346	51	67	2	4	$3\cdot17\cdot67$
105	10	70	1	4	$2^3\cdot5\cdot7$	347	13	67	2	2	$2^2\cdot13\cdot67$
106	5	70	1	4	$2^3\cdot5\cdot7$	348	19	187	2	4	$11\cdot17\cdot19$
107	1	70	2	4	$2^3\cdot5\cdot7$	349	19	195	4	8	$3\cdot5\cdot13\cdot19$
108	2	70	2	4	$2^3\cdot5\cdot7$	350	11	85	4	8	$2^2\cdot5\cdot11\cdot17$
109	15	19	2	4	$3\cdot5\cdot19$	351	5	187	4	8	$2^2\cdot5\cdot11\cdot17$
110	3	291	1	2	$3\cdot97$	352	22	43	2	2	$2^3\cdot11\cdot43$
111	1	73	1	2	$2^2\cdot73$	353	11	3795	2	16	$3\cdot5\cdot11\cdot23$
112	2	37	2	2	$2^3\cdot37$	354	42	483	1	8	$2^3\cdot3\cdot7\cdot23$
113	7	43	1	1	$7\cdot43$	355	7	555	4	8	$3\cdot5\cdot7\cdot37$
114	7	77	1	4	$2^2\cdot7\cdot11$	356	6	163	2	2	$2^3\cdot3\cdot163$
115	6	78	1	4	$2^3\cdot3\cdot13$	357	43	91	2	4	$7\cdot13\cdot43$
116	3	78	2	4	$2^3\cdot3\cdot13$	358	19	58	2	2	$2^3\cdot19\cdot29$
117	13	78	1	4	$2^3\cdot3\cdot13$	359	13	85	4	16	$2^2\cdot5\cdot13\cdot17$
118	1	78	2	4	$2^3\cdot3\cdot13$	360	11	403	2	2	$11\cdot13\cdot31$
119	2	78	2	4	$2^3\cdot3\cdot13$	361	19	235	2	4	$5\cdot19\cdot47$
120	6	13	2	4	$2^3\cdot3\cdot13$	362	2	1122	2	16	$2^3\cdot3\cdot11\cdot17$
121	19	323	1	2	$17\cdot19$	363	11	427	2	2	$7\cdot11\cdot61$
122	2	82	1	2	$2^3\cdot41$	364	2	595	4	8	$2^3\cdot5\cdot7\cdot17$
123	5	85	1	4	$2^2\cdot5\cdot17$	365	11	435	4	8	$3\cdot5\cdot11\cdot29$
124	1	85	2	4	$2^2\cdot5\cdot17$	366	10	123	4	8	$2^3\cdot3\cdot5\cdot41$
125	2	43	1	1	$2^3\cdot43$	367	43	115	2	2	$5\cdot23\cdot43$
126	15	115	1	2	$3\cdot5\cdot23$	368	2	627	4	16	$2^3\cdot3\cdot11\cdot19$
127	3	115	2	2	$3\cdot5\cdot23$	369	11	5115	2	16	$3\cdot5\cdot11\cdot31$
128	7	51	2	2	$3\cdot7\cdot17$	370	3	5187	2	16	$3\cdot7\cdot13\cdot19$
129	13	91	1	2	$2^2\cdot7\cdot13$	371	6	1302	1	16	$2^3\cdot3\cdot7\cdot31$
130	1	91	2	2	$2^2\cdot7\cdot13$	372	43	123	2	4	$3\cdot41\cdot43$
131	7	13	2	2	$2^2\cdot7\cdot13$	373	13	102	4	16	$2^3\cdot3\cdot13\cdot17$
132	3	93	1	2	$2^2\cdot3\cdot31$	374	11	483	4	8	$3\cdot7\cdot11\cdot23$
133	1	93	1	2	$2^2\cdot3\cdot31$	375	19	70	4	8	$2^3\cdot5\cdot7\cdot19$
134	5	19	2	4	$2^2\cdot5\cdot19$	376	7	190	4	8	$2^3\cdot5\cdot7\cdot19$
135	11	35	2	4	$5\cdot7\cdot11$	377	5	267	4	8	$2^2\cdot3\cdot5\cdot89$
136	1	97	1	2	$2^2\cdot97$	378	115	235	1	2	$5\cdot23\cdot47$
137	51	102	1	4	$2^3\cdot3\cdot17$	379	7	5467	1	4	$7\cdot11\cdot71$
138	6	102	1	4	$2^3\cdot3\cdot17$	380	7	795	4	8	$3\cdot5\cdot7\cdot53$
