Bull. Aust. Math. Soc. **82** (2010), 167–170
doi:10.1017/S0004972710000328

ON THE NUMBER OF LATIN RECTANGLES

DOUGLAS S. STONES

(Received 24 March 2010)

2000 *Mathematics subject classification*: primary 05B15; secondary 05A05, 11B50, 20D60, 20D45.

Keywords and phrases: Latin square, Latin rectangle, autotopism, automorphism, Alon–Tarsi conjecture, quasigroup, orthomorphism.

1. The number of Latin rectangles

A *Latin rectangle* L is a $k \times n$ array with symbols from $\mathbb{Z}_n$ such that each row and each column contains only distinct symbols. When $k = n$, L is called a *Latin square*. We say L is *reduced* if the first row is $(0, 1, \ldots, n-1)$ and the first column is $(0, 1, \ldots, k-1)^T$. The number of $k \times n$ Latin rectangles, denoted $L_{k,n}$, is related to the number of reduced $k \times n$ Latin rectangles, denoted $R_{k,n}$, by the formula $L_{k,n} = n!(n-1)!R_{k,n}/(n-k)!$. We also write $R_n = R_{n,n}$. McKay and Wanless [10] gave $R_{k,n}$ when $n \leq 11$.

The author's thesis [14] primarily investigates the number $R_{k,n}$. For example, we use a formula of Doyle [6, 12] to find $R_{4,n}$ for $n \leq 80$, $R_{5,n}$ for $n \leq 25$ and

- $R_{6,12} = 16\,790\,769\,154\,925\,929\,673\,725\,062\,021\,120$ and
- $R_{6,13} = 4\,453\,330\,421\,956\,050\,777\,867\,897\,829\,494\,620\,160$.

In general, the problem of finding $R_{k,n}$ is difficult and, furthermore, the literature contains many published errors (see [5, 9, 10, 12] for surveys of its history). In addition to tackling the enumeration problem computationally, we also find theoretical results for $R_{k,n}$. For example, we find the value of $R_{k,n}$ mod n for all k and n [19].

THEOREM 1.1. *If* $k \geq 1$ *and* $n \geq 1$, *then* $R_{k,n} \equiv ((-1)^{k-1}(k-1)!)^{n-1} \bmod n$.

Thesis submitted to Monash University, November 2009. Degree approved, February 2009. Main supervisor: Dr Ian M. Wanless. Secondary supervisor: Associate Professor Graham Farr. External supervisor: Professor Darryn Bryant.

Supported by the Faculty of Science Dean's Postgraduate Research Scholarship and Postgraduate Publications Award.

Theorem 1.1 implies the surprising fact that $R_n \bmod n$ is an indicator variable for primality of n. We also generalize recurrence congruences for $R_{3,n}$ by Riordan [11] and Carlitz [3] to arbitrary fixed k. The techniques were further developed to encompass the number of certain graph factorizations and the size of certain subsets of Latin hypercuboids (a very broad generalization of Latin rectangles).

2. Orthomorphisms and partial orthomorphisms

A *partial orthomorphism* of $\mathbb{Z}_n$ is an injection $\nu : S \to \mathbb{Z}_n$ for some $S \subseteq \mathbb{Z}_n$ such that $i \mapsto \nu(i) - i$ is also an injection [20]. An *orthomorphism* is a partial orthomorphism with $|S| = n$ [8]. Let z_n be the number of orthomorphisms σ of $\mathbb{Z}_n$ for which $\sigma(0) = 0$. We extend a result by Clark and Lewis [4] who found $z_n \bmod n$ for prime n [16].

THEOREM 2.1. *$R_{n+1} \equiv z_n \equiv -2 \bmod n$ for odd prime n and $R_{n+1} \equiv z_n \equiv 0 \bmod n$ for composite n.*

The enumeration of partial orthomorphisms is also linked to the value of $R_{k,n}$ [17]. We give new sufficient conditions for when a partial orthomorphism admits a completion to an orthomorphism and give a method for finding the number of partial orthomorphisms with $|S| = a$, for fixed a [17].

Let d be a divisor of n. If σ is an orthomorphism of $\mathbb{Z}_n$ such that $\sigma(i) \equiv \sigma(j) \bmod d$ whenever $i \equiv j \bmod d$ then we call σ a *d-compound* orthomorphism. We develop the theory of d-compound orthomorphisms and, in particular, two special subclasses, compatible and polynomial orthomorphisms [16].

3. The Alon–Tarsi conjecture

The sign of a Latin square is -1 if it has an odd number of rows and columns that are odd permutations, otherwise it is $+1$. Let R_n^{EVEN} and R_n^{ODD} be respectively the number of Latin squares of order n with sign $+1$ and -1. The Alon–Tarsi conjecture asserts that $R_n^{\mathrm{EVEN}} \neq R_n^{\mathrm{ODD}}$ when n is even [1]. In a 1997 paper, Drisko [7] proved that $R_{n+1}^{\mathrm{EVEN}} \not\equiv R_{n+1}^{\mathrm{ODD}} \bmod n$ for prime n and suggested some ideas for future research in the study of the Alon–Tarsi conjecture, which we show to be futile with the following theorem [18].

