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SPLITTING PROPERTIES IN ARCHIMEDEAN l-GROUPS

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1. Introduction and statement of the main results

Throughout this paper an l-group will always mean an archimedean lattice-ordered group and we shall confine our attention to such groups. An l-group *splits* if it is a cardinal summand of each l-group that contains it as an l-ideal. Suppose that G is an l-subgroup of an l-group H. Then G is *large* in H or H is an *essential extension* of G if for each l-ideal $L \neq 0$ of H, $L \cap G \neq 0$. G is *essentially closed* if it does not admit any proper essential extension. Conrad (1971) proved that each essentially closed l-group splits, but not conversely.

An l-group is *projectable* if each principal polar is a cardinal summand; *laterally complete* if each disjoint subset has a least upper bound; *ortho-complete* if it is both laterally complete and projectable. Each essentially closed l-group is orthocomplete. Jakubik (1974) proved that each orthocomplete l-group splits, and since a laterally complete vector lattice is orthocomplete (Veksler and Geiler (1972)) each laterally complete vector lattice splits. Bernau (1976) proved that each laterally complete l-group is orthocomplete. Thus each laterally complete l-group splits.

An essentially closed essential extension of G will be called an *essential closure* of G. Conrad (1971) proved that each l-group admits a unique essential closure G^e. Moreover, $G^e = G^{d\wedge L}$ where G^d is the divisible hull of G, $G^\wedge$ is the Dedekind–Mac Neille completion of G, and G^L is the lateral completion of G.

Consider the following properties of an l-group G.

(1) Each essential extension of G splits.

(2) $G^{d\wedge}$ splits.

(3) $G^\wedge$ splits.

(4) $G^\wedge$ is laterally complete.

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(5) G is an l-subgroup of a laterally complete l-group H, and G generates H as an l-ideal.
(6) Each disjoint subset of G is bounded.
(7) $G^{d\wedge} = G^{e}$
(8) $G^{d\wedge}$ is laterally complete.
(9) G generates G^{e} as an l-ideal.
(10) G splits.
(11) If G is an l-ideal of an l-group K, then $K = G'' \boxplus G'$.
(12) If $0 < x \in G^{e} \backslash G^{d\wedge}$, then $g \wedge nx \notin G$ for some $0 < g \in G$ and $n \in N$.
(13) G is the only l-subgroup of G^{e} that contains G as an l-ideal.
(14) If G is a large l-ideal of an l-group H, then $G = H$.

We prove (1) through (9) are equivalent, (10) through (14) are equivalent, and clearly (1) implies (10). We show by example that (10) does not imply (1). However, if G is complete clearly (10) implies (3), so (1) through (14) are equivalent. Thus we recover the results that a complete l-group splits if and only if it is laterally complete (Jakubik (1974)), and that for a complete vector lattice the following are equivalent: G is essentially closed; G splits; G is laterally complete (Conrad (1971)).

If G is a conditionally laterally complete l-group, then (1) through (14) are equivalent and each is equivalent to

(15) G is laterally complete.

Note that a laterally complete l-group satisfies (5). Thus from (1) we get a stronger version of the Jakubik–Bernau result, namely that each essential extension of a laterally complete l-group splits; from (7) we get that $G^{Ld\wedge} = G^{Le} = G^{e}$.

We generalize the Veksler–Geiler–Bernau theorem by showing that each essential extension of a laterally complete l-group is projectable.

We show that G^{L} is the subgroup of G^{e} that is generated by the joins of disjoint subsets of G.

Let S be the class of l-groups that satisfy (6), and hence (1) through (9). Then it is interesting to note that S is closed with respect to cardinal summands, cardinal products, and essential extensions; and if G is an l-subgroup of $H \in S$ and G generates H as an l-ideal, then $G \in S$.

Finally, we wish to acknowledge the contribution of Simon Bernau, who suggested numerous improvements to an earlier version of this paper. In particular, Theorem 3.2 is due to him.

