BULL. AUSTRAL. MATH. SOC. 46E40, 46G20
VOL. 33 (1986), 227-236.

ON ADJOINTS OF NON-LINEAR MAPPINGS

DINAMÉRICO P. POMBO JR.

Amply boundedness of collections of analytic mappings is proved to be equivalent to equicontinuity of the corresponding collections of adjoints, for certain classes of locally convex spaces which have good analytic properties.

In the linear theory of locally convex spaces it is well known that a collection of continuous linear mappings defined on an infrabarrelled locally convex space is equicontinuous if and only if the collection of their adjoints is strongly equicontinuous ([9], p. 138). The main purpose of this article is to obtain certain analytic versions of this result (Theorem 2, Corollary 5), using the notion of adjoint of an analytic mapping introduced by Aron and Schottenlöher ([2]). Using an argument of Grothendieck, we also prove that, under a polynomial condition, spaces of continuous m-homogeneous polynomials with values in spaces of continuous mappings can be identified with spaces of continuous mappings with values in spaces of scalar valued continuous m-homogeneous polynomials (Corollary 2). As we have observed, some of the preliminary results which are necessary do not depend on analyticity. For this reason, we shall define the adjoint of an arbitrary mapping in the natural way and derive some simple properties in this general context.

We shall adopt the notation and terminology of [5], [7], and [9]. Given two complex vector spaces E and F, $L_a(E;F)$ shall denote the

Received 7 June 1985. Research supported by CNPq (Brazil), when the author was visiting Kent State University, U.S.A.

complex vector space of all linear mappings from E into F. Given a Hausdorff topological space X and a Hausdorff locally convex space F over $\mathbb{C}$, $\mathcal{F}(X;F)$ shall denote the complex vector space of all mappings from X into F and $C(X;F)$ the vector subspace of all continuous mappings from X into F. Given a Hausdorff locally convex space E over $\mathbb{C}$, $P({}^mE;F)$ shall denote the complex vector space of all continuous m-homogeneous polynomials from E into F. When $m = 1$ we write $P({}^mE;F) = L(E;F)$. Given a non-void open subset U of E, $H(U;F)$ shall denote the complex vector space of all analytic mappings from U into F. When $F = \mathbb{C}$, it is not included in the notation for function spaces; thus $\mathcal{F}(X)$ stands for $\mathcal{F}(X;\mathbb{C})$, etc... Let $CS(F)$ denote the set of all continuous seminorms on F and let $x_o \in X$. A collection $\mathcal{X} \subset \mathcal{F}(X;F)$ is said to be amply bounded at $x_o \in X$ if, for each $\beta \in CS(F)$, there exists a neighbourhood V of x_o in X such that

$$\sup_{x \in V, f \in \mathcal{X}} \beta(f(x)) < +\infty .$$

$\mathcal{X}$ is said to be amply bounded if $\mathcal{X}$ is amply bounded at every point of X. If $\mathcal{X}$ is a collection of m-homogeneous polynomials from E into F, then $\mathcal{X}$ is equicontinuous if and only if $\mathcal{X}$ is amply bounded ([4], Theorem 1).

DEFINITION. If $f \in \mathcal{F}(X;F)$, its adjoint ${}^tf$ is the linear mapping from F' into $\mathcal{F}(X)$ defined by

$${}^tf(\phi) = \phi \circ f , \quad \text{for all} \quad \phi \in F' .$$

The mapping $f \in \mathcal{F}(X;F) \mapsto {}^tf \in L_a(F';\mathcal{F}(X))$ is obviously linear and if $f \in C(X;F)$ (respectively $H(U;F)$, $P({}^mE;F)$) then ${}^tf \in L_a(F';C(X))$ (respectively $L_a(F';H(U))$, $L_a(F';P({}^mE))$).

