A DECISION PROBLEM FOR TRANSFORMATIONS OF TREES

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For a given class of transformations on graphs there arises naturally the problem of deciding when one graph can be transformed into another by a finite sequence of such transformations. We consider, in this paper, the special case of this problem when the graphs are finite trees and the transformations consist of rearranging the order of the segments from a point and replacing subtrees by other trees according to a given set of pairs of interchangeable trees. This decision problem is, in fact, equivalent to the word problem for one-generator algebras in a certain variety of algebraic systems and we exhibit a procedure for solving the word problem for these algebras.

1. Transformation of trees. By a *tree*, we shall mean a finite rooted tree, and we assign a direction to each segment by directing, from the root, the segments with the root as one end, then assigning directions consistent with this to all other segments. In addition, we shall assume that an order is associated with the set of segments from each point.

If s_i is a segment from point P_i to P_{i+1}, $i = 1, 2, \ldots, n$, in a tree, then the sequence $P_1, s_1, P_2, \ldots, s_n, P_{n+1}$ is called the path from P_1 to P_{n+1}. A point with no segment from it is called a *free end* and the set of all paths from a point P to free ends is called the *subtree* with root P. If Q is the other end of a segment from P, then the subtree with root Q is called a *principal subtree* of the subtree with root P. The order associated with the segments from a point P induces a corresponding order on the set of principal subtrees of the subtree with root P.

Let T be a tree and S a subtree of T with root P. The replacement of S by a tree S', identifying P and the root of S', transforms T into a tree T' which we say is obtained from T by the substitution (S, S') at P. If S is a single point, then the substitution (S, S') at P consists of adding the tree S' at the free end P of T. If S' is a single point, then the substitution (S, S') at P consists of deleting the subtree S from T.

Let T be a tree and S a subtree of T with root P. Let $S_1, S_2, \ldots, S_k$ be the principal subtrees of S. If ϕ is a permutation of $\{1, 2, \ldots, k\}$, then the substitution for S at P of a tree having S_i as its ϕith principal subtree, $i = 1, 2, \ldots, k$, transforms T into a tree T' which we say is obtained from T by applying ϕ at P.

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Let R be a set of pairs of trees $\{(S_i, S_i'), i = 1, 2, \ldots\}$. An *$R$-transformation* of a tree T is a transformation (S_i, S_i') or (S_i', S_i) at a point in T. Two trees are called *R-equivalent* if one can be transformed into the other by a finite sequence of R-transformations.

Let Σ be a countably infinite set of permutation groups G_k, $k = 2, 3, 4, \ldots$, where G_k consists of permutations on $\{1, 2, \ldots, k\}$. A *Σ-transformation* of a tree T is an application of a permutation ϕ in G_k at a point in T having k segments from it. Two trees will be called *Σ-equivalent* if one can be obtained from the other by a finite set of Σ-transformations.

Two trees will be said to be *equivalent* with respect to the substitutions R and the permutations Σ if one can be transformed into the other by a finite sequence of transformations each of which is either an R-transformation or a Σ-transformation. We shall consider the problem of deciding for an arbitrary pair of trees and arbitrary R, Σ, whether the trees are equivalent with respect to R and Σ.

2. Tree algebras. Let Σ be a countably infinite set of permutation groups, G_k, $k = 2, 3, 4, \ldots$, where G_k consists of permutations of $\{1, 2, \ldots, k\}$. By a *Σ-commutative tree algebra*, we mean an algebra having a non-empty set of elements and a countably infinite set of operations Π_k, $k = 1, 2, 3, \ldots$, where Π_k assigns to each ordered sequence $x_1, x_2, \ldots, x_k$ of k elements of the algebra an element $\Pi_k\, x_1x_2 \ldots x_k$ in the algebra. Furthermore, we assume that, for each $k \geqslant 2$ and each permutation ϕ in G_k, the operation Π_k satisfies the identity

$$\Pi_k\, x_1x_2 \ldots x_k = \Pi_k\, x_{\phi 1}x_{\phi 2} \ldots x_{\phi k} \qquad \text{for all } x_1, x_2, \ldots, x_k.$$

We now describe a Σ-commutative tree algebra given by generators $g_1, g_2, g_3, \ldots$ and relations $S_i = S_i'$, $i = 1, 2, 3, \ldots$. A *word* in the generators is (i) a generator, (ii) a sequence $\Pi_k\, u_1u_2 \ldots u_k$ where $u_1, u_2, \ldots, u_k$ are words. The words $u_1, u_2, \ldots, u_k$ are called the *principal subwords* of $\Pi_k\, u_1, u_2 \ldots u_k$. A word u is called a *subword* of v if it is v itself or if it is a subword of a principal subword of v.

