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A Generalisation of Nagata's Theorem on Ruled Surfaces

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Abstract. We prove a generalisation of a theorem of Nagata on ruled surface to the case of the fiber bundle $E/P \to X$, associated to a principal G-bundle E. Using this we prove boundedness for the isomorphism classes of semi-stable G-bundles in all characteristics.

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1. Introduction

Let X be a smooth projective irreducible curve of genus g over an algebraically closed field k, G a connected reductive algebraic group over k and P a parabolic subgroup. For a principal G-bundle E over X, consider the associated G/P-bundle $\pi\colon E/P \longrightarrow X$. If σ is a section of π we denote by N_σ the normal bundle of $\sigma(X)$ in E/P. The first result proved in this paper is the following.

THEOREM 1.1. *There exist a section σ of $\pi\colon E/P \longrightarrow X$ such that*

$$\deg(N_\sigma) \leqslant g \cdot \dim(G/P),$$

where g is the genus of X and $\deg(N_\sigma)$ *denotes the degree of the normal bundle N_σ considered as a vector bundle on X.*

The above result was classically known in the case of $G = GL(2)$ and P a maximal parabolic, in the form of the theorem of M. Nagata [8] and C. Segre, which asserts that a ruled surface on X admits a section whose self intersection number is $\leqslant g$. It has also been proved for $G = GL(n)$ and P a maximal parabolic subgroup by Mukai and Sakai [12], and for G a classical group and P a maximal parabolic subgroup by Nitsure [9]. For a general survey of the topic in the case of vector bundles one may refer to Lange [7].

The main idea of our proof of the Theorem 1.1 is a 'no-ghosts theorem' for the Hilbert scheme of E/P, which asserts that every point of the Hilbert scheme which lies in an irreducible component containing the Hilbert point of a minimal section (i.e. for which $\deg(N_\sigma)$ is minimum), is itself the Hilbert point of a section (Proposition 2.3). We then adapt an argument of Mukai–Sakai to complete the proof of the theorem.

In the second part of the paper, we prove the following theorem:

THEOREM 1.2. *Let G be a connected reductive algebraic group and X a smooth projective irreducible curve over an algebraically closed field k of arbitrary characteristic. Then the set of isomorphism classes of semi-stable G-bundles on the curve X with a given degree is bounded. In particular, if G is semi-simple then semi-stable G-bundles form a bounded family.*

For a precise definition of degree see Section 3. In characteristic 0, the above theorem is due to Ramanathan [3]. For the classical groups, the result follows in all characteristics (except in characteristic 2 for $G = \mathrm{SO}(n)$) from the observation of Ramanan (see [13], Proposition 4.2) that a G-bundle is semi-stable if and only if the underlying vector bundle is so.

2. Minimal Sections

Let X be a smooth projective irreducible curve over an algebraically closed field k. Let G be a connected reductive algebraic group over k and P a parabolic subgroup of G.

Given a principal G bundle E over X, denote by $\pi\colon M \longrightarrow X$ the associated bundle E/P with G/P as fibres. If σ is a section of $\pi\colon M \longrightarrow X$, we denote by N_σ, the vector bundle on X obtained by pulling back by σ the normal bundle of $\sigma(X)$ in M. Observe that N_σ is the pullback $\sigma^*(T_\pi)$ where T_π is the tangent bundle along the fibres of $\pi\colon M \longrightarrow X$.

In the following lemma we prove that the degree $\deg(N_\sigma)$ of the vector bundle N_σ on X is bounded below.

LEMMA 2.1. *Given a principal G-bundle $E \longrightarrow X$, there exists a constant C such that $\deg(N_\sigma) \geqslant C$ for all sections of the associated bundle $\pi\colon M \longrightarrow X$.*

Proof. Let T_π be the tangent bundle along the fibres of π. As already observed, $N_\sigma \cong \sigma^*(T_\pi)$. If $\mathfrak{g}$ (resp. $\mathfrak{p}$) are the Lie algebras of G (resp. P) we have an exact sequence of P-modules

$$0 \longrightarrow \mathfrak{p} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{p} \longrightarrow 0.$$

Note that $\mathfrak{g}/\mathfrak{p}$ is the tangent space of G/P at e. On M, we have the principal P-bundle $E \longrightarrow M$, and the above short exact sequence of P-modules gives a short exact sequence of vector bundles on M. Pulling it back under σ gives us a short exact sequence of vector bundles on X, whose middle term is the adjoint bundle of E and the last term is $\sigma^*(T_\pi)$. This implies that $\sigma^*(T_\pi)$ is a quotient of a fixed vector bundle (independent of σ). □

It is known that $\pi\colon M \longrightarrow X$ admits sections. This follows from a theorem of Springer (see, for example, Ramanathan [3], 2.11, p. 306).

