ON STATIONARY PHASE INTEGRALS

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Let $f(x)$ and $g(x)$ be real functions defined on the interval $[a, b]$, with $f(x)$ at least twice continuously differentiable, $f'(x)$ monotone increasing, and $g(x)$ of bounded variation. We consider the exponential integral

$$I = \int_a^b g(x)e(f(x))\,dx, \tag{1}$$

where $e(t)$ denotes $\exp 2\pi i t$. The purpose of this note is to prove sharp forms of the well-known estimates:

A: If $f'(x)$ is nonzero on $[a, b]$, then I has order of magnitude

$$I = O(1/\min |f'(x)|). \tag{2}$$

The constant of proportionality depends on the function $g(x)$.

B: If $f'(x)$ changes sign at $x = c$ with $a < c < b$, then

$$I \sim \frac{g(c)e(f(c) + 1/8)}{\sqrt{f''(c)}}. \tag{3}$$

Bombieri and Iwaniec [**1**] remark that the methods of Hormänder [**5**] give an asymptotic expansion of the integral I when $f(x)$ and $g(x)$ are real-analytic. Such expansions were considered in great detail by van der Corput [**2**]. Our results correspond to the leading terms of the asymptotic series at the points a, b and c, to an accuracy corresponding to the expected order of magnitude of the second term in the asymptotic series. We do not need to assume the existence of high derivatives. Our estimates could have been obtained by Stokes and Kelvin when they first considered the integral (1), but they appear to be unknown to the highest authorities [**2, 3, 4, 6, 7, 8, 9, 10**]. Sharp estimates are useful in arguments involving sums of Fourier integrals or two-dimensional Fourier integrals, such as occur in the Poisson summation formula and its generalisations (see [**3**] and [**7**]). The spur for this work was the still more delicate summation formula of Voronoi, Hardy and Wilton.

Our estimates are in terms of size parameters T and U, and length parameters M and N, with $M \geq b - a$, for which certain derivatives of $f(x)$ and $g(x)$ satisfy

$$|f^{(r)}(x)| \leq B^r T/M^r, \qquad |g^{(s)}(x)| \leq B^s U/N^s \tag{4}$$

for some constant $B \geq 1$. For assertion (B) we also require that

$$f''(x) \geq T/B^2M^2. \tag{5}$$

This notation corresponds to the common situation of the Poisson summation formula [**7**] applied to

$$f(x) = TF(x/M), \qquad g(x) = UG(x/M)w(x),$$

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where $F(x)$ and $G(x)$ are bounded functions, and $w(x)$ is the smoothed characteristic function of an interval $[a_0, b_0]$ with

$$w(x)=0 \quad \text{for} \quad x \le a_0 - N,\ x \ge b_0 + N,$$

$$w(x)=1 \quad \text{for} \quad a_0 + N \le x \le b_0 - N.$$

Our bounds are additive on intervals, in the sense that when we dissect the interval $[a, b]$, then the leading terms of the asymptotic expansions at the internal points of dissection cancel one another. This is very useful if the orders of magnitude of the derivatives change on the range of integration. Improper integrals from 0 to ∞ can usually be treated by dissecting into subintervals of the form $[a, 2a]$.

The fundamental estimate for integrals of the form (1) is the rth derivative test

$$I = O\Big(\Big(\operatorname*{Var}_{[a,b]} g(x)\Big)\Big/\Big(\min_{[a,b]} |f^{(r)}(x)|^{1/r}\Big)\Big),$$

where Var is the functional defined by the total variation on an interval plus the modulus at either endpoint. We use this test with $r = 1$ or 2 to estimate residual terms.