PROPOSITION 1. *Let* θ *be a covering of* F *by bounded subsets. If* $f \in \mathcal{F}(X;F)$, *then* ${}^tf \in L(F'_\theta;\mathcal{F}_s(X))$, *where* F'_θ *denotes* F' *with the locally convex topology of* θ*-convergence and* $\mathcal{F}_s(X)$ *denotes* $\mathcal{F}(X)$ *with the locally convex topology of pointwise convergence* ([5], §1).

https://doi.org/10.1017/S0004972700003099 Published online by Cambridge University Press

Proof. Let us denote by F'_σ the vector space F' endowed with the weak topology $\sigma(F',F)$. Since θ is a covering of F, the identity mapping $F'_\theta \to F'_\sigma$ is obviously continuous. Hence it suffices to prove that ${}^t f \in L(F'_\sigma; \mathcal{F}_s(X))$. Let $x_1,\dots,x_m \in X$, $\varepsilon > 0$, and consider $W = \{g \in \mathcal{F}(X) ; |g(x_1)| \leqslant \varepsilon, \dots, |g(x_m)| \leqslant \varepsilon\}$, a basic neighbourhood of zero in $\mathcal{F}_s(X)$. Let $y_j = f(x_j)$, $j = 1,\dots,m$, and consider $V = \{\phi \in F' ; |\phi(y_1)| \leqslant \varepsilon, \dots, |\phi(y_m)| \leqslant \varepsilon\}$; V is a neighbourhood of zero in F'_σ and ${}^t f(V) \subset W$, proving the desired continuity of ${}^t f$.

PROPOSITION 2. $\mathfrak{X} \subset \mathcal{F}(X;F)$ *is equicontinuous (respectively amply bounded) if and only if for each equicontinuous subset* $\mathcal{Y}$ *of* F', *the collection* ${}^t\mathfrak{X}(\mathcal{Y}) = \{\phi \circ f ; \phi \in \mathcal{Y}, f \in \mathfrak{X}\}$ *is equicontinuous (respectively locally bounded).*

Proof. We shall prove the amply bounded case, the other one being analogous. Suppose $\mathfrak{X}$ amply bounded. Let $x_o \in X$ and take $\mathcal{Y} \subset F'$ equicontinuous. By equicontinuity, the mapping $\beta(y) = \sup_{\phi\in\mathcal{Y}} |\phi(y)|$ $(y \in F)$ is a continuous seminorm on F. Hence there exists V a neighbourhood of x_o in X so that

$$\sup_{f\in\mathfrak{X}, x\in V} \beta(f(x)) < +\infty .$$

Consequently,

$$\sup_{x\in V, \phi\in\mathcal{Y}, f\in\mathfrak{X}} |(\phi\circ f)(x)| < +\infty .$$

and ${}^t\mathfrak{X}(\mathcal{Y})$ is locally bounded at x_o. Conversely, let us fix $x_o \in X$, $\beta \in CS(F)$, and define $\mathcal{Y} = \{\phi \in F' ; |\phi| \leqslant \beta\}$. By the continuity of β, $\mathcal{Y}$ is equicontinuous. Hence, by hypothesis, there exists a neighbourhood V of x_o in X such that

$$\sup_{x\in V, \phi\in\mathcal{Y}, f\in\mathfrak{X}} |(\phi\circ f)(x)| < +\infty .$$

On the other hand, the Hahn-Banach theorem gives $\beta(y) = \sup_{\phi\in\mathcal{Y}} |\phi(y)|$, for each $y \in F$. Thus $\sup_{x \in V, f \in X} \beta(f(x)) < +\infty$, and X is amply bounded at x_o .