Suppose σ is a section of $\pi\colon M \longrightarrow X$. We say σ is a *minimal section* if $\deg(N_\sigma)$ is minimum. As sections exist, and as their degrees are bounded below by Lemma 2.1, there exists a minimal section.

We will now prove a lemma which is a crucial step in the proof of Theorem 1.1. Let Y be a one-dimensional projective scheme over k. If L is a line bundle (locally free sheaf of rank one) on Y, we define the degree of L by

$$\deg(Y, L) = \chi(Y, L) - \chi(Y, \mathcal{O}_Y).$$

Note that this is consistent with the usual definition of the degree of a line bundle on a non-singular projective curve.

It is well known that if L is ample on Y, then $\deg(Y, L) > 0$ (see for example Iitaka [10], 8.4). Observe that $\deg(Y, L)$ is the sum of $\deg(Y_i, L)$ where Y_i are the connected components of Y (regarded as open subschemes), where the zero dimensional components of Y contribute 0 to the degree.

LEMMA 2.2. *Let X be a smooth projective irreducible curve over k and Y a projective one dimensional scheme over k.*

Let $f : Y \longrightarrow X$ be a morphism. Assume that

(a) $\chi(X, \mathcal{O}_X) = \chi(Y, \mathcal{O}_Y)$.
(b) *For some line bundle L of degree 1 on X, we have $\chi(X, L) = \chi(Y, f^*(L))$.*

Then we have the following:

(i) *There exists a unique irreducible component D of Y which dominates X. Let D_{red} be the reduced subscheme structure on D induced from Y. Then $f|_{D_{red}} : D_{red} \longrightarrow X$ is an isomorphism.*
(ii) *Suppose that the component D given by (i) above is the only irreducible component of Y of dimension one. Then $f\colon Y \longrightarrow X$ is an isomorphism (in particular, Y has no zero-dimensional components).*
(iii) *Let ξ be a line bundle on Y. Suppose that Y has more than one irreducible component of dimension one. Let $D_1, D_2, \ldots, D_k$ be the one-dimensional irreducible components other than D and let $D_{i,red}$ be the corresponding reduced subscheme of Y. Suppose $\xi|_{D_{i,red}}$ is ample for all i. Then we have* $\deg(D_{red}, \xi) < \deg(Y, \xi)$.

Proof. We prove the proposition in several steps:

Step (1). $R^1 f_*(\mathcal{O}_Y)$ is a torsion sheaf, in particular, $H^1(X, R^1 f_*(\mathcal{O}_Y)) = 0$.

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Proof of Step (1). Let $S \subset X$ be the set of points of X over which the fibres of $Y \longrightarrow X$ are positive-dimensional. As Y is one-dimensional, it follows from the semi-continuity of the dimension of fibres that S is a finite set of points of X. If $U = X - S$, then we observe that $f|_{f^{-1}(U)}$ is quasi-finite and proper, hence it is a finite map. Therefore $R^1\psi_*(\mathcal{O}_{f^{-1}(U)}) = 0$. Now it is clear that $R^1 f_*(\mathcal{O}_Y)$ is supported over S, hence it is a torsion sheaf.

Step (2). $\deg(Y, f^*(L)) = \chi(X, f_*(\mathcal{O}_Y) \otimes L) - \chi(X, f_*(\mathcal{O}_Y))$

Proof of Step (2). We have $H^0(Y, \mathcal{O}_Y) = H^0(X, f_*(\mathcal{O}_Y))$, and $H^0(Y, f^*(L)) = H^0(X, f_* f^*(L)) = H^0(X, f_*(\mathcal{O}_Y) \otimes L)$ by the projection formula. Since $\dim(X) = 1$, the Leray spectral sequence gives us the following exact sequences

$$0 \longrightarrow H^1(X, f_*(\mathcal{O}_Y)) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^0(X, R^1 f_*(\mathcal{O}_Y)) \longrightarrow 0$$

and

$$0 \longrightarrow H^1(X, f_*(\mathcal{O}_Y) \otimes L) \longrightarrow H^1(Y, f^*(L)) \longrightarrow H^0(X, R^1 f_*(\mathcal{O}_Y) \otimes L) \longrightarrow 0.$$

Hence

$$\chi(Y, \mathcal{O}_Y) = \chi(X, f_*(\mathcal{O}_Y)) - h^0(X, R^1 f_*(\mathcal{O}_Y))$$

and

$$\chi(Y, f^*(L)) = \chi(X, f_*(\mathcal{O}_Y) \otimes L) - h^0(X, R^1 f_*(\mathcal{O}_Y) \otimes L).$$