NOTATION. R will always denote the naturally ordered additive group of reals and N the set of natural numbers. We shall denote the direct sum of two

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l-groups by $A \oplus B$ and the cardinal sum by $A \boxplus B$. The cardinal sum (product) of the l-groups $\{A_\lambda \mid \lambda \in \Lambda\}$ will be denoted by ΣA_λ (ΠA_λ). If A is a subset of an l-group G, then $A' = \{g \in G \mid |g| \wedge |a| = 0 \text{ for all } a \in A\}$ is the *polar* of A. $[x]$ will denote the cyclic group generated by x. $A \subset B$ will always mean proper containment.

2. The equivalence of (10) through (14)

We use the fact that if X is an l-ideal of an l-group H and Y is an l-subgroup of H, then $X + Y$ is an l-subgroup of H. For if $h = x + y \in X + Y$, then $X + h = X + y$; thus $X + (h \vee 0) = X + (y \vee 0)$ and hence $h \vee 0 = a + (y \vee 0) \in X + Y$.

LEMMA 2.1. *Let G be an l-group.*

(a) *$G^{d\wedge}$ is the l-ideal of G^e generated by G.*

(b) *$G^{d\wedge}$ is the minimal complete vector lattice in which G is large.*

PROOF. (a). G^d is dense in G^e so $G^{d\wedge}$ is the l-ideal of G^e generated by G^d (Conrad and McAlister (1969)). But any l-ideal of G^e is divisible, so if it contains G it must contain G^d.

(b). Suppose G is a large l-subgroup of the complete vector lattice K. Then G^d is dense in K, so $G^{d\wedge}$ is an l-ideal of K. However, $G^{d\wedge}$ is a complete vector lattice since it is an l-ideal of G^e.

THEOREM 2.2. *Statements* (10) *through* (14) *are equivalent.*

PROOF. (10) $\Rightarrow$ (11). Clear. (11) $\Rightarrow$ (12). If (12) is false, then there exists $0 < x \in G^e \backslash G^{d\wedge}$ such that $g \wedge nx \in G$ for all $0 < g \in G$ and $n \in N$. Then we claim that $G \oplus [x]$ is an l-subgroup of G^e; it suffices to show that $(g + mx) \wedge 0 \in G \oplus [x]$ for all $g \in G$ and integers m. If $m \geqq 0$, then

$$g^- + [(g + mx) \wedge 0] = (g^+ + mx) \wedge g^- = mx \wedge g^- \in G$$

and if $m < 0$ then

$$g^- - mx + [(g + mx) \wedge 0] = g^+ \wedge (g^- - mx) = g^+ \wedge (-mx) \in G.$$

Thus in either case $(g + mx) \wedge 0 \in G \oplus [x]$, so $G \oplus [x]$ is an l-subgroup of G^e. Furthermore G is an l-ideal of $G \oplus [x]$. For if not, then $0 < g + nx < h \in G$ for some $n \neq 0$, so $-g < nx < -g + h$. Then $nx \in G^{d\wedge}$, so $x \in G^{d\wedge}$. Now, let $L = \{f \in C[0,1] : f(\frac{1}{2}) = 0\}$, a maximal l-ideal of the l-group $C[0,1]$. Let $K = (G \boxplus L) \oplus [(x, \bar{1})] \subset G \oplus [x] \boxplus C[0,1] = S$, where $\bar{1}(y) = 1$ for all $y \in [0,1]$. Then $G \boxplus L$ is an l-ideal of S and $[(x, \bar{1})]$ is an l-subgroup of S, so by the fact above K is an l-subgroup of S. Then G is an l-ideal of K and $G = G''$, but $K \neq G \boxplus G'$, so (11) is false.

(12) $\Rightarrow$ (13). If G is a proper l-ideal of $K \subseteq G^e$ and $0 < x \in K \backslash G$, then $x \in G^e \backslash G^{d\wedge}$ and $nx \wedge g \in G$ for all $0 < g \in G$ and $n \in \boldsymbol{N}$.

(13) $\Rightarrow$ (14). Clear.