THEOREM 1. *Let $f(x)$ be a real function, three times continuously differentiable, and let $g(x)$ be a real function, twice continuously differentiable on the interval $[a, b]$, with the estimate* (4) *holding for $r = 2$ and* 3, *and $s = 0, 1$ and* 2. *Suppose that $f'(x)$ and $f''(x)$ do not vanish on the interval $[a, b]$. Then we have*

$$I = \int_a^b g(x)e(f(x))\,dx = \frac{g(b)e(f(b))}{2\pi i f'(b)} - \frac{g(a)e(f(a))}{2\pi i f'(a)}$$

$$+ O\left(\frac{B^3TU}{M^2}\left(1 + \frac{M}{N} + \frac{M^2}{N^2}\frac{\min|f'(x)|}{T/M}\right)\frac{1}{\min|f'(x)|^3}\right).$$

THEOREM 2. *Let $f(x)$ be a real function, four times continuously differentiable, and let $g(x)$ be a real function, three times continuously differentiable on the interval $[a, b]$, satisfying the estimates* (5) *and* (4) *for $r = 2, 3$ and* 4 *and $s = 0, 1$ and* 2. *Suppose that $f'(x)$ changes sign from negative to positive at $x = c$ with $a < c < b$. If T is sufficiently large in terms of B, then we have*

$$\int_a^b g(x)e(f(x))\,dx = \frac{g(c)e(f(c)+1/8)}{\sqrt{f''(c)}} + \frac{g(b)e(f(b))}{2\pi i f'(b)} - \frac{g(a)e(f(a))}{2\pi i f'(a)}$$

$$+ O\left(\frac{B^4M^4U}{T^2}\left(\frac{1}{(c-a)^3} + \frac{1}{(b-c)^3}\right)\right) + O\left(\frac{B^{13}MU}{T^{3/2}}\left(1 + \frac{M}{B^4N}\right)^2\right).$$

Proof of Theorem 1. We integrate by parts to get

$$\int_a^b g(x)e(f(x))\,dx = \left[\frac{g(x)e(f(x))}{2\pi i f'(x)}\right]_a^b - \int_a^b \frac{e(f(x))}{2\pi i}\frac{d}{dx}\frac{g(x)}{f'(x)}\,dx$$

$$= \left[\frac{g(x)e(f(x))}{2\pi i f'(x)}\right]_a^b + \left[\frac{(f''(x)g(x)-f'(x)g'(x))e(f(x))}{(2\pi i)^2 f'^3(x)}\right]_a^b$$

$$- \int_a^b \frac{e(f(x))}{(2\pi i)^2}\frac{d}{dx}\frac{(f''(x)g(x)-f'(x)g'(x))}{f'^3(x)}\,dx,$$

and estimate the integrals by the first derivative test. We have

$$\max\left|\frac{d}{dx}\frac{g(x)}{f'(x)}\right| = \max\left|\frac{g'}{f'}-\frac{gf''}{f'^2}\right| \le \frac{BU}{N}\max\frac{1}{|f'|}+\frac{B^2TU}{M^2}\max\frac{1}{f'^2}.$$

The total variation is

$$\int_a^b \left|\frac{d^2}{dx^2}\frac{g}{f'}\right|dx = \int_a^b \left|\frac{g''}{f''}-\frac{2f''g'}{f'^2}-\frac{f^{(3)}g}{f'^2}+\frac{2f''^2g}{f'^3}\right|dx$$

$$\le \max|g''|\int_a^b\frac{1}{|f'|}\,dx + \max|f^{(3)}g|\int_a^b\frac{1}{f'^2}\,dx + \max\,(|f'g'|+|f''g|)\int_a^b\frac{2f''}{|f'|^3}\,dx$$

$$\le \frac{B^2MU}{N^2}\max\frac{1}{|f'|}+\left(\frac{B^3TU}{M^2}+\frac{B^2TU}{M^2}+\frac{B^2TU}{MN}\right)\max\frac{1}{|f'|^2}.$$