Note that as $R^1 f_*(\mathcal{O}_Y)$ is torsion by step (1), we have

$$h^0(X, R^1 f_*(\mathcal{O}_Y)) = h^0(X, R^1 f_*(\mathcal{O}_Y) \otimes L).$$

Hence

$$\begin{aligned} \deg(Y, f^*(L)) &= \chi(Y, f^*(L)) - \chi(Y, \mathcal{O}_Y) \\ &= \chi(X, f_*(\mathcal{O}_Y) \otimes L) - \chi(X, f_*(\mathcal{O}_Y)). \end{aligned}$$

Step (3). Rank $f_*(\mathcal{O}_Y) = 1$, in particular, f is dominant.

Proof of Step (3). If T is the torsion submodule of $f_*(\mathcal{O}_Y)$, we have the short exact sequence

$$0 \longrightarrow T \longrightarrow f_*(\mathcal{O}_Y) \longrightarrow Q \longrightarrow 0,$$

Q being locally free. Since we have

$$\begin{aligned} \deg(Y, f^*(L)) &= \chi(Y, f^*(L)) - \chi(Y, \mathcal{O}_Y) \\ &= \chi(X, L) - \chi(X, \mathcal{O}_X) \text{(by (a) and (b) of the lemma)} \\ &= 1, \end{aligned}$$

it follows from step (2) that

$$1 = \chi(X, f_*(\mathcal{O}_Y) \otimes L) - \chi(X, f_*(\mathcal{O}_Y)).$$

From the short exact sequence $0 \longrightarrow T \otimes L \longrightarrow f_*(\mathcal{O}_Y) \otimes L \longrightarrow Q \otimes L \longrightarrow 0$ we see that

$$\chi(X, f_*(\mathcal{O}_Y) \otimes L) - \chi(X, f_*(\mathcal{O}_Y)) = \chi(X, Q \otimes L) - \chi(X, Q),$$

as $\chi(X, T \otimes L) = \chi(X, T)$ since T is a torsion sheaf.

Thus $\chi(X, Q \otimes L) - \chi(X, Q) = 1$, in particular we have $Q \neq 0$. Note that this implies that f is dominant. Let r be the rank of Q. Since $\deg(L) = 1$, Riemann–Roch on X gives

$$\begin{aligned}&\chi(X, Q \otimes L) - \chi(X, Q)\\ &\quad = (r + \deg(Q) + r(1-g)) - (\deg(Q) + r(1-g))\\ &\quad = r.\end{aligned}$$

Thus $r = 1$.

Step (4). Proof of (i)

Since $(f, f^\sharp) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ is dominant (by step (3)) and X is reduced, the corresponding homomorphism $f^\sharp : \mathcal{O}_X \longrightarrow f_*(\mathcal{O}_Y)$ is injective (see EGA [2], Proposition 5.4.3, p. 284). Since $\mathrm{rank}(f_*(\mathcal{O}_Y)) = 1$ (by step (3)), we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f_*(\mathcal{O}_Y) \longrightarrow T' \longrightarrow 0$$

where T' is torsion. Let $V = X - \mathrm{Supp}(T')$ and U a non-empty open subscheme of X such that $f^{-1}(U) \longrightarrow U$ is finite (see step (1)). Then $f' : f^{-1}(U \cap V) \longrightarrow U \cap V$, $(f' = f|_{f^{-1}(U \cap V)})$ is finite (and, hence, affine) and $f'_*(f^{-1}(U \cap V), \mathcal{O}_{f^{-1}(U \cap V)}) = \mathcal{O}_{U \cap V}$. Hence f' is an isomorphism. Let Y_0 be the schematic image (see EGA [2], 6.10, pp. 324–325) of the open inclusion $f^{-1}(U \cap V) \hookrightarrow Y$. Since $f^{-1}(U \cap V)$ is reduced, Y_0 is the reduced structure on $\overline{f^{-1}(U \cap V)}$ induced by Y. Then Y_0 is also irreducible and, hence, by Zariski's main theorem, $f|_{Y_0} : Y_0 \longrightarrow X$ is an isomorphism. Since $f' : f^{-1}(U \cap V) \longrightarrow U \cap V$ is an isomorphism, we see that Y_0 is the only component of Y which dominates X. In the notation of the statement (i) of the lemma, we have $D_{red} = Y_0$.