139	2	51	2	4	$2^3\cdot3\cdot17$	381	35	163	2	2	$5\cdot7\cdot163$
140	1	102	2	8	$2^3\cdot3\cdot17$	382	11	130	4	8	$2^3\cdot5\cdot11\cdot13$
141	2	102	2	8	$2^3\cdot3\cdot17$	383	5	1435	2	8	$2^2\cdot5\cdot7\cdot41$
142	7	427	1	1	$7\cdot61$	384	1	1435	4	8	$2^2\cdot5\cdot7\cdot41$
143	15	435	1	4	$3\cdot5\cdot29$	385	22	67	2	2	$2^3\cdot11\cdot67$
144	10	11	2	4	$2^3\cdot5\cdot11$	386	15	403	4	8	$3\cdot5\cdot13\cdot31$
145	10	22	2	4	$2^3\cdot5\cdot11$	387	11	555	4	16	$3\cdot5\cdot11\cdot37$
146	5	22	2	4	$2^3\cdot5\cdot11$	388	37	43	2	2	$2^2\cdot37\cdot43$
147	3	37	2	4	$2^2\cdot3\cdot37$	389	15	427	4	8	$3\cdot5\cdot7\cdot61$
148	6	19	2	2	$2^3\cdot3\cdot19$	390	19	85	4	16	$2^2\cdot5\cdot17\cdot19$
149	2	114	1	4	$2^3\cdot3\cdot19$	391	10	163	2	2	$2^3\cdot5\cdot163$
150	6	57	1	4	$2^3\cdot3\cdot19$	392	11	595	4	8	$5\cdot7\cdot11\cdot17$
151	2	57	2	8	$2^3\cdot3\cdot19$	393	43	163	1	1	$43\cdot163$
152	5	115	1	2	$2^2\cdot5\cdot23$	394	57	93	1	8	$2^2\cdot3\cdot19\cdot31$
153	1	115	2	2	$2^2\cdot5\cdot23$	395	19	7315	2	16	$5\cdot7\cdot11\cdot19$
154	7	483	1	2	$3\cdot7\cdot23$	396	10	187	4	8	$2^3\cdot5\cdot11\cdot17$
155	3	483	1	2	$3\cdot7\cdot23$	397	37	51	4	8	$2^2\cdot3\cdot17\cdot37$
156	3	163	1	1	$3\cdot163$	398	19	403	2	2	$13\cdot19\cdot31$
157	1	123	2	4	$2^2\cdot3\cdot41$	399	19	102	4	8	$2^3\cdot3\cdot17\cdot19$
158	1	130	4	8	$2^3\cdot5\cdot13$	400	11	7755	2	16	$3\cdot5\cdot11\cdot47$
159	1	133	1	2	$2^2\cdot7\cdot19$	401	33	177	1	8	$2^2\cdot3\cdot11\cdot59$
160	7	133	1	2	$2^2\cdot7\cdot19$	402	43	187	2	4	$11\cdot17\cdot43$
161	19	133	1	2	$2^2\cdot7\cdot19$	403	35	58	4	8	$2^3\cdot5\cdot7\cdot29$
162	2	67	1	1	$2^3\cdot67$	404	67	123	2	2	$3\cdot41\cdot67$
163	15	555	1	4	$3\cdot5\cdot37$	405	19	435	4	8	$3\cdot5\cdot19\cdot29$
164	3	555	2	4	$3\cdot5\cdot37$	406	51	163	2	2	$3\cdot17\cdot163$
165	51	187	1	2	$3\cdot11\cdot17$	407	11	190	4	8	$2^3\cdot5\cdot11\cdot19$
166	3	187	2	2	$3\cdot11\cdot17$	408	43	195	4	8	$3\cdot5\cdot13\cdot43$
167	11	51	2	2	$3\cdot11\cdot17$	409	13	163	2	2	$2^2\cdot13\cdot163$
168	11	13	2	2	$2^2\cdot11\cdot13$	410	15	8835	1	16	$3\cdot5\cdot19\cdot31$
169	7	154	1	4	$2^3\cdot7\cdot11$	411	7	2233	2	16	$2^2\cdot7\cdot11\cdot29$
170	7	22	2	4	$2^3\cdot7\cdot11$	412	19	483	4	8	$3\cdot7\cdot19\cdot23$

The fields $N_4 = \mathbb{Q}(\sqrt{-m_1}, \sqrt{-m_2})$ with $\exp\left(\mathrm{Cl}(N_4)\right) \leqslant 2$.