We now apply an idea of Grothendieck to the non-linear case ([8], p. 167). Let T be a Hausdorff locally compact topological space and consider $C(T)$ endowed with the compact-open topology. We have the following

THEOREM 1. *$C(X;C(T))$ is algebraically isomorphic to the vector subspace H of $C(T;C_s(X))$ of all $h \in C(T;C_s(X))$ which map compact subsets of T into equicontinuous subsets of $C(X)$, where $C_s(X)$ denotes $C(X)$ with the topology of pointwise convergence.*

Proof. For each $t \in T$, let $\psi(t) \in (C(T))'$ be defined by $\psi(t)(f) = f(t)$, if $f \in C(T)$. The mapping $\psi : t \in T \mapsto \psi(t) \in (C(T))'$ is continuous, if $(C(T))'$ is endowed with the weak topology $\sigma((C(T))',C(T))$. Moreover, the topology of $C(T)$ is the topology of uniform convergence on the equicontinuous sets $\psi(K)$ of $(C(T))'$, where K varies in the collection of compact subsets of T (see [8], p. 167, for the details). Now, for each $g \in C(X;C(T))$, define $w(g) = {}^t g \circ \psi$. By Proposition 1, $w(g) \in C(T;C_s(X))$, and, by Proposition 2, $w(g)$ maps compact subsets of T into equicontinuous subsets of $C(X)$ (thus $w(g)$ maps compact subsets of T into compact subsets of $C(X)$ for the compact open topology, by Ascoli's Theorem). The mapping

$$w : g \in C(X;C(T)) \mapsto w(g) \in C(T;C_s(X))$$

is obviously linear and injective, and we have just observed that $w(C(X;C(T))) \subset H$.

Let $h \in H$ and define $g : X \to F(T)$ by $g(x)(t) = h(t)(x)$, for $x \in X$, $t \in T$. We must prove that $g(x) \in C(T)$ and $g \in C(X;C(T))$. In fact, if $x \in X$, the evaluation mapping $\delta_x : C(X) \to C$ is continuous for the simple topology and $g(x) = \delta_x \circ h$. By composition, $g(x) \in C(T)$. Now, fix $x_o \in X$, $K \subset T$ compact and $\varepsilon > 0$. By the equicontinuity of $h(K)$ at x_o , we can find a neighbourhood V of x_o in X such that the relations $x \in V$, $t \in K$ imply

https://doi.org/10.1017/S0004972700003099 Published online by Cambridge University Press

$$|h(t)(x) - h(t)(x_o)| \leqslant \varepsilon .$$

Thus,

$$\sup_{t\in K} |g(x)(t) - g(x_o)(t)| \leqslant \varepsilon ,$$

for all $x \in V$, proving the continuity of g at x_o. Finally, by definition, $\omega(g) = h$, and the theorem is proved.

Let θ be a collection of bounded subsets of the Hausdorff locally convex space E and denote by $\mathcal{B}$ the vector space of all mappings from T into $P(^mE)$ which map compact subsets of T into bounded subsets of $P(^mE)$ for the topology of θ-convergence on $P(^mE)$. Consider on $\mathcal{B}$ the locally convex topology defined by the family of seminorms

$$p_{K,B} : h \in \mathcal{B} \mapsto p_{K,B}(h) = \sup_{t\in K} (\sup_{x\in B} |h(t)(x)|) \in \mathbb{R}_+ ,$$

where K varies in the collection of all compact subsets of T and B varies in θ. We obtain the following

COROLLARY 1. (a) *The restriction of* ω *to* $P(^mE;C(T))$ *establishes a vector space isomorphism between* $P(^mE;C(T))$ *and*

$H^{(m)} = \{h \in C(T;P_s(^mE))$; h *maps compact subsets of* T *into equicontinuous subsets of* $P(^mE)\}$.

(b) $H^{(m)}$ *is a vector subspace of* $\mathcal{B}$, *and* $P_\theta(^mE;C(T))$ $(P(^mE;C(T))$ *with the topology of* θ*-convergence) is isomorphic to* $H^{(m)}$, *if* $H^{(m)}$ *is endowed with the locally convex topology induced by* $\mathcal{B}$.

