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ON GENERALIZED BOREL SETS

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Abstract

A certain natural extension $\mathscr{B}$ of the Borel σ-algebra is studied in generalized weakly θ-refinable spaces. It is shown that a set belongs to $\mathscr{B}$ whenever it belongs to $\mathscr{B}$ locally. From this it is derived that if $\aleph_\alpha$ is an uncountable regular cardinal which is not two-valued measurable, then the space of all ordinals less than ω_α is more complicated than a union of less than $\aleph_\alpha$ weakly θ-refinable subspaces.

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Given a set A, we shall denote by $|A|$ the cardinality of A and by $\exp A$ the family of all subsets of A. Throughout, by $\aleph$ we shall denote an *uncountable* cardinal.

DEFINITION 1. Let Z be a set. A family $\mathscr{A} \subset \exp Z$ is called an $\aleph$-*algebra* in Z if
(i) $Z \in \mathscr{A}$;
(ii) $A \in \mathscr{A} \Rightarrow Z - A \in \mathscr{A}$;
(iii) $(\{A_\alpha : \alpha \in T\} \subset \mathscr{A}$ and $|T| < \aleph) \Rightarrow \bigcup \{A_\alpha : \alpha \in T\} \in \mathscr{A}$.

DEFINITION 2. Let $\mathscr{A}$ be an $\aleph$-algebra in a set Z. A function $\mu : \mathscr{A} \to [0, +\infty]$ is called an $\aleph$-*measure* on $\mathscr{A}$ if $\mu(\varnothing) = 0$ and

$$\mu(\bigcup \{A_\alpha : \alpha \in T\}) = \sum \{\mu(A_\alpha) : \alpha \in T\}$$

for each disjoint family $\{A_\alpha : \alpha \in T\} \subset \mathscr{A}$ with $|T| < \aleph$.

Thus in our terminology, a σ-additive measure on a σ-algebra will be called an $\aleph_1$-measure on an $\aleph_1$-algebra.

Let $\mathscr{A}$ be an $\aleph$-algebra in a set Z and let μ be an $\aleph$-measure on $\mathscr{A}$. We shall say that μ is *complete* if $A \in \mathscr{A}$ whenever there is a $B \in \mathscr{A}$ such that $A \subset B$ and $\mu(B) = 0$. We shall say that μ is *saturated* if $A \in \mathscr{A}$ whenever $A \cap B \in \mathscr{A}$ for each $B \in \mathscr{A}$ with $\mu(B) < +\infty$.

An uncountable cardinal $\aleph$ is called *measurable* if there is a set Z with $|Z| = \aleph$ and an $\aleph$-measure μ on $\exp Z$ such that $\mu(Z) = 1$ and $\mu(\{z\}) = 0$ for each $z \in Z$.

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If the measure μ takes only values 0 and 1, the cardinal $\aleph$ is called *two-valued measurable*. The basic properties of measurable and two-valued measurable cardinals which do not involve axiomatic set theory are proved in Ulam (1930); more recent results can be found, for example, in Dickmann (1975, Chapter 0, Section 4).

Unless specified otherwise, throughout, X will be an arbitrary topological space. By $\mathscr{G}$ we shall denote the family of all open subsets of X. Let $Y \subset X$. A collection $\{A_\alpha: \alpha \in T\} \subset \exp X$ is called *separated* in Y if $\{A_\alpha: \alpha \in T\} \subset \exp Y$ and there is a family $\{G_\alpha: \alpha \in T\} \subset \mathscr{G}$ such that $\{G_\alpha \cap Y: \alpha \in T\}$ is a disjoint collection and $A_\alpha \subset G_\alpha$ for each $\alpha \in T$.

DEFINITION 3. An $\aleph$-algebra $\mathscr{A}$ in X is called *complete* (abbreviated as c$\aleph$-algebra) if $\bigcup \{A_\alpha: \alpha \in T\} \in \mathscr{A}$ for every collection $\{A_\alpha: \alpha \in T\} \subset \mathscr{A}$ which is separated in some $Y \in \mathscr{A}$.