Nr.	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}	Nr.	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}
171	3	627	1	2	$3 \cdot 11 \cdot 19$	413	6	403	4	8	$2^3 \cdot 3 \cdot 13 \cdot 31$
172	11	627	1	2	$3 \cdot 11 \cdot 19$	414	13	187	4	8	$2^2 \cdot 11 \cdot 13 \cdot 17$
173	19	627	1	2	$3 \cdot 11 \cdot 19$	415	19	130	4	8	$2^3 \cdot 5 \cdot 13 \cdot 19$
174	15	43	2	2	$3 \cdot 5 \cdot 43$	416	37	67	2	4	$2^2 \cdot 37 \cdot 67$
175	3	651	1	4	$3 \cdot 7 \cdot 31$	417	43	58	2	2	$2^3 \cdot 29 \cdot 43$
176	1	163	1	1	$2^2 \cdot 163$	418	115	435	2	8	$3 \cdot 5 \cdot 23 \cdot 29$
177	11	165	2	8	$2^2 \cdot 3 \cdot 5 \cdot 11$	419	43	235	2	2	$5 \cdot 43 \cdot 47$
178	19	35	2	4	$5 \cdot 7 \cdot 19$	420	22	115	4	8	$2^3 \cdot 5 \cdot 11 \cdot 23$
179	2	85	4	8	$2^3 \cdot 5 \cdot 17$	421	6	427	4	8	$2^3 \cdot 3 \cdot 7 \cdot 61$
180	3	58	2	2	$2^3 \cdot 3 \cdot 29$	422	19	555	4	8	$3 \cdot 5 \cdot 19 \cdot 37$
181	15	235	1	2	$3 \cdot 5 \cdot 47$	423	10	267	4	8	$2^3 \cdot 3 \cdot 5 \cdot 89$
182	3	235	2	2	$3 \cdot 5 \cdot 47$	424	67	163	1	1	$67 \cdot 163$
183	1	177	1	2	$2^2 \cdot 3 \cdot 59$	425	19	595	4	16	$5 \cdot 7 \cdot 17 \cdot 19$
184	3	177	1	2	$2^2 \cdot 3 \cdot 59$	426	2	1435	4	8	$2^3 \cdot 5 \cdot 7 \cdot 41$
185	11	715	2	4	$5 \cdot 11 \cdot 13$	427	43	267	2	2	$3 \cdot 43 \cdot 89$
186	3	723	1	2	$3 \cdot 241$	428	11	11715	2	16	$3 \cdot 5 \cdot 11 \cdot 71$
187	2	91	2	2	$2^3 \cdot 7 \cdot 13$	429	67	187	2	4	$11 \cdot 17 \cdot 67$
188	11	67	1	1	$11 \cdot 67$	430	13	3315	4	32	$2^2 \cdot 3 \cdot 5 \cdot 13 \cdot 17$
189	5	37	2	4	$2^2 \cdot 5 \cdot 37$	431	1	3315	8	32	$2^2 \cdot 3 \cdot 5 \cdot 13 \cdot 17$
190	6	93	1	4	$2^3 \cdot 3 \cdot 31$	432	43	78	4	8	$2^3 \cdot 3 \cdot 13 \cdot 43$
191	2	93	2	4	$2^3 \cdot 3 \cdot 31$	433	19	715	4	8	$5 \cdot 11 \cdot 13 \cdot 19$
192	1	187	2	4	$2^2 \cdot 11 \cdot 17$	434	13	267	4	8	$2^2 \cdot 3 \cdot 13 \cdot 89$
193	19	190	2	4	$2^3 \cdot 5 \cdot 19$	435	115	123	4	8	$3 \cdot 5 \cdot 23 \cdot 41$
194	10	19	2	4	$2^3 \cdot 5 \cdot 19$	436	22	163	2	2	$2^3 \cdot 11 \cdot 163$
195	5	190	1	4	$2^3 \cdot 5 \cdot 19$	437	43	85	4	8	$2^2 \cdot 5 \cdot 17 \cdot 43$