Proof. (a) Obviously, $\omega(P(^mE;C(T))) \subset H^{(m)}$. Conversely, if $h \in H^{(m)}$, we can apply Theorem 1 to obtain $g \in C(E;C(T))$ so that $\omega(g) = h$. Thus, it remains to verify that g is an m-homogeneous polynomial from E into $C(T)$. In fact, for each $t \in T$, there exists a unique continuous symmetric m-linear mapping from E^m into $\mathbb{C}$, A_t , such that $A_t(x,\ldots,x) = h(t)(x)$, if $x \in E$. Define $A : E^m \to F(T)$ by $A(x_1,\ldots,x_m)(t) = A_t(x_1,\ldots,x_m)$, for $x_1,\ldots,x_m \in E$, $t \in T$. Arguing as in the proof of Theorem 1 and using [11], Proposition 2.6, we verify that $A(x_1,\ldots,x_m) \in C(T)$. Finally, it is easy to see that A is a symmetric m-linear mapping from E^m into $C(T)$ such that $A(x,\ldots,x) = h(x)$, if $x \in E$, proving (a).

(b) Let $P \in P(^mE;C(T))$, $K \subset T$ compact and $B \in \theta$. By the continuity of P , we have

$$\sup_{x\in B}\,(\sup_{t\in K}|P(x)(t)|) < +\infty .$$

But,

$$\sup_{x\in B}\,(\sup_{t\in K}|P(x)(t)|) = \sup_{x\in B}\,(\sup_{t\in K}|\omega(P)(t)(x)|) = \sup_{t\in K}\,(\sup_{x\in B}|\omega(P)(t)(x)|) .$$

Thus, $H^{(m)} \subset B$, and the locally convex space isomorphism is also established.

COROLLARY 2. *If E is a polynomially barrelled locally convex space ([1], Definition 5), then $H^{(m)} = C(T;P_s(^mE))$. Consequently, $P_\theta(^mE;C(T))$ and $C(T;P_s(^mE))$ (with the topology indicated as before) are isomorphic as locally convex spaces.*

Proof. Let $h \in C(T;P_s(^mE))$. For each $K \subset T$ compact, $h(K)$ is a compact, hence bounded subset of $P(^mE)$ for the simple topology. By [11], Proposition 3.23, $h(K)$ is an equicontinuous subset of $P(^mE)$. Thus, $h \in H^{(m)}$, and the proof is complete.

We could have considered the Banach space $C_o(T)$ of all continuous complex valued mappings on T which vanish at infinity endowed with the supremum norm, instead of $C(T)$. By similar arguments we get:

THEOREM 1'. *$C(X;C_o(T))$ is algebraically isomorphic to the vector subspace of $C_o(T;C_s(X))$ of all such continuous mappings which map T into an equicontinuous subset of $C(X)$.*

Obviously, we would have the analogues of Corollaries 1 and 2 corresponding to this case.

Now, let U be a non-void open subset of the Hausdorff locally convex space E , and denote by τ_o (respectively τ_{of}) the topology on $C(U;F)$ of uniform convergence on the compact (respectively finite

https://doi.org/10.1017/S0004972700003099 Published online by Cambridge University Press

dimensional compact) subsets of U. Obviously, $\tau_{of} \leqslant \tau_o$, and it can be proven that they coincide in the polynomial case ([11], Proposition 2.7).

PROPOSITION 3. *$X \subset C(U;F)$ is τ_o-bounded (respectively, τ_{of}-bounded) if and only if ${}^tX(\phi)$ is τ_o-bounded (respectively, τ_{of}-bounded) in $C(U)$, for all $\phi \in F'$.*

Proof. If $\phi \in F'$, then $\beta = |\phi| \in CS(F)$. Thus X τ_o-bounded (respectively, τ_{of}-bounded) in $C(U;F)$ implies immediately ${}^tX(\phi)$ τ_o-bounded (respectively, τ_{of}-bounded) in $C(U)$. Conversely, if K is a compact (respectively, finite dimensional compact) subset of U, our hypothesis ensures that $X(K)$ is $\sigma(F,F')$-bounded in F, hence a bounded subset of F for its original topology. Thus X is τ_o-bounded (respectively, τ_{of}-bounded) in $C(U;F)$.