Clearly, $\exp X$ is a c$\aleph$-algebra in X, and the intersection of any nonempty family of c$\aleph$-algebras in X is again a c$\aleph$-algebra in X. Thus we can define the *Borel c$\aleph$-algebra* in X as the smallest c$\aleph$-algebra $\mathscr{B}_\aleph$ in X containing $\mathscr{G}$. The elements of $\mathscr{B}_\aleph$ will be called *c$\aleph$-Borel subsets* of X.

The next two propositions indicate that c$\aleph$-Borel subsets occur quite naturally.

PROPOSITION 1. *Let $\mathscr{A}$ be an $\aleph$-algebra in X containing $\mathscr{G}$ and let μ be a complete and saturated $\aleph$-measure on $\mathscr{A}$. If X contains no discrete subspace of measurable cardinality, then $\mathscr{A}$ is complete and so $\mathscr{B}_\aleph \subset \mathscr{A}$.*

PROOF. Let $\{A_\alpha \neq \emptyset: \alpha \in T\} \subset \mathscr{A}$ be separated in some $Y \in \mathscr{A}$ and let $A = \bigcup \{A_\alpha: \alpha \in T\}$. Choose $B \in \mathscr{A}$ with $\mu(B) < +\infty$ and $\{G_\alpha: \alpha \in T\} \subset \mathscr{G}$ such that $\{G_\alpha \cap Y: \alpha \in T\}$ is a disjoint family and $A_\alpha \subset G_\alpha \cap Y$ for each $\alpha \in T$. Let $T_0 = \{\alpha \in T: \mu(G_\alpha \cap Y \cap B) = 0\}$ and $B_0 = \bigcup \{G_\alpha \cap Y \cap B: \alpha \in T_0\}$. Suppose that $\mu(B_0) > 0$. Because the sets $G_\alpha \cap Y \cap B$ are open in $Y \cap B$ and disjoint, we can define an $\aleph$-measure ν on $\exp T_0$ by letting

$$\nu(T') = \frac{1}{\mu(B_0)} \mu\left(\bigcup \{G_\alpha \cap Y \cap B: \alpha \in T'\}\right)$$

for each $T' \subset T_0$. Since $\aleph > \aleph_0$, it follows from Dickman (1975, Lemma 0.4.12, p. 36) that T_0 contains a set T_1 of measurable cardinality. Choosing $x_\alpha \in A_\alpha$ for each $\alpha \in T_1$, we obtain a discrete subspace $X_1 = \{x_\alpha: \alpha \in T_1\}$ of X with $|X_1| = |T_1|$. This contradiction shows that $\mu(B_0) = 0$. By the completeness of μ,

$$\bigcup \{A_\alpha \cap B: \alpha \in T_0\} \in \mathscr{A}.$$

Because $\mu(B) < +\infty$, we have $|T - T_0| \leqslant \aleph_0 < \aleph$. Hence

$$A \cap B = \left(\bigcup \{A_\alpha \cap B: \alpha \in T_0\}\right) \cup \left(\bigcup \{A_\alpha \cap B: \alpha \in T - T_0\}\right)$$

belongs to $\mathscr{A}$. Since μ is saturated, $A \in \mathscr{A}$.

REMARK 1. From the previous proof it is clear that if μ is a two-valued measure, we can replace "measurable" by "two-valued measurable" in Proposition 1: we only need to apply Dickmann (1975, Theorem 0.4.25(4), p. 39).

A set $A \subset X$ is called *$\aleph$-Lindelöf* if every open cover of A contains a subcover whose cardinality is less than $\aleph$. Thus an ordinary Lindelöf set is $\aleph_1$-Lindelöf. We shall denote by $\mathscr{F}_\aleph$ the family of all closed $\aleph$-Lindelöf subsets of X.

Let $\mathscr{A}$ be an $\aleph$-algebra in X containing $\mathscr{G}$. An $\aleph$-measure μ on $\mathscr{A}$ is called *inner regular* if

$$\mu(A) = \sup\{\mu(C): C \in \mathscr{F}_\aleph, C \subset A\}$$

for each $A \in \mathscr{A}$ with $\mu(A) < +\infty$.