Let W be a set of pairs of words $\{(s_i, s_i'), i = 1, 2, 3, \ldots\}$ in the generators $g_1, g_2, g_3, \ldots$. Replacing a subword s_i or s_i' of a word u by the other word in the pair (s_i, s_i') is called a *W-transformation* of u and we write $u \leftrightarrow_W v$ to indicate that u can be transformed into v by one W-transformation. Two words are called *W-equivalent* if one can be transformed into the other by a finite sequence of W-transformations. We indicate this by $u =_W v$.

If $\Pi_k\, u_1u_2 \ldots u_k$ is a subword of a word u and ϕ is a permutation in G_k, then replacing $\Pi_k\, u_1u_2 \ldots u_k$ by $\Pi_k\, v_1v_2 \ldots v_k$, where $u_i = v_{\phi i}$, is called a *Σ-transformation* of u and we write $u \leftrightarrow_\Sigma v$ to indicate that v is obtained from u by one Σ-transformation. We say that u is *Σ-equivalent* to v and write this as $u =_\Sigma v$ if u can be transformed into v by a finite sequence of Σ-transformations.

We define two words u, v to be equivalent, $u = v$, if one can be transformed into the other by a finite sequence of transformations, each of which is either a W-transformation or a Σ-transformation. We define operations Π_k on the classes of equivalent words $\{w\}$ by $\Pi_k \{u_1\}\{u_2\} \ldots \{u_k\} = \{\Pi_k u_1 u_2 \ldots u_k\}$. The Σ-commutative tree algebra having the classes of equivalent words as elements, and with the operations defined as above, is called the Σ-commutative tree algebra generated by $g_1, g_2, g_3, \ldots$ with defining relations $\{s_i = s_i',$ $i = 1, 2, 3, \ldots\}$.

By the word problem for Σ-commutative tree algebras, we mean the question of the existence of an algorithm for deciding for any Σ-commutative tree algebra given by a finite set of generators and defining relations whether any two words in the algebra are equivalent. We shall show in the next section that a special case of this problem is equivalent to the decision problem for trees described earlier.

3. An algebraic representation for trees. Let A be a Σ-commutative tree algebra, generated by one element a. We first set up a one–one correspondence between the set of words in a and the set of finite rooted trees.

(i) To the tree consisting of a single point, we assign the generator a.

(ii) To the tree T having $T_1, T_2, \ldots, T_k$ as principal subtrees, we assign the word $\Pi_k u_1 u_2 \ldots u_k$, where $u_1, u_2, \ldots, u_k$ are the words assigned to $T_1, T_2, \ldots, T_k$ respectively.

Let $\{(S_i, S_i'), i = 1, 2, 3, \ldots\}$ be a set R of pairs of trees and let W be the set $\{(s_i, s_i'), i = 1, 2, 3, \ldots\}$ of pairs of words in a, where s_i corresponds to S_i, s_i' to S_i' under the correspondence described above. Let U, V be trees and let u, v be the words in a corresponding to these trees. The following theorem shows that our transformation problem for trees is equivalent to a special case of the word problem for Σ-commutative tree algebras.

THEOREM 1. *U, V are equivalent with respect to R and Σ if and only if $u = v$ in the Σ-commutative tree algebra generated by a and with defining relations* $\{s_i = s_i', i = 1, 2, 3, \ldots\}$.

Proof. If S is a subtree of a tree T, we define the place of S in T as a sequence of positive integers such that (i) the place of T in T is the empty sequence, (ii) the place in T of a subtree S of the principal subtree T_i of T is the place of S in T_i preceded by i. Similarly, we define the place of a subword s of a word t. The following statements are all proved by simple inductions on the number of terms in the places of the subtrees involved and we omit the details.

(i) If the trees S, T correspond to the words s, t, then S is a subtree of T at the place p in T if and only if s is a subword of t at the place p in t.