THEOREM 3.1. *If $2 \leq t \leq n$, then $R_{n+1}^{\mathrm{EVEN}} \not\equiv R_{n+1}^{\mathrm{ODD}} \bmod t$ if and only if $t = n$ is prime.*

4. Autotopisms and subsquares

We also investigate symmetries of Latin squares; see [5] for the relevant definitions. Autotopisms and automorphisms play a key role in finding divisors of R_n. Moreover, Latin squares that admit automorphisms typically contain partial orthomorphisms. Let L be a Latin square of order n and let $\mathrm{Atop}(L)$ be the autotopism group of L. We bound the maximum cardinality of $\mathrm{Atop}(L)$, enabling us to find divisors of R_n for large n. A similar method gives a bound on the maximum number of $k \times k$ subsquares in a Latin square, for general k.

https://doi.org/10.1017/S0004972710000328 Published online by Cambridge University Press

THEOREM 4.1. *If L is a Latin square of order n, then*

$$|\mathrm{Atop}(L)| \le n^2 \prod_{t=1}^{\lfloor \log_2 n \rfloor} (n - 2^{t-1}).$$

THEOREM 4.2. *The number of $k \times k$ subsquares in a Latin square of order n is $O(n^{\lceil \log_2(\lfloor k/2 \rfloor + 1) \rceil + 2})$.*

Finally, we find new strong necessary conditions for when an isotopism is an autotopism of some Latin square [15]. We use Ξ_n to denote the set of permutations $\alpha \in S_n$, such that (α, α, α) is an automorphism of some Latin square of order n.

THEOREM 4.3. *Suppose that $\alpha \in S_n$ has precisely m nontrivial cycles of length d. If α has no fixed points, then $\alpha \in \Xi_n$ if and only if m is even or d is odd. If α has at least one fixed point, then $\alpha \in \Xi_n$ if and only if $n \le 2md$.*

THEOREM 4.4. *Suppose $\alpha \in S_n$ consists of a d_1-cycle, a d_2-cycle and d_∞ fixed points. If $d_1 = d_2$ then $\alpha \in \Xi_n$ if and only if $0 \le d_\infty \le 2d_1$. If $d_1 > d_2$ then $\alpha \in \Xi_n$ if and only if:*

(a) *d_2 divides d_1;*
(b) *$d_1 \ge \lceil n/2 \rceil$;*
(c) *$d_2 \ge d_\infty$; and*
(d) *if d_2 is even then $d_\infty > 0$.*

Theorems 4.3 and 4.4 generalize a theorem of Wanless [20] (see also [2]) for cyclic automorphisms. The techniques developed in [15] were subsequently used to study the parity of the number of quasigroups [13].

References

[1] N. Alon and M. Tarsi, 'Colorings and orientations of graphs', *Combinatorica* **12**(2) (1992), 125–134.
[2] D. Bryant, M. Buchanan and I. M. Wanless, 'The spectrum for quasigroups with cyclic automorphisms and additional symmetries', *Discrete Math.* **304**(4) (2009), 821–833.
[3] L. Carlitz, 'Congruences connected with three-line Latin rectangles', *Proc. Amer. Math. Soc.* **4**(1) (1953), 9–11.
[4] D. Clark and J. T. Lewis, 'Transversals of cyclic Latin squares', *Congr. Numer.* **128** (1997), 113–120.
[5] J. Denés and A. D. Keedwell, *Latin Squares and their Applications* (Academic Press, New York, 1974).
[6] P. G. Doyle, 'The number of Latin rectangles', arXiv:math/0703896v1 [math.CO], 15 pp.
[7] A. A. Drisko, 'On the number of even and odd Latin squares of order $p + 1$', *Adv. Math.* **128**(1) (1997), 20–35.
[8] A. B. Evans, *Orthomorphism Graphs of Groups* (Springer, Berlin, 1992).
[9] B. D. McKay, A. Meynert and W. Myrvold, 'Small Latin squares, quasigroups and loops', *J. Combin. Des.* **15** (2007), 98–119.
[10] B. D. McKay and I. M. Wanless, 'On the number of Latin squares', *Ann. Comb.* **9** (2005), 335–344.
[11] J. Riordan, 'A recurrence relation for three-line Latin rectangles', *Amer. Math. Monthly* **59**(3) (1952), 159–162.

https://doi.org/10.1017/S0004972710000328 Published online by Cambridge University Press

[12] D. S. Stones, 'The many formulae for the number of Latin rectangles', *Electron. J. Combin.* **17** (2010), A1.
[13] D. S. Stones, 'The parity of the number of quasigroups', submitted.
[14] D. S. Stones, 'On the number of Latin rectangles'. PhD Thesis, Monash University, 2010.
[15] D. S. Stones, P. Vojtěchovský and I. M. Wanless, 'Autotopisms and automorphisms of Latin squares', in preparation (working title).
[16] D. S. Stones and I. M. Wanless, 'Compound orthomorphisms of the cyclic group', *Finite Fields Appl.* **16** (2010), 277–289.
[17] D. S. Stones and I. M. Wanless, 'A congruence connecting Latin rectangles and partial orthomorphisms', submitted.
[18] D. S. Stones and I. M. Wanless, 'How not to prove the Alon–Tarsi conjecture', submitted.
[19] D. S. Stones and I. M. Wanless, 'Divisors of the number of Latin rectangles', *J. Combin. Theory Ser. A* **117**(2) (2010), 204–215.
[20] I. M. Wanless, 'Diagonally cyclic Latin squares', *European J. Combin.* **25** (2004), 393–413.

DOUGLAS S. STONES, School of Mathematical Sciences, Monash University, Vic 3800, Australia
e-mail: the_empty_element@yahoo.com

https://doi.org/10.1017/S0004972710000328 Published online by Cambridge University Press