PROPOSITION 4. *Let us denote by F'_b the vector space F' endowed with the strong topology and consider $X \subset C(U;F)$. Then, ${}^tX$ is equicontinuous from F'_b into $(C(U),\tau_o)$ (respectively, $(C(U),\tau_{of})$) if and only if for each compact (respectively, finite dimensional compact) subset K of U there exists a bounded subset B of F such that $X(K) \subset B$.*

Proof. We shall prove the τ_o-case, the τ_{of}-case being similar to the first one. Let us suppose ${}^tX$ equicontinuous from F'_b into $(C(U),\tau_o)$. Given a compact subset K of U, we can find a bounded subset B_1 of E and $\lambda > 0$ such that the relations $\phi \in F'$, $\sup_{y \in B_1} |\phi(y)| \leqslant \lambda$ imply

$$(\Delta) \qquad \sup_{x \in K} |\phi(f(x))| \leqslant 1 \text{, for all } f \in X .$$

Let $B = \frac{1}{\lambda}$ (closed balanced convex hull of B_1); B is a bounded subset of E and $B^o \subset (X(K))^o$, by (Δ). Hence, $X(K) \subset (X(K))^{oo} \subset B^{oo} = B$, the last equality being a consequence of the Bipolar Theorem (where we are

https://doi.org/10.1017/S0004972700003099 Published online by Cambridge University Press

taking polars with respect to the duality $(F,F'))$. Conversely, if K is a compact subset of U , and $\varepsilon > 0$, consider $W = \{h \in C(U)$; $\sup_{x\in K} |h(x)| \leqslant \varepsilon\}$ a basic neighbourhood of zero in $(C(U),\tau_o)$. Take B a bounded subset of F so that $X(K) \subset B$. Then, $V = \{\phi \in F'$; $\sup_{y\in B} |\phi(y)| \leqslant \varepsilon\}$ is a neighbourhood of zero in F'_b such that ${}^tX(V) \subset W$, proving the equicontinuity of ${}^tX$.

COROLLARY 3. *${}^tX$ is equicontinuous from F'_b into $(C(U),\tau_o)$ (respectively, $(C(U),\tau_{of})$) if and only if X is τ_o-bounded (respectively, τ_{of}-bounded) in $C(U;F)$.*

COROLLARY 4. *If ${}^tX$ is pointwise bounded, $C(U)$ being endowed with τ_o (respectively, τ_{of}) , then ${}^tX$ is equicontinuous from F'_b into $(C(U),\tau_o)$ (respectively, $(C(U),\tau_{of})$) .*

Proof. Follows immediately from Proposition 3 and Corollary 3.

We can now state our main result

THEOREM 2. *Let E be a holomorphically infrabarrelled locally convex space ([10], Definition 9). For any collection $X \subset H(U;F)$, the following conditions are equivalent:*

(i) X is τ_o-bounded in $H(U;F)$;

(ii) X is amply bounded ;

(iii) ${}^tX$ is pointwise bounded, if $H(U)$ is endowed with τ_o ;

(iv) ${}^tX$ is equicontinuous from F'_b into $(H(U),\tau_o)$.

Proof. (i) $\Rightarrow$ (ii) since E is holomorphically infrabarrelled. (ii) $\Rightarrow$ (iii): by Proposition 2, ${}^tX(\phi)$ is locally bounded, for all $\phi \in F'$. Thus, ${}^tX(\phi)$ is τ_o-bounded in $H(U)$, by a straightforward compacity argument.

(iii) $\Rightarrow$ (iv): immediate consequence of Corollary 4.

(iv) $\Rightarrow$ (i): immediate consequence of Corollary 3.