(ii) Let the trees U, V, S, S' correspond to the words u, v, s, s'. Then V is obtained from U by the substitution (S_i, S_i') at the place p in U, if and only if v is obtained from u by the substitution of s_i' for s_i at the place p in u.

(iii) Let the trees U, V, S correspond to the words u, v, s. Then V is obtained from U by the application of the permutation ϕ to the subtree S of U at place p in U, if and only if v is obtained from u by the application of ϕ to the subword s of u at place p in u.

The theorem follows by repeated applications of (ii) and (iii) above.

4. The word problem for tree algebras. If A is a Σ-commutative tree algebra given by generators and relations, we say that the defining relations of A are a set of partial operation tables if the following conditions hold:

(i) Each defining relation is of the form

$$\Pi_k\, x_1x_2 \ldots x_k = x_p,$$

where the x_i are generators.

(ii) There are no two defining relations $\Pi_k\, x_1x_2 \ldots x_k = x_p$, $\Pi_k\, x_1x_2 \ldots x_k = x_q$ with x_p, x_q different generators.

(iii) If $\Pi_k\, x_1x_2 \ldots x_k = x_p$ is a defining relation, then

$$\Pi_k\, x_{\phi 1}x_{\phi 2} \ldots x_{\phi k} = x_p$$

is a defining relation, for each ϕ in the permutation group G_k of Σ.

The word problem will be solved for finitely generated tree algebras whose defining relations are a finite set of partial operation tables. That this is equivalent to a solution for tree algebras with an arbitrary finite set of defining relations follows from the next theorem, which is a special case of a theorem proved in **(1)** (the theorem proved in **(1)** actually assumes that the set of operations and the set of identities which the algebra satisfies are finite, but it is easily seen that methods used in **(1)** carry over to finitely generated tree algebras).

THEOREM 2. *Let A be a Σ-commutative tree algebra given by a finite set of defining relations. Then we can construct, in a finite number of steps, a Σ-commutative tree algebra A^* isomorphic to A and such that the defining relations of A^* are a finite set of partial operation tables.*

Let A be a Σ-commutative tree algebra whose defining relations are a set of partial operation tables. We introduce a further notation for transformations in addition to those given earlier. A W-transformation of a word u in the generators of A which involves replacing a subword of u of the form $\Pi_k\, x_1x_2 \ldots x_k$ by x_p, where $\Pi_k\, x_1x_2 \ldots x_k = x_p$ is a defining relation of A, will be called a *W-contraction* of u. We write $u \rightarrow_W v$ to indicate that v is obtained from u by a W-contraction. The inverse of a contraction of a word will be called a *W-expansion.* A word for which no further W-contractions are possible will be said to be W-contracted.

We list first a sequence of lemmas which lead to a solution of the word problem for A and return later to the proofs of these lemmas.

LEMMA 1. (i) *If u, v, z are words in A such that $u \leftrightarrow_\Sigma z \leftrightarrow_W v$, then either $u \leftrightarrow_W v$ or there is a word t such that $u \leftrightarrow_W t \leftrightarrow_\Sigma v$.*

(ii) *If u, v, z are words in A such that $u \leftrightarrow_W z \leftrightarrow_\Sigma v$, then either $u \leftrightarrow_W v$ or there is a word t such that $u \leftrightarrow_\Sigma t \leftrightarrow_W v$.*

LEMMA 2. *If $u = v$ in A, then there are words s, t such that $u =_W s =_\Sigma v$ and $u =_\Sigma t =_W v$.*

LEMMA 3. *If $u \rightarrow_W \ldots \rightarrow_W s$, $u \rightarrow_W \ldots \rightarrow_W t$ are two sequences of W-contractions such that s, t are W-contracted words, then s and t are identical.*

We shall denote this unique W-contracted word obtained from a word u by $\bar{u}$.