196	10	190	1	4	$2^3 \cdot 5 \cdot 19$	438	3	14763	2	16	$3 \cdot 7 \cdot 19 \cdot 37$
197	1	190	2	4	$2^3 \cdot 5 \cdot 19$	439	91	163	2	2	$7 \cdot 13 \cdot 163$
198	2	190	2	4	$2^3 \cdot 5 \cdot 19$	440	19	795	4	8	$3 \cdot 5 \cdot 19 \cdot 53$
199	7	763	1	2	$7 \cdot 109$	441	58	67	2	4	$2^3 \cdot 29 \cdot 67$
200	1	193	1	2	$2^2 \cdot 193$	442	67	235	2	2	$5 \cdot 47 \cdot 67$
201	13	195	2	8	$2^2 \cdot 3 \cdot 5 \cdot 13$	443	11	1435	4	8	$5 \cdot 7 \cdot 11 \cdot 41$
202	1	195	4	8	$2^2 \cdot 3 \cdot 5 \cdot 13$	444	43	403	2	4	$13 \cdot 31 \cdot 43$
203	35	115	1	2	$5 \cdot 7 \cdot 23$	445	43	102	4	8	$2^3 \cdot 3 \cdot 17 \cdot 43$
204	7	115	2	2	$5 \cdot 7 \cdot 23$	446	37	123	4	16	$2^2 \cdot 3 \cdot 37 \cdot 41$
205	5	205	1	8	$2^2 \cdot 5 \cdot 41$	447	43	427	2	2	$7 \cdot 43 \cdot 61$
206	7	30	4	8	$2^3 \cdot 3 \cdot 5 \cdot 7$	448	115	163	2	2	$5 \cdot 23 \cdot 163$
207	5	43	2	2	$2^2 \cdot 5 \cdot 43$	449	123	163	2	4	$3 \cdot 41 \cdot 163$
208	7	123	2	2	$3 \cdot 7 \cdot 41$	450	51	403	4	8	$3 \cdot 13 \cdot 17 \cdot 31$
209	6	37	2	4	$2^3 \cdot 3 \cdot 37$	451	22	235	4	8	$2^3 \cdot 5 \cdot 11 \cdot 47$
210	15	915	1	8	$3 \cdot 5 \cdot 61$	452	58	91	4	8	$2^3 \cdot 7 \cdot 13 \cdot 29$
211	10	115	1	2	$2^3 \cdot 5 \cdot 23$	453	19	1155	8	32	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
212	2	115	2	2	$2^3 \cdot 5 \cdot 23$	454	43	130	4	8	$2^3 \cdot 5 \cdot 13 \cdot 43$
213	7	33	4	8	$2^2 \cdot 3 \cdot 7 \cdot 11$	455	67	85	4	8	$2^2 \cdot 5 \cdot 17 \cdot 67$
214	5	235	1	2	$2^2 \cdot 5 \cdot 47$	456	22	267	4	8	$2^3 \cdot 3 \cdot 11 \cdot 89$
215	19	51	2	4	$3 \cdot 17 \cdot 19$	457	37	163	2	2	$2^2 \cdot 37 \cdot 163$
216	2	123	2	4	$2^3 \cdot 3 \cdot 41$	458	10	627	8	32	$2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 19$
217	13	19	2	2	$2^2 \cdot 13 \cdot 19$	459	43	595	4	8	$5 \cdot 7 \cdot 17 \cdot 43$
218	11	91	2	2	$7 \cdot 11 \cdot 13$	460	67	403	2	2	$13 \cdot 31 \cdot 67$
219	15	67	2	2	$3 \cdot 5 \cdot 67$	461	19	1435	4	8	$5 \cdot 7 \cdot 19 \cdot 41$
220	1	253	1	2	$2^2 \cdot 11 \cdot 23$	462	37	187	4	8	$2^2 \cdot 11 \cdot 17 \cdot 37$