Step (5). Proof of (ii)

Suppose now that Y has only one irreducible component D of dimension 1. Let D_{red} be the reduced subscheme of Y with support D. Then we have a short exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{D_{red}} \longrightarrow 0.$$

Since $f^{-1}(U \cap V)$ is reduced and the other components, if any, are zero-dimensional, we see that I is supported at finitely many points. Now by hypothesis, $\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) = \chi(D_{red}, \mathcal{O}_{D_{red}})$ as $D_{red} \longrightarrow X$ is an isomorphism. Since

$$\chi(Y, \mathcal{O}_Y) = \chi(Y, \mathcal{O}_{D_{red}}) + h^0(Y, I) = \chi(X, \mathcal{O}_X) + h^0(Y, I),$$

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we see that $h^0(Y, I) = 0$, and since I is torsion, $I = 0$. Thus in this case $f : Y \longrightarrow X$ is an isomorphism.

Step (6). Proof of (iii)

Suppose that Y has other one-dimensional components apart from D.

Let $D_1, \ldots, D_k$ $(k \geqslant 1)$ be the other one dimensional components by $P_1, \ldots, P_l$ the 0-dimensional components. Let $Y' = Y - \{P_1, \ldots, P_l\}$, considered as an open subscheme of Y. Let $W = Y' - \{$points of intersection of two distinct components$\}$, considered as an open subscheme of Y. Let W^s be the schematic closure of W in Y'. Similarly define D_i^s for any component to be the schematic closure in Y' of $D_i - \{$points of intersection of D_i with the other components$\}$. Observe that $D^s = D_{red}$ (see step (4)). Note that D^s and D_i^s are closed subschemes of W^s. We have a short exact sequence

$$0 \longrightarrow T_1 \longrightarrow \mathcal{O}_{Y'} \longrightarrow \mathcal{O}_{W^s} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{W^s} \longrightarrow \mathcal{O}_{D^s_{red}} \oplus \mathcal{O}_{D^s_1} \oplus \ldots \oplus \mathcal{O}_{D^s_k} \longrightarrow T_2 \longrightarrow 0.$$

where T_1 and T_2 are supported at finite number of points.

For the line bundle ξ on Y, we have

$$\begin{aligned}\deg(Y, \xi) &= \deg(Y', \xi) = \deg(W^s, \xi)\\ &= \deg(D^s_{red}, \xi) + \sum_{i=1}^{k} \deg(D^s_i, \xi).\end{aligned}$$

Now $(D_i^s)_{red}$ is the same as the reduced scheme structure $D_{i,red}$ induced on D_i by Y'. Since by hypothesis, $\xi|_{D_{i,red}}$ is ample, $\xi|_{D_i^s}$ is ample too as can be seen. Hence $\deg(D_i^s, \xi) > 0$ for each i. Thus, as $D_{red} = D^s$, we get

$$\deg(D_{red}, \xi) = \deg(D^s, \xi) < \deg(Y, \xi).$$

This completes the proof of the Proposition 2.2. □

We now go back to proving the Theorem 1.1. The above Lemma is used in the proof of the following proposition.

PROPOSITION 2.3. *Let σ be a minimal section of π: $M \longrightarrow X$ as defined earlier. Let $\mathcal{H}$ be the Hilbert scheme of closed subschemes of M (we may restrict ourselves to* $\mathrm{Hilb}^P(M)$ *where P is the Hilbert polynomial of σ, with respect to an ample line bundle). Let Y be the closed subscheme of M, represented by a point of $\mathcal{H}$ which lies in an irreducible component containing the Hilbert point of $\sigma_0(X)$. Then $\pi|_Y : Y \longrightarrow X$ is an isomorphism.*

Proof. Let L be a line bundle of degree 1 on X. Let η be the line bundle $\det(T_\pi)$ on M, where T_π is the tangent bundle along the fibres of π. Consider the diagram

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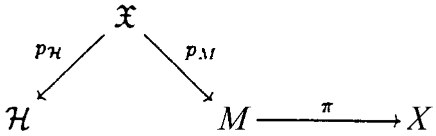


where $p_{\mathcal{H}} : \mathfrak{X} \longrightarrow \mathcal{H}$ is the universal family which is a flat morphism. By considering the line bundles $\mathcal{O}_{\mathfrak{X}}$, $p_M^*\pi^*(L)$ and $p_M^*(\eta)$ and using the fact that Euler characteristics are locally constant in a flat family of coherent sheaves, we see that

$$\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) \quad \text{and} \quad \chi(Y, f^*(L)) = \chi(X, L) = 1,$$

where $f = \pi|_Y$ and $\chi(Y, \xi) = \chi(X, \sigma_0^*(\eta))$, where $\xi = \eta|_Y$.