(14) $\Rightarrow$ (10). Let G be an l-ideal of H. Then G is l-isomorphic to $G \oplus G'/G' \subseteq H/G'$. If $G' < G' + h \in H/G'$, $h \wedge g > 0$ for some $0 < g \in G$. Hence $G' + h \geqq G' + (h \wedge g) \in G \oplus G'/G'$, since $h \wedge g \in G$. Thus $G \oplus G'/G'$ is large in H/G', so $G \oplus G'/G' = H/G'$. Therefore $G \oplus G' = H$.

3. The structure of G^L

The proof that a laterally complete l-group is projectable (Bernau (1976)) can be slightly altered to obtain stronger results.

LEMMA 3.1 (*Bernau*). *Let x and y be positive elements in the l-group G, and for $n = 0, 1, 2, \cdots$ let $w_n = ((n+2)x - y)^+ \wedge (y - nx)^+$. Then $w_n \wedge w_m = 0$ if $|m - n| \geqq 2$. Let u be an upper bound for $\{(2n+1)w_{2n} : n = 0, 1, 2, \cdots\}$ and v be an upper bound for $\{(2n+2)w_{2n+1} : n = 0, 1, 2, \cdots\}$. Then $y - y \wedge (u + v) \in x'$. In addition if $y \in x''$ and u and v are least upper bounds, then $y = u + v$.*

PROOF. The first part of the lemma is proved by Bernau (1976) with the unnecessary (and unused) assumption that u and v are least upper bounds. We have only to show that if u, v are least upper bounds, $u + v \leqq y$. For this we note that

$$(n+1)w_n = [y + (n+2)((n+1)x - y)]^+ \wedge [y + n(y - (n+1)x)]^+$$
$$= [y + [(n+2)((n+1)x - y) \wedge n(y - (n+1)x)]]^+ \leqq y.$$

Now, $u + v = \bigvee_{m,n}((2n+2)w_{2n+1} + (2m+1)w_{2m})$ and since $w_n \wedge w_m = 0$ if $|m - n| \geqq 2$ it is enough to show that $(n+1)w_n + (n+2)w_{n+1} \leqq y$ for $n = 0, 1, 2, \cdots$. Since $a^+ + b^+ = (a+b)^+ \vee a^+ \vee b^+$ we have only to show that

$$(n+1)[((n+2)x - y) \wedge (y - nx)] + (n+2)[((n+3)x - y) \wedge (y - (n+1)x)] \leqq y$$

However, this last quantity is less than or equal to $(n+1)((n+2)x - y) + (n+2)(y - (n+1)x) = y$.

REMARK. Note that each $w_n \in x''$, so if we can choose $u, v \in x''$, then

$$y = [y \wedge (u+v)] + [y - (y \wedge (u+v))] \in x'' \oplus x'.$$

Thus, if each disjoint subset of x'' is bounded in x'', then $G = x'' \boxplus x'$.

COROLLARY (Bernau (1976)). *A laterally complete l-group is orthocomplete.*

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In Conrad (1973) it is shown that $G^{0\wedge} = G^{\wedge 0}$, where G^0 is the orthocompletion of G. Thus, since $G^0 = G^L$, we have

COROLLARY (Bernau (1976)). $G^{L\wedge} = G^{\wedge L}$.

THEOREM 3.2. *$G^0 = G^L$ is the subgroup of G^e generated by the joins of disjoint subsets of G.*

PROOF. If $0 < w \in G^0$, then there exists a collection $\{M_\alpha\}$ of disjoint polars of G, and a collection $\{y_\alpha\}$ of positive elements of G, so that $w = \vee_\alpha [M_\alpha] y_\alpha$. Here $[M_\alpha] y_\alpha$ is the projection of y_α, considered as an element of G^0, on the unique polar of G^0 which corresponds to M_α (Bernau (1966)). For each α, $M_\alpha = \vee_\lambda x''_{\alpha\lambda}$, for some disjoint collection $\{x_{\alpha\lambda}\}$ of elements in G. Thus $w = \vee_\lambda [x''_\lambda] y_\lambda$, for some disjoint collection $\{x_\lambda\}$ in G and $\{y_\lambda\} \subseteq G$.