221	11	253	1	2	$2^2 \cdot 11 \cdot 23$	463	67	435	4	8	$3 \cdot 5 \cdot 29 \cdot 67$
222	6	43	2	2	$2^3 \cdot 3 \cdot 43$	464	163	187	2	2	$11 \cdot 17 \cdot 163$
223	2	258	1	4	$2^3 \cdot 3 \cdot 43$	465	43	715	4	8	$5 \cdot 11 \cdot 13 \cdot 43$
224	7	37	2	4	$2^2 \cdot 7 \cdot 37$	466	43	795	4	8	$3 \cdot 5 \cdot 43 \cdot 53$
225	2	133	2	4	$2^3 \cdot 7 \cdot 19$	467	67	555	4	8	$3 \cdot 5 \cdot 37 \cdot 67$
226	1	267	2	4	$2^2 \cdot 3 \cdot 89$	468	58	163	2	2	$2^3 \cdot 29 \cdot 163$
227	3	273	2	8	$2^2 \cdot 3 \cdot 7 \cdot 13$	469	163	235	2	2	$5 \cdot 47 \cdot 163$
228	15	57	2	8	$2^2 \cdot 3 \cdot 5 \cdot 19$	470	37	267	4	8	$2^2 \cdot 3 \cdot 37 \cdot 89$
229	7	163	1	1	$7 \cdot 163$	471	67	595	4	8	$5 \cdot 7 \cdot 17 \cdot 67$
230	11	1155	2	8	$3 \cdot 5 \cdot 7 \cdot 11$	472	163	267	2	2	$3 \cdot 89 \cdot 163$
231	7	301	1	4	$2^2 \cdot 7 \cdot 43$	473	187	235	4	8	$5 \cdot 11 \cdot 17 \cdot 47$
232	3	403	2	4	$3 \cdot 13 \cdot 31$	474	123	403	4	8	$3 \cdot 13 \cdot 31 \cdot 41$
233	3	1227	1	2	$3 \cdot 409$	475	187	267	4	8	$3 \cdot 11 \cdot 17 \cdot 89$
234	11	1243	1	2	$11 \cdot 113$	476	123	427	4	8	$3 \cdot 7 \cdot 41 \cdot 61$
235	11	115	2	4	$5 \cdot 11 \cdot 23$	477	43	1435	4	8	$5 \cdot 7 \cdot 41 \cdot 43$
236	19	67	1	1	$19 \cdot 67$	478	19	3315	8	32	$3 \cdot 5 \cdot 13 \cdot 17 \cdot 19$
237	3	427	2	4	$3 \cdot 7 \cdot 61$	479	163	403	2	2	$13 \cdot 31 \cdot 163$
238	7	322	1	4	$2^3 \cdot 7 \cdot 23$	480	163	427	2	4	$7 \cdot 61 \cdot 163$
239	7	187	2	2	$7 \cdot 11 \cdot 17$	481	235	427	4	8	$5 \cdot 7 \cdot 47 \cdot 61$
240	10	330	1	8	$2^3 \cdot 3 \cdot 5 \cdot 11$	482	43	3315	8	32	$3 \cdot 5 \cdot 13 \cdot 17 \cdot 43$

The fields $N_4 = \mathbb{Q}(\sqrt{-m_1}, \sqrt{-m_2})$ with $\exp\left(\mathrm{Cl}(N_4)\right) \leqslant 2$.

Nr.	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}	Nr.	m_1	m_2	$h_{N_4^+}$	h_{N_4}	f_{N_4}
241	2	330	2	8	$2^3 \cdot 3 \cdot 5 \cdot 11$	483	163	1435	4	8	$5 \cdot 7 \cdot 41 \cdot 163$
242	11	330	2	8	$2^3 \cdot 3 \cdot 5 \cdot 11$						

Table 5

The fields $N_8 = \mathbb{Q}(\sqrt{-m_1}, \sqrt{-m_2}, \sqrt{-m_3})$ with $\exp\left(\mathrm{Cl}(N_8)\right) \leqslant 2$.