COROLLARY 5. *Let E be a holomorphically barrelled locally convex space ([10], Definition 6). For any collection $X \subset H(U;F)$, the* *following conditions are equivalent:*

(i) X *is* τ_o*-bounded in* $H(U;F)$;

(ii) X *is amply bounded* ;

(iii) ${}^tX$ *is pointwise bounded, if* $H(U)$ *is endowed with* τ_o ;

(iv) ${}^tX$ *is equicontinuous from* F'_b *into* $(H(U),\tau_o)$;

(v) X *is* τ_{of}*-bounded in* $H(U;F)$;

(vi) ${}^tX$ *is pointwise bounded, if* $H(U)$ *is endowed with* τ_{of} .

Proof. Since every holomorphically barrelled space is holomorphically infrabarrelled, then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv), and, obviously, (i) $\Rightarrow$ (v). As in the proof of Theorem 2, (ii) $\Rightarrow$ (vi) by Proposition 2.

(vi) $\Rightarrow$ (iii): if ${}^tX(\phi)$ is τ_{of}-bounded in $H(U)$, then ${}^tX(\phi)$ is locally bounded, since E is holomorphically barrelled. Hence ${}^tX(\phi)$ is τ_o-bounded in $H(U)$, proving (iii).

(v) $\Rightarrow$ (ii) since E is holomorphically barrelled.

We end by furnishing examples of certain types of locally convex spaces which were considered in the text.

EXAMPLES. Every metrizable space is holomorphically infrabarrelled ([3], Propositions 6 and 54) and every DFM space (strong dual of a Fréchet-Montel space) is holomorphically infrabarrelled ([6], Proposition 6 and [3], Proposition 52); every Baire space and every Silva space is holomorphically barrelled, hence polynomially barrelled ([3], Propositions 37 and 41); every barrelled DF space is polynomially barrelled ([1], p. 41, Remark (b)).

References

[1] J. Aragona, "Holomorphically significant properties of spaces of holomorphic germs", *Advances in Holomorphy* (North-Holland Math. Studies 34 (1979), 31-46).

[2] R. Aron and M. Schottenlöher, "Compact holomorphic mappings on Banach spaces and the approximation property", *J. Funct. Anal.* 21 (1976), 7-30.

https://doi.org/10.1017/S0004972700003099 Published online by Cambridge University Press

[3] J.A. Barroso, M.C. Matos and L. Nachbin, "On holomorphy versus linearity in classifying locally convex spaces", *Infinite Dimensional Holomorphy and Applications* (North-Holland Math. Studies 12 (1977), 31-74)

[4] J. Bochnak and J. Siciak, "Polynomials and multilinear mappings in topological vector spaces", *Studia Math.* 39 (1971), 59-76.

[5] N. Bourbaki, *Topologie générale*, Chap. 10, (Hermann, (1967).

[6] S. Dineen, "Holomorphic functions on strong duals of Fréchet-Montel spaces", *Infinite Dimensional Holomorphy and Applications* (North-Holland Math. Studies 12 (1977), 147-166).

[7] S. Dineen, *"Complex Analysis in Locally Convex Spaces"*, (North-Holland Math. Studies 57 (1981)).

[8] A. Grothendieck, *Espaces vectoriels topologiques*, (Publicação da Soc. Mat. de São Paulo, 3ª edição, 1966).

[9] G. Köthe, *Topological vector spaces II*, (Springer-Verlag, 1979).

[10] L. Nachbin, "Some holomorphically significant properties of locally convex spaces", in D. de Figueiredo (ed.) *Functional Analysis* (Marcel Dekker, 1976, 251-277).

[11] D.P. Pombo, Jr., "On polynomial classification of locally convex spaces", *Studia Math.* 78 (1984), 39-57.

Instituto de Matemática,
Universidade Federal do Rio de Janeiro,
Caixa Postal 68530,
21944 - Rio de Janeiro, RJ - Brasil,

https://doi.org/10.1017/S0004972700003099 Published online by Cambridge University Press