Define $w_n(\lambda) = ((n+2)x_\lambda - y_\lambda)^+ \wedge (y_\lambda - nx_\lambda)^+$. By Lemma 3.1, $w_n(\lambda) \wedge w_m(\lambda) = 0$ if $\lambda \neq u$, or if $| n - m | \geqq 2$. Let $u = \vee_{n,\lambda} (2n+1) w_{2n}(\lambda)$ and $v = \vee_{n,\lambda} (2n+2) w_{2n+1}(\lambda)$. Consider

$$[x''_\lambda](w - (u + v)) = [x''_\lambda] y_\lambda - \bigvee_n (2n+1) w_{2n}(\lambda) - \bigvee_n (2n+2) w_{2n+1}(\lambda).$$

This last quantity is unchanged if we replace y_λ by $[x''_\lambda] y_\lambda$ in the expressions for $w_n(\lambda)$. Hence, by Lemma 3.1, $[x''_\lambda](w - (u + v)) = 0$ for all λ. Since w is an element of the G^0 polar corresponding to x''_λ, we see that $w = u + v$.

REMARK. Our original version of this theorem was that $G(1)$ generates G^e as an l-ideal, where $G(1)$ is the l-subgroup of G^L that is generated by the joins of disjoint subsets of G. But now since $G(1) = G^L$, this follows from Theorem 4.1. We wish to thank Simon Bernau for this proof of Theorem 3.2.

4. The equivalence of (1) through (9)

THEOREM 4.1. *Statements* (1) *through* (9) *are equivalent.*

PROOF. (1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (3) $G^{\wedge d \wedge} = G^{d\wedge} = G^{d\wedge d\wedge}$ and (12) holds for $G^{d\wedge}$, and hence for $G^\wedge$.

(3) $\Rightarrow$ (4). $G^\wedge$ is an l-ideal in $G^{\wedge L}$ (Jakubik (1963)), and so by (14) $G^\wedge = G^{\wedge L}$.

(4) $\Rightarrow$ (5). Clear.

(5) $\Rightarrow$ (6). Clear.

(6) $\Rightarrow$ (7). If $\{g_\lambda\}$ is a disjoint subset of G^d, there exist positive integers m_λ so that $\{m_\lambda g_\lambda\}$ is a disjoint subset of G. Hence $\vee g_\lambda \in G^{d\wedge}$, so by Theorem 3.2 $G^{dL} \subseteq G^{d\wedge}$. Thus $G^e = G^{d\wedge L} = G^{dL\wedge} \subseteq G^{d\wedge}$.

(7) $\Rightarrow$ (8). Clear.

(8) $\Rightarrow$ (9). $G^{d\wedge} = G^{d\wedge L} = G^e$.

(9) $\Rightarrow$ (1). Clearly G satisfies (13) and hence G splits; but any essential extension of G also satisfies (13) and so splits.

REMARK. The equivalence of (1), (2), (7), (8) and (9) does not depend on Section 3.

COROLLARY 1. *G^L generates G^e as an l-ideal.*

COROLLARY 2 (*Conrad* (1971)). *If $\Sigma T_\lambda \subseteq G \subseteq \Pi T_\lambda$, when each T_λ is a subgroup of R, and G generates ΠT_λ as an l-ideal, then G splits.*

PROOF. This follows from the fact that $G^e = \Pi R_\lambda$.

COROLLARY 3. *Each essential extension of a laterally complete l-group is projectable.*

PROOF. Let G be an essential extension of a laterally complete l-group H, and consider $0 < x, y \in G$. Now $H \subseteq G \subseteq H^e$, and H generates H^e as an l-ideal so there exists $h \in H$ so that $h > y > 0$. Let $\{x_\lambda \mid \lambda \in \Lambda\}$ be a maximal disjoint subset of $x'' \cap H$ and let $z = \bigvee x_\lambda$. Then $x' = z'$ and so $x'' = z''$. By the corollary to Lemma 3.1 H is projectable, so there exist $h_1, h_2 \in H$ such that $h_1 \in z'$, $h_2 \in z''$ and $h = h_1 + h_2$. Then

$$\begin{aligned} y = y \wedge h &= y \wedge (h_1 + h_2) = y \wedge (h_1 \vee h_2) = (y \wedge h_1) \vee (y \wedge h_2) \\ &= (y \wedge h_1) + (y \wedge h_2) \in z' \boxplus z'' = x' \boxplus x'', \end{aligned}$$

so G is projectable.