Hence

$$\operatorname*{Var}_{[a,b]}\frac{d}{dx}\frac{g}{f'} = O\left(\frac{B^3TU}{M^2}\frac{1}{\min f'^2}\left(1+\frac{M}{N}+\frac{M^2}{N^2}\frac{\min|f'|}{T/M}\right)\right),$$

and the result follows by the first derivative test. □

LEMMA 1. *Let* λ *and* m *be positive real numbers. Then*

$$\int_{-m}^{m} e(\lambda y^2)\,dy = \frac{1+i}{2\sqrt{\lambda}} - \frac{J}{2\pi i\lambda} + \frac{e(\lambda m^2)}{2\pi i\lambda m},$$

where

$$J = \int_m^\infty \frac{e(\lambda y^2)}{y^2}\,dy.$$

Proof. The complex error integral

$$\operatorname*{Limit}_{m\to\infty}\int_{-m}^{m} e(\lambda y^2))\,dy = \frac{1+i}{2\sqrt{\lambda}}$$

is well known. We have

$$\int_m^\infty e(\lambda y^2))\,dy = \left[\frac{e(\lambda y^2)}{4\pi i\lambda y}\right]_m^\infty + \frac{1}{4\pi i\lambda}\int_m^\infty \frac{e(\lambda y^2)}{y^2}\,dy$$
$$= -\frac{e(\lambda m^2)}{4\pi i\lambda m} + \frac{J}{4\pi i\lambda}.$$

The integral from $-\infty$ to $-m$ takes the same value. □

LEMMA 2. *Let $G(y)$ be a real function satisfying*

$$|G^{(s)}(y)| \le A\,|y|^{2-s}$$

for $s = 0, 1, 2$ on the range $a \le y < b$, where $a < 0 < b$. Then for $\lambda > 0$ we have

$$\int_a^b G(y)e(\lambda y^2))\,dy = \frac{G(b)e(\lambda b^2)}{4\pi i\lambda b} - \frac{G(a)e(\lambda a^2)}{4\pi i\lambda a} + O\left(\frac{A}{\lambda^{3/2}}\right).$$

Proof. We divide the range of integration. Choose c by

$$c = 1/\sqrt{\lambda}.$$

Then by the second derivative test

$$\int_{\max(a,-c)}^{\min(b,c)} G(y)e(\lambda y^2)\,dy = O\left(\frac{Ac^2}{\sqrt{\lambda}}\right) = O\left(\frac{A}{\lambda^{3/2}}\right).$$

Next we have

$$\int_n^{\min(b,2n)} G(y)e(\lambda y^2)\,dy = \left[\frac{G(b)e(\lambda b^2)}{4\pi i\lambda y}\right]_n^{\min(b,2n)} - \int_n^{\min(b,2n)} \frac{1}{4\pi i\lambda}\left(\frac{G'(y)}{y} - \frac{G(y)}{y^2}\right)e(\lambda y^2)\,dy.$$

The total variation of $G'(y)/y - G(y)/y^2$ for $n \le y \le 2n$ is $O(A)$, so by the first derivative test we have

$$\int_n^{\min(b,2n)} \frac{1}{4\pi i\lambda}\left(\frac{G'(y)}{y} - \frac{G(y)}{y^2}\right)e(\lambda y^2)\,dy = O\left(\frac{A}{\lambda^2 n}\right).$$

Summing n through numbers of the form $2^r c$, with $r = 0, 1, 2, \ldots$, we get a contribution

$$O(A/\lambda^2 c) = O(A/\lambda^{3/2}).$$

The integrated terms cancel except at the top and bottom of the range, and

$$\frac{G(c)e(\lambda c^2)}{4\pi i\lambda c} = O\left(\frac{A}{\lambda^{3/2}}\right)$$

once more. We treat the range $a \le y \le -c$ similarly. □

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LEMMA 3. *Let $f(x)$ be a real function, four times continuously differentiable, and let $g(x)$ be a real function twice continuously differentiable on the interval $[a,b]$, satisfying the estimates* (5) *and* (4) *for $r=2$, 3 and 4 and $s=0$, 1 and 2. Suppose that $f'(x)$ changes sign from negative to positive at a point $x=c$ with $a<c<b$. Let $\lambda=f''(c)/2$. There is an absolute constant $\delta>0$, such that, if $a\le a_1<c$ and $c<b_1\le b$ with*

$$f(a_1)=f(b_1)=f(c)+\lambda m^2,$$

and

$$0<m\le\delta M/B^5, \tag{6}$$

then we have

$$\int_{a_1}^{b_1} g(x)e(f(x))\,dx=\frac{g(c)e(f(c)+1/8)}{\sqrt{f''(c)}}+\frac{g(b_1)e(f(b_1))}{2\pi i f'(b_1)}$$
$$-\frac{g(a_1)e(f(a_1))}{2\pi i f'(a_1)}+O\left(\frac{B^{13}MU}{T^{3/2}}\left(1+\frac{M}{B^4N}\right)^2\right)+O\left(\frac{B^4M^4U}{m^3T^2}\right).$$