Nr.	m_1	m_2	m_3	$h_{N_8^+}$	h_{N_8}	f_{N_8}	Nr.	m_1	m_2	m_3	$h_{N_8^+}$	h_{N_8}	f_{N_8}
1	1	2	3	1	1	$2^3 \cdot 3$	51	3	11	51	1	1	$3 \cdot 11 \cdot 17$
2	1	2	5	1	1	$2^3 \cdot 5$	52	2	7	11	1	2	$2^3 \cdot 7 \cdot 11$
3	1	2	7	1	2	$2^3 \cdot 7$	53	7	11	14	1	2	$2^3 \cdot 7 \cdot 11$
4	1	3	5	1	1	$2^2 \cdot 3 \cdot 5$	54	1	7	22	2	8	$2^3 \cdot 7 \cdot 11$
5	1	3	7	1	1	$2^2 \cdot 3 \cdot 7$	55	2	7	22	2	8	$2^3 \cdot 7 \cdot 11$
6	1	2	11	1	1	$2^3 \cdot 11$	56	3	11	19	1	1	$3 \cdot 11 \cdot 19$
7	3	7	15	1	1	$3 \cdot 5 \cdot 7$	57	1	11	15	2	8	$2^2 \cdot 3 \cdot 5 \cdot 11$
8	2	3	10	1	1	$2^3 \cdot 3 \cdot 5$	58	1	11	17	1	2	$2^2 \cdot 11 \cdot 17$
9	3	6	15	1	2	$2^3 \cdot 3 \cdot 5$	59	2	5	19	2	4	$2^3 \cdot 5 \cdot 19$
10	3	5	6	2	4	$2^3 \cdot 3 \cdot 5$	60	1	10	19	2	4	$2^3 \cdot 5 \cdot 19$
11	1	3	11	1	1	$2^2 \cdot 3 \cdot 11$	61	1	13	15	2	8	$2^2 \cdot 3 \cdot 5 \cdot 13$
12	1	2	17	1	4	$2^3 \cdot 17$	62	3	5	13	2	8	$2^2 \cdot 3 \cdot 5 \cdot 13$
13	1	5	7	1	1	$2^2 \cdot 5 \cdot 7$	63	10	15	35	1	8	$2^3 \cdot 3 \cdot 5 \cdot 7$
14	1	3	13	1	2	$2^2 \cdot 3 \cdot 13$	64	3	6	35	2	8	$2^3 \cdot 3 \cdot 5 \cdot 7$
15	3	11	15	1	2	$3 \cdot 5 \cdot 11$	65	1	3	73	2	8	$2^2 \cdot 3 \cdot 73$
16	2	3	7	1	1	$2^3 \cdot 3 \cdot 7$	66	3	19	51	1	2	$3 \cdot 17 \cdot 19$
17	3	6	7	1	2	$2^3 \cdot 3 \cdot 7$	67	2	3	82	1	2	$2^3 \cdot 3 \cdot 41$
18	1	2	21	1	4	$2^3 \cdot 3 \cdot 7$	68	3	5	51	2	8	$2^2 \cdot 3 \cdot 5 \cdot 17$
19	2	6	7	2	4	$2^3 \cdot 3 \cdot 7$	69	2	3	43	1	2	$2^3 \cdot 3 \cdot 43$
20	1	6	7	2	4	$2^3 \cdot 3 \cdot 7$	70	1	7	37	1	2	$2^2 \cdot 7 \cdot 37$
21	3	6	21	1	4	$2^3 \cdot 3 \cdot 7$	71	1	3	91	2	8	$2^2 \cdot 3 \cdot 7 \cdot 13$
22	1	3	17	1	2	$2^2 \cdot 3 \cdot 17$	72	5	15	19	2	8	$2^2 \cdot 3 \cdot 5 \cdot 19$
23	1	5	11	1	2	$2^2 \cdot 5 \cdot 11$	73	1	7	43	1	2	$2^2 \cdot 7 \cdot 43$
24	1	3	19	1	1	$2^2 \cdot 3 \cdot 19$	74	1	17	19	2	8	$2^2 \cdot 17 \cdot 19$
25	3	6	11	1	1	$2^3 \cdot 3 \cdot 11$	75	3	10	11	2	8	$2^3 \cdot 3 \cdot 5 \cdot 11$
26	2	3	11	1	2	$2^3 \cdot 3 \cdot 11$	76	2	11	15	2	8	$2^3 \cdot 3 \cdot 5 \cdot 11$
27	2	6	11	2	4	$2^3 \cdot 3 \cdot 11$	77	7	15	91	2	8	$3 \cdot 5 \cdot 7 \cdot 13$
28	2	3	22	2	4	$2^3 \cdot 3 \cdot 11$	78	2	11	34	1	2	$2^3 \cdot 11 \cdot 17$