5. The implication (10) $\Rightarrow$ (1)

In the next section we give an example of an l-group that satisfies (10) but not (1). In this section we provide conditions for (10) to imply (1).

PROPOSITION 5.1. *For an l-group, the following are equivalent:*

(a) G does not split,

(b) There exists $0 < x \in G^e \backslash G^{d\wedge}$ such that $G \oplus [x]$ is an l-subgroup of G^e.

(c) There exists $0 < x \in G^e \backslash G^{d\wedge}$ such that $g \wedge nx \in G$ for all $0 < g \in G$ and $n \in \mathbf{N}$.

PROOF. (a) and (c) are the negations of (10) and (12), and so are equivalent by Theorem 2.2.

(a) $\Rightarrow$ (b). Since G does not satisfy (13), G is a proper l-ideal of an l-subgroup K of G^e. If $0 < x \in K \backslash G$ then $G \oplus [x]$ is an l-subgroup of K with G as an l-ideal, so $x \in G^e \backslash G^{d\wedge}$.

(b) $\Rightarrow$ (a). It suffices to show that G is an l-ideal of $G \oplus [x]$. If not, then $0 < g + nx < h \in G$ for some $n \neq 0$, so $-g < nx < -g + h$. Thus $nx \in G^{d\wedge}$, and so $x \in G^{d\wedge}$, which is a contradiction.

As a corollary, we obtain

PROPOSITION 5.2. *For an l-group G, the following are equivalent*:

(a) (10) *implies* (1).

(b) *If $G^{d\wedge} \subset G^e$, then $G \oplus [x]$ is an l-subgroup of G^e for some $0 < x \in G^e \backslash G^{d\wedge}$.*

(c) *If $G^{d\wedge} \subset G^e$, then there exists $0 < x \in G^e \backslash G^{d\wedge}$ such that $g \wedge nx \in G$ for all $0 < g \in G$ and $n \in N$.*

THEOREM 5.3 *For a conditionally laterally complete l-group G,* (1) *through* (14) *are equivalent and each is equivalent to*

(15) *G is laterally complete.*

PROOF. If $G^{d\wedge} \subset G^e$ then there is a disjoint subset $\{g_\delta \mid \delta \in \Delta\}$ of G that is not bounded in G. Then $x = \vee g_\delta \in G^e \backslash G^{d\wedge}$. For $0 < g \in G$ and $n \in N$, we have

$$g \wedge nx = g \wedge n(\vee g_\delta) = g \wedge (\vee ng_\delta) = \vee(g \wedge ng_\delta)$$

and since $\{g \wedge ng_\delta \mid \delta \in \Delta\}$ is a disjoint subset of G that is bounded by g, it follows that $g \wedge nx \in G$. Thus, by Proposition 5.2, (10) implies (1) and so (1) through (14) are equivalent. Finally, for a conditionally laterally complete l-group, (6) is clearly equivalent to (15).

Note that a complete l-group is conditionally laterally complete and so there is an obvious corollary which contains and generalizes some of the results in Jakubik (1974) and Conrad (1971).

6. Examples and some additional results

6.1. A projectable vector lattice that splits but is not conditionally laterally complete.

Let H be the cardinal product $\Pi_{i=1}^{\infty} \mathbf{Z}_i$ of integers and let G be the vector lattice in $\Pi_{i=1}^{\infty} \mathbf{R}_i$ generated by H. Then $\Sigma \mathbf{R}_i \subset G \subset \Pi \mathbf{R}_i$. G is an essential extension of $\Pi \mathbf{Z}_i$, so it is projectable by Corollary 3 to Theorem 4.1, and it satisfies (6). Thus G is conditionally laterally complete if and only if it is laterally complete, but $(\Sigma \mathbf{R}_i)^L = \Pi \mathbf{R}_i$, so G is not laterally complete.