Now apply Lemma 2.2 to the morphism f. Using the notation of that proposition, if D is the only irreducible component of dimension one which dominates X, then by (ii) of the proposition, $f : Y \longrightarrow X$ is an isomorphism. Suppose there were other one-dimensional components $D_1, D_2, \ldots, D_k$. Now $D_{i,red}$ is contained as a closed subschemes of a fibre of f. Since the restriction of η to any fibre of the map M is ample we conclude that $\xi|_{D_{i,red}}$ is ample. Let τ be the section of $M \longrightarrow X$ defined by the inverse of the isomorphism $f|_{D_{red}} : D_{red} \longrightarrow X$. We would then have

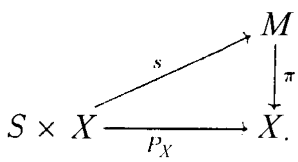
$$\deg(X, \tau^*(T_\pi)) = \deg(D_{red}, \xi) < \deg(Y, \xi) = \deg(X, \sigma^*(T_\pi)),$$

by (iii) of the Lemma 2.2 if there are other components. But this would contradict that σ is a minimal section. Hence D is the only component of Y. This completes the proof of the proposition. □

Remark. The Hilbert scheme $\mathcal{H}$ has an open subscheme $\Pi(M/X)$, which consists of Hilbert points of all sections of $\pi : M \longrightarrow X$ (see FGA [1], TDTE, IV, SS 4c, pp. 19–20).

LEMMA 2.4. *Suppose that S is an irreducible component of $\Pi(M/X)$ which is proper over k. Then* $\dim(S) \leqslant \dim(G/P)$.

Proof. The restriction $s : S \times X \longrightarrow M$ of the universal morphism $\Pi(M/X) \times X \longrightarrow M$ makes the following diagram commute.



We claim that s is a finite morphism. As by assumption S is proper over k, the morphism s is proper, it is enough to check that the fibres of s are zero-dimensional. Suppose $y_0 \in M$, such that $\dim(s^{-1}(y_0)) \geqslant 1$. Then $s^{-1}(y_0)$ is of the form $B_0 \times \{x_0\}$ where $x_0 = \pi(y_0)$, $B_0 \subset S$. We can find a closed sub-variety B of B_0 with $\dim(B) \geqslant 1$, with $s(B \times \{x_0\}) = y_0$. Consider the morphism $s|_{B\times X} : B \times X \longrightarrow M$.

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Now B is complete, since S is so. Since $s(B \times \{x_0\}) = y_0$, by rigidity Lemma (Mumford [5], II.4, p. 43) s factors through X, that is, there exists a morphism $\phi : X \longrightarrow M$ such that $s = \phi \circ p_X$. This is a contradiction as $\dim(B) \geqslant 1$ (compare Mukai and Sakai [12], pp. 254–255). □

LEMMA 2.5. *Let S be an irreducible component of the Hilbert scheme $\mathcal{H}$ which contains the Hilbert point of a minimal section σ. Then S lies in $\Pi(M/X)$, and* $\dim(S) \leqslant \dim(G/P)$.

Proof. As S is closed in $\mathcal{H}$, and $\mathcal{H}$ is proper over k, it follows that S is proper over k. By Proposition 2.3, every point of S is the Hilbert point of some section of $\pi : M \longrightarrow X$, hence S is contained in the open subscheme $\Pi(M/X)$ of $\mathcal{H}$. Therefore, S is an irreducible component of $\Pi(M/X)$. Hence, by Proposition 2.4, we have the desired conclusion.

Proof of the Theorem 1.1. Let σ be a minimal section, and N_σ be the normal bundle of $\sigma(X)$ in M. By deformation theory, it is known that the dimension of the Hilbert scheme $\mathcal{H}$ at a point $\sigma(X)$ satisfies the inequality (see Mori [11], Proposition 3)

$$\dim_{[\sigma(X)]}(\mathcal{H}) \geqslant h^0(X, N) - h^1(X, N).$$

By Lemma 2.5 we have $\dim_{[\sigma(X)]}(\mathcal{H}) \leqslant \dim(G/P)$. On the other hand by Riemann–Roch we have

$$h^0(X, N_\sigma) - h^1(X, N_\sigma) = \deg(N_\sigma) + \dim(G/P)(1 - g).$$

Hence it follows that $\deg(N_\sigma) \leqslant g \cdot \dim(G/P)$. This completes the proof of the Theorem 1.1. □