Proof. We work with the Taylor series

$$f(x)=f(c)+\lambda(x-c)^2+\lambda_3(x-c)^3+O\left(\frac{B^4T\,|x-c|^4}{M^4}\right),$$

and

$$g(x)=\mu+\mu_1(x-c)+O\left(\frac{B^2U\,|x-c|^2}{N^2}\right),$$

and the corresponding expansions of $f'(x)$ and $f''(x)$ as far as the term in λ_3, and of $g'(x)$ as far as the term in μ_1.

We define a new variable y by

$$\lambda y^2=f(x)-f(c),$$

where y has the sign of $x-c$. Hence

$$\frac{dx}{dy}=\frac{2\lambda y}{f'(x)},$$

and

$$y^2=(x-c)^2\left(1+\frac{\lambda_3}{\lambda}(x-c)+O\left(B^6\left|\frac{x-c}{M}\right|^2\right)\right). \tag{7}$$

Thus

$$y=(x-c)\left(1+\frac{\lambda_3}{\lambda}(x-c)+O\left(B^6\left|\frac{x-c}{M}\right|^2\right)\right)^{1/2}$$
$$=(x-c)+\frac{\lambda_3}{2\lambda}(x-c)^2+O\left(\frac{B^{10}\,|x-c|^3}{M^2}\right). \tag{8}$$

We must invert (8) to express $x-c$ in terms of y. We suppose that δ in (6) is so small that (8) implies

$$|x-c|/2\le|y|\le 3\,|x-c|/2.$$

Now (8) gives $x-c$ in terms of y, $(x-c)^2$ and higher powers of $x-c$, and (7) gives $(x-c)^2$ in terms of y^2 and higher powers of $x-c$. Substitution gives

$$(x-c)=y-\frac{\lambda_3}{2\lambda}y^2+O\left(\frac{B^{10}|y|^3}{M^2}\right).$$

We can now substitute in the Taylor series for $f(x)$, $f'(x)$ and $f''(x)$ with remainder term involving $f^{(4)}(x)$, and the Taylor series for $g(x)$ and $g'(x)$ with remainder term involving $g''(x)$, to obtain the expansions

$$\frac{2\lambda y g(x)}{f'(x)}=\mu+\kappa y+G(y), \tag{9}$$

for some coefficient κ which we do not need to estimate, with

$$G(y)=O\left(\frac{y^2U}{M^2}\left(B^5+\frac{BM}{N}\right)^2\right), \tag{10}$$

and

$$\frac{d}{dy}\frac{2\lambda y g(x)}{f'(x)}=\kappa+O\left(\frac{|y|\,U}{M^2}\left(B^5+\frac{BM}{N}\right)^2\right), \tag{11}$$

$$\frac{d^2}{dy^2}\frac{2\lambda y g(x)}{f'(x)}=O\left(\frac{U}{M^2}\left(B^5+\frac{BM}{N}\right)^2\right). \tag{12}$$

We have to calculate the expansions independently, since we cannot differentiate under an order of magnitude sign in (10). An alternative method is to expand y and its derivatives as power series in $x-c$, write the left hand side of (12) as

$$\frac{d^2}{dy^2}g(x)\frac{dx}{dy},$$

and substitute the identities

$$\frac{d^2x}{dy^2}=-\frac{y''}{(y')^3},\qquad \frac{d^3x}{dy^3}=\frac{3(y'')^2-y'y^{(3)}}{(y')^5}$$

to obtain the estimate (12), from which (10) and (11) follow by integration. Some process of series expansion seems to be necessary in evaluating the integral (1); our aim is to simplify the argument by separating the expansion step from the integration.