LEMMA 4. *If $u =_W v$, then $\bar{u}$, $\bar{v}$ are identical.*

LEMMA 5. *If $u \leftrightarrow_\Sigma v$, then u is W-contracted if and only if v is W-contracted.*

LEMMA 6. *If $u =_\Sigma v$, then u is W-contracted if and only if v is W-contracted.*

LEMMA 7. *If $u = v$, then $\bar{u} =_\Sigma \bar{v}$.*

That is, two words in A are equivalent if and only if their W-contracted forms are Σ-equivalent. Thus, to decide whether two words u, v in A are equivalent, we first calculate their W-contracted forms u, v. Now a word has only a finite number of words which are Σ-equivalent to it and these can be enumerated. Thus, in a finite number of steps, we can decide if u, v are Σ-equivalent and hence decide if $u = v$ in A.

THEOREM 3. *The word problem for Σ-commutative tree algebras is solvable.*

In view of Theorem 1, we may also claim the following.

THEOREM 4. *There is an algorithm for deciding whether two trees are equivalent relative to a given set R of substitutions and a given set Σ of permutation groups.*

We now return to the proofs of the lemmas listed above. We use frequently the *length* of a word in the generators of A, where this is defined by:

(i) The length of a generator is 1.

(ii) If the words $u_1, u_2, \ldots, u_k$ are of length $n_1, n_2, \ldots, n_k$, respectively, then the length of $\Pi_k\, u_1 u_2 \ldots u_k$ is $1 + n_1 + n_2 + \ldots + n_k$.

Proof of Lemma 1. The proof is by induction on the length of u. We shall prove part (i) only, since the procedure for the proof of (ii) is similar. If the length of u is one, then clearly (i) is true. Assume the truth of (i) for words of length less than n and consider the case where u is of length n.

Let u be $\Pi_k\, u_1 u_2 \ldots u_k$. If the transformation $u \leftrightarrow_\Sigma z$ occurs in a principal subword u_i transforming it into u_i' and the transformation $z \leftrightarrow_W v$ occurs in a different principal subword u_j transforming it into u_j', then we may take t as $\Pi_k\, u_1 \ldots u_i \ldots u_j' \ldots u_k$.

If the transformation $u \leftrightarrow_\Sigma z$ occurs in the principal subword u_i of u transforming it into u_i' and if the transformation $z \leftrightarrow_W v$ occurs in u_i' transforming

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it into u_i'', then, by our inductive hypothesis, either there is a word s and transformations $u_i \leftrightarrow_W s \leftrightarrow_\Sigma u_i''$ or else there is a transformation $u_i \leftrightarrow_W u_i''$. Thus, either there is a t, namely $\Pi_k u_1 \ldots u_{i-1} s u_{i+1} \ldots u_k$, such that $u \leftrightarrow_W t \leftrightarrow_\Sigma v$ or else v is $\Pi_k u_1 \ldots u_i'' \ldots u_k$ and there is a transformation $u \leftrightarrow_W v$.

If $u \leftrightarrow_\Sigma z$ involves all the principal subwords of u so that z is

$$\Pi_k u_{\phi 1} u_{\phi 2} \ldots u_{\phi k}$$

and $z \leftrightarrow_W v$ occurs in one of the principal subwords $u_{\phi i}$ of z, transforming it into $u_{\phi i}'$ then we may take t as $\Pi_k u_1 u_2 \ldots u_j' \ldots u_k$, where $j = \phi i$.

If $u \leftrightarrow_\Sigma z$ involves all the principal subwords of u so that z is $\Pi_k u_{\phi 1} u_{\phi 2} \ldots u_{\phi k}$ and $z \leftrightarrow_W v$ also involves all principal subwords of u, then each principal subword is a generator and v is a generator. It follows from property (iii) of a set of defining relations in the form of partial operation tables that there is a transformation $u \leftrightarrow_W v$.

This completes the consideration of all cases and the lemma now follows by induction on the length of u.

Proof of Lemma 2. This follows by repeated applications of Lemma 1. We omit details of the proof since the induction and use of Lemma 1 is straightforward.

Proof of Lemma 3. The proof is similar to that of (**2**, Theorem 2.1) and proceeds by induction on the length of u. If this is 1, the statement of the lemma is vacuously true. Assume the truth of the statement for words of length less than n and consider the case where u is of length n. If both sequences of W-contractions begin with the same W-contraction $u \rightarrow_W v$, then, since $v \rightarrow_W \ldots \rightarrow_W s$, $v \rightarrow_W \ldots \rightarrow_W t$, by our inductive hypothesis s and t are the same word.