Remark 2.6. In the case of a Borel subgroup it is easier to prove the existence of a section σ such that $\deg(N_\sigma) \leqslant C$, where C is a constant which depends on the genus of the curve and the group G, but not on the particular G-bundle. In fact by a result of Harder ([6], satz 2.2.6) there exists a reduction σ to B, a Borel subgroup, such that if L_{α_i} is the line bundle associated to a simple root α_i we have $\deg(L_{\alpha_i}) \geqslant 2g$. Now $\det(N_\sigma)$ is the line bundle associated to the character of B defined by $(-\sum_{\alpha>0} \alpha)$, sum over all positive roots. Now

$$\left(-\sum_{\alpha>0} \alpha\right) = \left(-\sum m_i \alpha_i\right),$$

where α_i's are simple roots taken with respect to a fixed maximal torus contained in B and $m_i > 0$ depending only on the group G. Hence

$$\begin{aligned} \deg(N_\sigma) &= -\sum m_i \deg(L_{\alpha_i}) \\ &\leqslant \left(\sum m_i\right) \cdot 2g. \end{aligned}$$

This remark is sufficient for the applications we have in mind.

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3. Boundedness for Semi-Stable G-Bundles

In this section we use the results of the previous section to prove the boundedness of semi-stable G-bundles of a fixed degree on a smooth projective curve X over an algebraically closed field k of arbitrary characteristic, where G is a connected reductive algebraic group over k.

For any algebraic group G, a set $\mathcal{S}$ of principal G-bundles on X is called *bounded* if there exists a scheme $\mathcal{S}$ of finite type over k, and a family of principal G-bundles parametrised by $\mathcal{S}$, such that each element of $\mathcal{S}$ is isomorphic on X to the G-bundle on X obtained by restriction of the given family to some point of $\mathcal{S}$.

PROPOSITION 3.1. *Let B be a Borel subgroup of the reductive group G and $T = B/B_u$, where B_u the unipotent radical of B. Let $\mathcal{B}_T$ be a bounded set of T-bundles on X, and let $\mathcal{B}$ be a set of G-bundles on X such that each member of $\mathcal{B}$ admits a reduction of structure group to B such that the associated T-bundle is isomorphic to a member of $\mathcal{B}_T$. Then $\mathcal{B}$ is a bounded set of G-bundles.*

Proof. We first prove it in the case of $G = GL(n)$ and $B =$ upper triangular matrices. We may identify principal $GL(n)$-bundles with their associated vector bundles. By hypothesis, each vector bundle E in $\mathcal{B}$ admits a full flag $0 \subset E_1 \subset E_2 \subset \ldots \subset E_n = E$ such that the degrees of the line bundles $E_i/E_{i-1} (i = 1, \ldots, n)$ are bounded. Since line bundles of a given degree form a bounded family and extensions of vector bundles in bounded families form a bounded family (see FGA [1], 4, Proposition 1.2, p. 221), the proposition is proved in this case.

In the general case, we embed G as a closed subgroup of $GL(n)$ for some n. Let B_1 (resp. B) be a Borel subgroup of $GL(n)$ (resp. G) with $B \subset B_1$. Since B_u is contained in $B_{1,u}$, we have an induced homomorphism of T into T_1, where $T = B/B_u$ and $T_1 = B_1/B_{1,u}$.

Let $\mathcal{B}'$ be the set of $GL(n)$ bundles obtained from $\mathcal{B}$ by extension of structure group via $G \hookrightarrow GL(n)$. From the commutative diagram

$$\begin{array}{ccc}
G & \longrightarrow & GL(n) \\
\uparrow & & \uparrow \\
B & \longrightarrow & B_1 \\
\downarrow & & \downarrow \\
T & \longrightarrow & T_1
\end{array}$$

we see that each bundle in $\mathcal{B}'$ has a reduction to B_1, such that the corresponding T_1 bundle is obtained by extension of structure group from an element of the set $\mathcal{B}_T$. Since by hypothesis $\mathcal{B}_T$ is a bounded set, by what has been proved above for $GL(n)$-bundles, $\mathcal{B}_T$ is a bounded set.

Let $\mathcal{P} \longrightarrow X \times W$ be a family of principal $GL(n)$-bundles on X parametrised by a scheme W of finite type over k, such that up to isomorphism all the bundles in

$\mathcal{B}'$ occur in this family. Using $\mathcal{P}$ we shall now construct a family of G-bundles on X parametrised by a scheme S of finite type over k, such that every bundle in $\mathcal{B}$ occurs in this family.