REMARK. Jakubik (1974) claims that his example (2) satisfies this condition, but it is not a group.

All the examples in the literature of l-groups that split are essential extensions of laterally complete l-groups and hence projectable.

6.2. An l-group G such that each disjoint subset is bounded but is not projectable. Then G splits but does not contain a large laterally complete l-subgroup.

For each $0 < x \in H = \Pi_{i=1}^{\infty} \mathbf{R}_i$ pick an element $\bar{x} \geqq x$ in H such that $\bar{x}_{2k+1} = \bar{x}_{2k+2}$ for $k = 0, 1, 2, \cdots$. Let G be the l-subgroup of H generated by $Y = \{\bar{x} \mid 0 < x \in H\}$, $\Sigma \mathbf{R}_i$ and $g = (1, 0, 1, 0, \cdots)$. Then G generates H as an l-ideal and hence splits. Pick an $\bar{x}$ which is unbounded and such that $\bar{x}_i > 0$ for all $i \in \mathbf{N}$. Suppose (by way of contradiction) that G is projectable. Then since $G = g'' \boxplus g'$,

$$z = (\bar{x}_1, 0, \bar{x}_3, 0, \cdots) \in G$$

and so $z = \bigvee_I \bigwedge_J (\beta_{ij} + n_{ij} g)$, where each β_{ij} belongs to the subgroup of H generated by Y and $\Sigma \mathbf{R}_i$, and I and J are finite.

$$\begin{aligned} \bar{x}_{2k+1} = z_{2k+1} = [\bigvee_I \bigwedge_J (\beta_{ij} + n_{ij} g)]_{2k+1} &= \bigvee_I \bigwedge_J ((\beta_{ij})_{2k+1} + n_{ij}) \\ &= (\beta_{i_0 j_0})_{2k+1} + n_{i_0 j_0}, \end{aligned}$$

since $\mathbf{R}_{2k+1}$ is totally ordered. Now $0 = z_{2k+2} = \bigvee_I \bigwedge_J (\beta_{ij})_{2k+2}$ so for each i, $\bigwedge_J (\beta_{ij})_{2k+2} \leqq 0$ and hence in particular $\bigwedge_J (\beta_{i_0 j})_{2k+2} \leqq 0$. Thus $(\beta_{i_0 j_1})_{2k+2} \leqq 0$ for some j_1. Since

$$\bar{x}_{2k+1} = \bigwedge_J ((\beta_{i_0 j})_{2k+1} + n_{i_0 j}) = (\beta_{i_0 j_0})_{2k+1} + n_{i_0 j_0}$$

we have

$$n_{i_0 j_1} + (\beta_{i_0 j_1})_{2k+2} = n_{i_0 j_1} + (\beta_{i_0 j_1})_{2k+1} \geqq \bar{x}_{2k+1},$$

and hence $n_{i_0 j_1} \geqq \bar{x}_{2k+1}$. Thus for each k we have $i_k \in I$ and $j_k \in J$ such that $n_{i_k j_k} \geqq \bar{x}_{2k+1}$. But $I \cup J$ is finite and the $\bar{x}_{2k+1}$ are unbounded, so this is impossible. Therefore $z \notin G$, so G is not projectable.

6.3. An l-subgroup G of $H = \Pi_{i=1}^{\infty} \mathbf{R}_i$ such that G generates H as an l-ideal but G is not large in H.

Let G be the l-subgroup of H generated by the subset Y defined above.

6.4. An l-subgroup of $H = \Pi_{x \in [0,1]} \mathbf{R}_x$ that splits, but not every essential extension splits. Let

$$G = \{g \in H \mid g(x) = \alpha \text{ for some fixed } \alpha$$
$$\text{for all but countably many of the } x\text{'s}\}.$$

Note that G is the direct sum of the l-ideal of all functions in H with countable support and the l-subgroup of all the constant functions, so G is an l-subgroup of H.