Let u be $\Pi_k u_1 u_2 \ldots u_k$ and let the two sequences of W-contractions begin with $u \rightarrow_W v$, $u \rightarrow_W z$, respectively, where v and z are different words. We show that there is a word u' which can be obtained from both v and z by W-contractions. If $u \rightarrow_W v$ occurs in u_i transforming u_i into u_i' and $u \rightarrow_W z$ occurs in u_j, where $i \neq j$, transforming u_j into u_j', then we may take u' as $\Pi_k u_1 \ldots u_i' \ldots u_j' \ldots u_k$ and we have $v \rightarrow_W u'$, $z \rightarrow_W u'$. If both contractions $u \rightarrow_W v$, $u \rightarrow_W z$ occur in the same principal subword u_i, say $u_i \rightarrow_W u_i'$, $u_i \rightarrow_W u_i''$, then by our inductive hypothesis there is a W-contracted word $\bar{u}_i$ such that $u_i \rightarrow_W u_i' \rightarrow_W \ldots \rightarrow_W \bar{u}_i$ and $u_i \rightarrow_W u_i'' \rightarrow_W \ldots \rightarrow_W \bar{u}_i$. Thus, we may take u' as $\Pi_k u_1 \ldots \bar{u}_i \ldots u_k$ and we have two sequences of W-contractions, $u \rightarrow_W v \rightarrow_W \ldots \rightarrow_W u'$, $u \rightarrow_W z \rightarrow_W \ldots \rightarrow_W u'$, where the contractions all occur in the ith principal subword of u.

Let us denote the unique contracted form of a word y of length less than n by $\bar{y}$. By our inductive hypothesis, $\bar{v}$ and $\bar{u}'$ are identical and similarly $\bar{w}$ and $\bar{u}'$ are identical. But $\bar{v}$ is s and $\bar{w}$ is t. Hence, s and t are identical. The lemma now follows by induction.

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Proof of Lemma 4. The proof is similar to that of (**2**, Theorem 2.2) and proceeds by the use of Lemma 3 and a straightforward induction on the number of W-transformations connecting u and v.

Proof of Lemma 5. The proof is by induction on the length of u. If this is 1, the statement is certainly true. Assume the truth of the statement for words of length less than n and consider the case where u is of length n. Let u be $\Pi_k\, u_1u_2 \ldots u_k$ and v be $\Pi_k\, v_1v_2 \ldots v_k$. Either $v_i = u_{\phi i}$, $i = 1, 2, \ldots, k$, or else the transformation $u \leftrightarrow_\Sigma v$ occurs in some principal subword u_j of u, and u_i and v_i are identical for $i \neq j$. In the first case, a W-contraction is possible in a principal subword of u if and only if one is possible in a principal subword of v, or there is a W-contraction involving all principal subwords of u if and only if there is one involving all principal subwords of v in view of property (iii) of a set of defining relations in the form of partial operation tables. In the second case, it follows from our inductive hypothesis that u_j is W-contracted if and only if v_j is W-contracted. Since u_i and v_i are the same for all other i, it follows that u is W-contracted if and only if v is W-contracted. The lemma now follows by induction.

Proof of Lemma 6. This follows from Lemma 5 and is a simple induction on the number of Σ-transformations connecting u and v.

Proof of Lemma 7. We have $u = \bar{u}$ and $v = \bar{v}$. Hence $\bar{u} = \bar{v}$. By Lemma 2, there is a word s such that $\bar{u} =_W s =_\Sigma \bar{v}$. By Lemma 4, $\bar{u}$ and $\bar{s}$ are identical. But, by Lemma 6, since $s =_\Sigma \bar{v}$, s is W-contracted. Hence s and $\bar{s}$ are identical. Thus $\bar{u} =_\Sigma \bar{v}$.

This concludes the proof of Theorem 3. We note that Theorem 3 may also be obtained as a consequence of the results in (**1**) since it is easily verified that a partial tree algebra can be embedded isomorphically in a tree algebra. The proof we have given here, however, has the advantage that it yields information on the structure of tree algebras given by generators and relations.

It is possible to extend the results of this section and obtain a solution of the word problem for tree algebras satisfying further identities which correspond to natural types of transformations on trees. We mention as one example of this the identity $\Pi_1 x = x$, which corresponds to the insertion and removal of segments in a tree.

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