By the results of Grothendieck (see FGA, 221, 4.c), there exists a W-scheme $S = \Pi_{W\times X/W}((\mathcal{P}/G)/W \times X)$ which has the following universal property: for any W-scheme $U \longrightarrow W$, the set of sections of $(\mathcal{P}/G)_U \longrightarrow X \times_W U$ is in bijective correspondence with the set of sections of S_U over U. In particular, for $w \in W$, the fibres of $S \longrightarrow W$ consists of the sections of the fibre bundle $\mathcal{P}_w/G \longrightarrow X$, where $\mathcal{P}_w = \mathcal{P}|w \times X$, and these are exactly the reductions of the $GL(n)$ bundle $\mathcal{P}_w$ to G.

Therefore, the universal section of $(\mathcal{P}/G)_S \longrightarrow X \times_W S$ gives a family of G-bundles parametrised by S, in which each bundle from the set $\mathcal{B}$ occurs. Finally, as G and $GL(n)$ are reductive groups, $GL(n)/G$ is affine, and there is a representation of $GL(n)$ on a vector space V which gives a $GL(n)$-equivariant closed embedding of $GL(n)/G \hookrightarrow V$. Now it is clear that the scheme S is a closed subscheme of the scheme $S' = \Pi_{W\times X/W}(\tilde{V}/W \times X)$, where $\tilde{V}$ is the vector bundle associated to $\mathcal{P}$ by the representation of $GL(n)$ on V. Hence, S is of finite type over k (see Ramanathan [4], Remark 4.8.2, p. 425). This completes the proof of the Proposition 3.1. □

Let G be a connected reductive group. Let $\mathcal{X}^*(G) = \mathrm{Hom}(G, k^*)$. Let Z be the center of G and Z^0 its connected component of identity. Then $G = Z^0 \cdot [G, G]$ and $Z^0 \cap [G, G]$ is finite. Thus $\mathcal{X}^*(G)$ is a subgroup of $\mathcal{X}^*(Z^0)$ of maximal rank.

If E is a G-bundle on X, we have a homomorphism $d_E : \mathcal{X}^*(G) \longrightarrow \mathbb{Z}$ given by $\chi \mapsto \deg(E_\chi)$, where E_χ is the line bundle associated to E by χ.

DEFINITION 3.2. *We shall call the element* $d_E \in \mathrm{Hom}(\mathcal{X}^*(G), \mathbb{Z})$ *the degree of* E. When $G = GL(n)$, the above definition reduces to the usual definition of the degree of the associated rank n vector bundle, as $\mathcal{X}^*(GL(n)) = \mathbb{Z}$. Also note that if G is semi-simple then d_E is zero as $\mathrm{Hom}(\mathcal{X}^*(G), \mathbb{Z}) = 0$. We have the following:

LEMMA 3.3. *Let* $T = GL(1)^l$ *be a torus and* $L \subset \mathcal{X}^*(T)$ *be a subgroup of maximal rank. For a* T*-bundle* F *on* X*, let* $d_F : \mathcal{X}^*(T) \longrightarrow \mathbb{Z}$ *be the homomorphism as above, and* $d'_F : L \longrightarrow \mathbb{Z}$ *be the restriction of* d_F *to* L*. If* $\mathcal{S}$ *is a set of* T*-bundles on* X *such that the set* $\{d'_F | F \in \mathcal{S}\}$ *is a finite set, then* $\mathcal{S}$ *is a bounded set of* T*-bundles.*

Proof. We reduce the proof to the case where $L = \mathcal{X}^*(T)$ as follows. If $L \subset \mathcal{X}^*(T)$ is an arbitrary subgroup of maximal rank, then there exists a basis $\{\chi_1, \ldots, \chi_l\}$ of $\mathcal{X}^*(T)$ such that $\{\lambda_1\chi_1, \ldots, \lambda_l\chi_l\}$ forms a basis for L, with $\lambda_i \in \mathbb{Z}$, $\lambda_i \neq 0$ for each i. Since $d_F(\chi_i) = \lambda_i^{-1} d_F(\lambda_i\chi_i)$, the result is true for L if it is true for $\mathcal{X}^*(T)$.

Hence we can assume that $L = \mathcal{X}^*(T)$. Let $\{\chi_1, \ldots, \chi_l\}$ be a basis of $\mathcal{X}^*(T)$. Then by our hypothesis the set $N_0 = \{d'_F(\chi_i) | F \in \mathcal{S}, 1 \leqslant i \leqslant l\}$ is a finite set of integers. Thus the set $\mathcal{S}$ can be considered as a subset of the set of all l-tuples ($l = \dim(T)$)

$$\{(L_1, \ldots, L_l) | L_i \in \mathrm{Pic}(X) \quad \text{with} \quad \deg(L) \in N_0\}.$$