29	1	6	22	1	4	$2^3 \cdot 3 \cdot 11$	79	5	7	11	2	8	$2^2 \cdot 5 \cdot 7 \cdot 11$
30	3	7	39	1	2	$3 \cdot 7 \cdot 13$	80	1	11	35	2	8	$2^2 \cdot 5 \cdot 7 \cdot 11$
31	2	7	10	1	1	$2^3 \cdot 5 \cdot 7$	81	1	21	57	1	8	$2^2 \cdot 3 \cdot 7 \cdot 19$
32	2	5	7	2	4	$2^3 \cdot 5 \cdot 7$	82	2	7	58	1	2	$2^3 \cdot 7 \cdot 29$
33	1	7	10	2	4	$2^3 \cdot 5 \cdot 7$	83	2	11	19	1	2	$2^3 \cdot 11 \cdot 19$
34	1	2	35	2	4	$2^3 \cdot 5 \cdot 7$	84	2	7	33	4	32	$2^3 \cdot 3 \cdot 7 \cdot 11$
35	5	10	35	1	4	$2^3 \cdot 5 \cdot 7$	85	7	15	19	2	8	$3 \cdot 5 \cdot 7 \cdot 19$
36	1	7	11	1	2	$2^2 \cdot 7 \cdot 11$	86	3	19	35	2	8	$3 \cdot 5 \cdot 7 \cdot 19$
37	1	6	13	1	4	$2^3 \cdot 3 \cdot 13$	87	3	11	195	2	8	$3 \cdot 5 \cdot 11 \cdot 13$
38	2	3	13	2	4	$2^3 \cdot 3 \cdot 13$	88	6	7	78	2	8	$2^3 \cdot 3 \cdot 7 \cdot 13$
39	1	7	13	1	1	$2^2 \cdot 7 \cdot 13$	89	1	15	37	2	8	$2^2 \cdot 3 \cdot 5 \cdot 37$
40	7	11	35	1	2	$5 \cdot 7 \cdot 11$	90	3	5	37	2	8	$2^2 \cdot 3 \cdot 5 \cdot 37$
41	3	6	51	1	2	$2^3 \cdot 3 \cdot 17$	91	1	33	51	2	8	$2^2 \cdot 3 \cdot 11 \cdot 17$
42	2	3	34	1	2	$2^3 \cdot 3 \cdot 17$	92	5	11	13	2	8	$2^2 \cdot 5 \cdot 11 \cdot 13$
43	1	6	17	1	8	$2^3 \cdot 3 \cdot 17$	93	11	22	35	2	8	$2^3 \cdot 5 \cdot 7 \cdot 11$
44	1	2	51	4	8	$2^3 \cdot 3 \cdot 17$	94	3	51	195	2	16	$3 \cdot 5 \cdot 13 \cdot 17$
45	2	6	34	1	8	$2^3 \cdot 3 \cdot 17$	95	11	19	187	1	2	$11 \cdot 17 \cdot 19$
46	2	10	11	1	2	$2^3 \cdot 5 \cdot 11$	96	5	11	85	2	8	$2^2 \cdot 5 \cdot 11 \cdot 17$
47	5	10	11	2	4	$2^3 \cdot 5 \cdot 11$	97	7	19	70	2	8	$2^3 \cdot 5 \cdot 7 \cdot 19$
48	2	3	19	1	2	$2^3 \cdot 3 \cdot 19$	98	3	13	37	4	32	$2^2 \cdot 3 \cdot 13 \cdot 37$
49	2	6	19	2	4	$2^3 \cdot 3 \cdot 19$	99	5	19	85	4	32	$2^2 \cdot 5 \cdot 17 \cdot 19$
50	1	7	19	1	1	$2^2 \cdot 7 \cdot 19$							

Table 6

The fields $N_{16} = \mathbb{Q}(\sqrt{-m_1}, \sqrt{-m_2}, \sqrt{-m_3}, \sqrt{-m_4})$ with $\exp\left(\mathrm{Cl}(N_{16})\right) \leqslant 2$.

$Nr.$	m_1	m_2	m_3	m_4	$h_{N_{16}^+}$	$h_{N_{16}}$	$f_{N_{16}}$
1	1	2	3	5	1	2	$2^3 \cdot 3 \cdot 5$
2	1	3	5	7	1	4	$2^2 \cdot 3 \cdot 5 \cdot 7$
3	2	3	7	10	1	4	$2^3 \cdot 3 \cdot 5 \cdot 7$

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