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(a) G is a projectable vector lattice and $G \supsetneq \Sigma \mathbf{R}_x$. Thus $G^L = G^e = H$. Moreover, if $\{g_n \mid n \in \mathbf{N}\}$ is a countable subset of G that is bounded in H then $\bigvee g_n \in G$.

(b) G does not generate G^e as an l-ideal.

PROOF. Partition $[0,1]$ into a countable number of subsets each with the same cardinality as $[0,1]$. Denote these by $X_1, X_2, \cdots$. Define $h(x) = n$ if $x \in X_n$. Then h is not bounded by any element of G.

(c) G splits.

PROOF. Suppose $G \subsetneq K \subsetneq G^e$ where G is an l-ideal of K and pick $0 < k \in K$. Then $g \wedge k \in G$ for each $0 < g \in G$. For $n \in \mathbf{N}$ define $\bar{n} \in G$ by $\bar{n}(x) = n$ for all $x \in [0,1]$. Then $(\bar{n} \wedge k)(x) = \alpha$ for all but a countable number of the x's. If $\alpha < n$, then $k(x) = \alpha$ for all but a countable number of the x's, and so $k \in G$. Now if $\alpha = n$, then $k(x) \geqq n$ for all but a countable number of the x's, so $k(x) < n$ for at most a countable number of the x's.

Let $Y_n = \{x \in [0,1] \mid k(x) < n\}$. Then $[0,1] = \bigcup_n Y_n$, which is countable, a contradiction. Therefore $G = K$, and so by Theorem 4.1 G splits.

REMARKS. If T is an l-subgroup of G that contains $\Sigma \mathbf{R}_x$ and generates G *as an* l-ideal, then it follows that T splits. In particular, one can restrict the elements of G to be integral valued.

6.5. If G^d splits then so does G.

PROOF. If G is an l-ideal of $K \subsetneq G^e$, then G^d is an l-ideal of K^d so $G^d = K^d$. Thus if $0 < k \in K \subsetneq G^d$, then $nk \in G$ for some $n > 0$, so $k \in G$. Therefore $G = K$ and so G splits.

6.6. If G splits and $G = A \boxplus B$ then A splits.

PROOF. If A is an l-ideal of an l-group H, then G is an l-ideal of $K = H \boxplus B$, so $K = G \boxplus C = A \boxplus B \boxplus C$. Thus $H = A \boxplus C$.

6.7 (Jakubik (1974)). If $\{A_\lambda \mid \lambda \in \Lambda\}$ is a set of l-groups that split, then $\Pi_\Lambda A_\lambda$ splits.

PROOF. Suppose by way of contradiction that

$$\Pi A_\lambda \subset H \subsetneq (\Pi A_\lambda)^e = \Pi A_\lambda^e$$

where ΠA_λ is an l-ideal of H. Let H_λ be the image of the projection of H into A_λ^e. Then for some λ we have

$$A_\lambda \subset H_\lambda \subsetneq A_\lambda^e$$

and A_λ is an l-ideal of H_λ, but this contradicts the fact that A_λ splits.

6.8. ΠA_λ splits if and only if each A_λ splits.

Each l-group G is contained in a unique minimal projectable l-group G^P in which G is large (Conrad (1973)).

6.9. If each essential extension of G^P splits, then each essential extension of G splits.

PROOF. Conrad (1973) shows that $G^{dP} = G^{Pd}$ and $G^{P\wedge} = G^{\wedge P} = G^\wedge$. Thus

$$G^{d\wedge} = G^{dP\wedge} = G^{Pd\wedge} = G^{Pe} = G^e.$$

6.10. For an l-group G the following are equivalent.

(a) G is essentially closed.

(b) If G is an l-subgroup of an l-group H and G is large in G'', then $H = G \boxplus G'$.

PROOF. (a $\Rightarrow$ b). Since G is essentially closed, $G = G''$ and so is an l-ideal of H. Then since G splits, $H = G \boxplus G'$.

(b $\Rightarrow$ a). G is large in G^e and $G \subseteq G'' = G^e$. Then $G^e = G \boxplus G'$, and since $G' = 0$, $G = G^e$ is essentially closed.

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