Hence $\mathcal{S}$ is a bounded set. □

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PROPOSITION 3.4. *Let G be a connected semi-simple group. Then the family of semi-stable G-bundles on X is bounded.*

Proof. Let $\mathcal{S}$ be the set of G-bundles such that every element is semi-stable. We shall show that each member E of $\mathcal{S}$ admits a reduction of structure group to B such that the associated T-bundles E_T (as E varies in $\mathcal{S}$) form a bounded family. We then apply the Proposition 3.1 to complete the proof. For any principal G-bundle E, by Remark 2.6, we can choose a reduction σ of the structure group to B such that $\deg(N_\sigma) \leqslant C$, where C is a constant independent of E. To show that the set of associated T-bundles $\{E_T\}$ is bounded, we will show that there is a subgroup L of $\mathcal{X}^*(T)$ of maximal rank with the property that $\{(d_{E_T}|_L) \mid E \in \mathcal{S}\}$ is a finite set and then use Lemma 3.3.

Let $\Lambda_1, \ldots, \Lambda_l$ be the set of fundamental weights with respect to a maximal torus contained in B and the positive roots being contained in the Lie algebra of B. Let m be a positive integer with the property that $m\Lambda_i$ is a character of T for every i. Let L be the subgroup of $\mathcal{X}^*(T)$ generated by $\{m\Lambda_i \mid 1 \leqslant i \leqslant l\}$. Then we observe that L is of maximal rank. Now the line bundle $\det(N_\sigma)^{\otimes m}$ is associated to the character

$$-2\sum_{i=1}^{l}(m\Lambda_i) = -\sum_{\alpha>0\,\mathrm{root}} m\alpha.$$

Hence for each E_T as above we have the condition

$$\sum_{i=1}^{l} d_{E_T}(m\Lambda_i) = -\deg(\det(N_\sigma)^{\otimes m})/2 \geqslant -mC/2,$$

where $d_{E_T}(m\Lambda_i)$ is the degree of the line bundle associated to E_T by the character $m\Lambda_i$. On the other hand, if E is semi-stable then for any reduction of structure group to B the degree of the line bundle associated to a dominant character of B is $\leqslant 0$ (see Ramanathan [3]). Thus we have $d_{E_T}(m\Lambda_i) \leqslant 0$. This together with the above inequality implies that $-mC/2 \leqslant d_{E_T}(m\Lambda_i) \leqslant 0$ for each i. Hence $\{(d_{E_T}|_L) \mid E \in \mathcal{S}\}$ is a finite set. This completes the proof. □

Proof of the Theorem 1.2. Let $\mathcal{S}'$ be the set of semi-stable G-bundles with a fixed degree. For each element E of $\mathcal{S}'$ we choose a reduction σ of structure group to B with $\deg(N_\sigma) \leqslant C$, C independent of E. We shall show that the associated T-bundles form a bounded family and apply Proposition 3.1. This will be shown by proving that there is a subgroup L of maximal rank in $\mathcal{X}^*(T)$ such that $\{(d_{E_T}|_L) \mid E \in \mathcal{S}'\}$ is a finite set and then using the Lemma 3.3.

Note that $T' = T/Z^0$ is a maximal torus of $G' = G/Z^0$, contained in its Borel subgroup $B' = B/Z^0$. As we have the isomorphism $G/B \cong G'/B'$, it follows that the G'-bundle E' obtained from E by extension of structure group is semi-stable, and σ gives rise to a reduction σ' of structure group of E to B'. We also observe that any dominant character vanishes on Z^0.

Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X}^*(T/Z^0) & \longrightarrow & \mathcal{X}^*(T) & \longrightarrow & \mathcal{X}^*(Z^0) & \longrightarrow & 0 \\ & & & & \uparrow & \nearrow & & & \\ & & & & \mathcal{X}^*(G) & & & & \end{array}$$

where the row is exact. As already remarked, $\mathcal{X}^*(G)$ is a subgroup of maximal rank in $\mathcal{X}^*(Z^0)$. Hence the subgroup L generated by $\mathcal{X}^*(G)$ and $\mathcal{X}^*(G/Z^0)$ is of maximal rank in $\mathcal{X}^*(T)$. The set $\{d_{E_T}|_L \mid E \in \mathcal{S}'\}$ is finite because $d_{E_T}|_{\mathrm{im}(\mathcal{X}^*(G))}$ is fixed while $\{d_{E_T}|_{\mathrm{im}(T/Z^0)} \mid E \in \mathcal{S}'\}$ is a finite set as shown in Lemma 3.4, since G' is semi-simple and E' is semi-stable. Now the theorem follows from the Lemma 3.3 and Proposition 3.1. □

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