After all this, then we can write

$$\begin{aligned}\int_{a_1}^{b_1} g(x)e(f(x))\,dx &= \int_{y=-m}^{m} g(x)e(f(x))\frac{dx}{dy}\,dy\\ &= \int_{-m}^{m}\frac{2\lambda y g(y)}{f'(x)}e(f(c)+\lambda y^2)\,dy\\ &= \int_{-m}^{m}(\mu+\kappa y+G(y))e(f(c)+\lambda y^2)\,dy.\end{aligned}$$

By Lemma 1

$$\int_{-m}^{m} e(\lambda y^2)\, dy = \frac{e(1/8)}{\sqrt{(2\lambda)}} - \frac{J}{2\pi i\lambda} + \frac{e(\lambda m^2)}{2\pi i\lambda m}, \tag{13}$$

where

$$J = \int_{m}^{\infty} \frac{e(\lambda y^2)}{y^2}\, dy = O\left(\frac{1}{\lambda m^3}\right) = O\left(\frac{B^2 M^2}{m^3 T}\right)$$

by the first derivative test. The term in κ integrates to zero.

Comparing (9), (10), (11) and (12), we see that $G(y)$ satisfies the hypotheses of Lemma 2 with a, b replaced by $-m$, m, and with

$$A = \frac{U}{M^2}\left(B^5 + \frac{BM}{N}\right)^2.$$

Combining (13) with the result of Lemma 2, we have

$$\begin{aligned}\int_{-m}^{m} &(\mu + \kappa y + G(y))e(f(c) + \lambda y^2)\, dy \\ &= \frac{\mu e(f(c) + 1/8)}{\sqrt{\lambda}} + \left[\frac{(\mu + \kappa y + G(y))e(f(c) + \lambda y^2)}{4\pi i\lambda y}\right]_{-m}^{m} + O\left(\frac{\mu\, |J|}{\lambda}\right) \\ &\quad + O\left(\frac{U}{\lambda^{3/2} M^2}\left(B^5 + \frac{BM}{N}\right)^2\right) \\ &= \frac{g(c)e(f(c) + 1/8)}{\sqrt{f''(c)}} + \left[\frac{g(x)e(f(x))}{2\pi i f'(x)}\right]_{a_1}^{b_1} + O\left(\frac{B^4 M^4 U}{m^3 T^2} + \frac{B^{13} MU}{T^{3/2}}\left(1 + \frac{M}{B^4 N}\right)^2\right),\end{aligned}$$

as asserted. □

Proof of Theorem 2. We have to optimise the choice of m between Theorem 1 applied to the ranges a to a_1 and b_1 to b, and Lemma 3. If we ignore the conditions $a \le a_1$ and $b_1 \le b$, then we can choose

$$m = M/T^{1/6}. \tag{14}$$

This choice may or may not satisfy the conditions $a \le a_1$, or $b_1 \le b$. If either condition is not satisfied, then we reduce the value of m so that

$$a = a_1, \qquad m \asymp c - a, \tag{15}$$

or

$$b_1 = b, \qquad m \asymp b - c. \tag{16}$$

The terms in Lemma 3 containing negative powers of m are larger than they would be at the optimum. To cover these cases, we must include in the upper bound the values of these terms with the choices (15) and (16), which give the terms in $1/(c-a)^3$ and $1/(b-c)^3$ in the theorem. Otherwise we use Theorem 1 on the ranges a to a_1 and b_1 to b. For $b_1 \le x \le b$ we have

$$f'(x) \ge f'(b_1) \ge (b_1 - c)\frac{T}{B^2 M^2} \asymp \frac{mT}{B^2 M^2},$$

so the error term in Theorem 1 is

$$O\left(\frac{B^3TU}{M^2}\left(1+\frac{M^3}{N^2}\frac{m}{B^2M}\right)\frac{B^6M^6}{m^3T^3}\right)=O\left(\frac{B^9M^4U}{m^3T^2}\left(1+\frac{mM}{B^2N^2}\right)\right).$$

This term absorbs the second error term in Lemma 3, and is itself dominated by the first error term in Lemma 3 when (14) holds. The range $a \leq x \leq a_1$ is treated similarly. □

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