PROPOSITION 2. *Let $\mathscr{A}$ be an $\aleph$-algebra in X containing $\mathscr{G}$ and let μ be a complete and saturated $\aleph$-measure on $\mathscr{A}$. If μ is inner regular, then $\mathscr{A}$ is complete and so $\mathscr{B}_\aleph \subset \mathscr{A}$.*

PROOF. Using the same notation as in the proof of Proposition 1, it clearly suffices to show that $\mu(B_0) = 0$. If $C \in \mathscr{F}_\aleph$ and $C \subset B_0$, then

$$C \subset \bigcup\{G_\alpha \cap Y \cap B: \alpha \in S\}$$

where $S \subset T_0$ with $|S| < \aleph$. Hence $\mu(C) = 0$ for each $C \in \mathscr{F}_\aleph$ for which $C \subset B_0$. By the inner regularity of μ, $\mu(B_0) = 0$.

The Borel $\aleph$-algebra in X is defined as the smallest $\aleph$-algebra in X containing $\mathscr{G}$. Thus the Borel $\aleph$-algebra in X is contained in $\mathscr{B}_\aleph$ but, in general, it is not complete. If X is a free union of subspaces X_α, then it is easy to see that the Borel $c\aleph$-algebra in X is isomorphic to the direct product of the Borel $c\aleph$-algebras in X_α's. This is not correct if the Borel $c\aleph$-algebras are replaced by the Borel $\aleph$-algebras. The situation is well illustrated by the following example.

EXAMPLE 1. Let T be the discrete space of all countable ordinals and let $X = T \times [0,1]$. According to Natanson (1957, Chapter 15, Section 2), for each $\alpha \in T$ there is a set $A_\alpha \subset [0,1]$ whose characteristic function belongs to the Baire class α. Thus the set $A = \bigcup\{(\alpha) \times A_\alpha: \alpha \in T\}$ is not a Borel subset of X. Obviously, $A \in \mathscr{B}_{\aleph_1}$.

A set $A \subset X$ is called *locally $c\aleph$-Borel* if for each $x \in X$ there is a neighborhood U of x such that $A \cap U \in \mathscr{B}_\aleph$. The family of all locally $c\aleph$-Borel subsets of X will be denoted by $\mathscr{L}_\aleph$. Obviously, $\mathscr{B}_\aleph \subset \mathscr{L}_\aleph$ and, in general, this inclusion is proper (see the Corollary to Proposition 3). If $\mathscr{B}_\aleph = \mathscr{L}_\aleph$, the space X is called *$\aleph$-saturated.*

If $\mathscr{V} \subset \exp X$ and $x \in X$, let $\operatorname{st}(x, \mathscr{V}) = \{V \in \mathscr{V}: x \in V\}$.

DEFINITION 4. The space X is called *$\aleph$-weakly θ-refinable* if each open cover of X has an open refinement $\mathscr{V} = \bigcup\{\mathscr{V}_\alpha: \alpha \in T\}$ such that $|T| < \aleph$ and for each $x \in X$ there is an $\alpha_x \in T$ such that $\operatorname{st}(x, \mathscr{V}_{\alpha_x})$ is nonempty and finite.

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We note that X is weakly θ-refinable in the sense of Bennett and Lutzer (1972) if and only if it is $\aleph_1$-weakly θ-refinable.

THEOREM. *Let X be $\aleph$-weakly θ-refinable. Then X is $\aleph$-saturated.*

PROOF. Let $A \in \mathscr{L}_\aleph$. For each $x \in X$ choose an open neighborhood U_x of x so that $A \cap U_x \in \mathscr{B}_\aleph$. Let $\mathscr{V} = \bigcup\{\mathscr{V}_\alpha : \alpha \in T\}$ be an open refinement of $\{U_x : x \in X\}$ such that $|T| < \aleph$ and given $x \in X$, there is an $\alpha_x \in T$ for which $\mathrm{st}(x, \mathscr{V}_{\alpha_x})$ is nonempty and finite. Because the sets $\{x \in X : |\mathrm{st}(x, \mathscr{V}_\alpha)| \geq k\}$, $\alpha \in T$, $k = 1, 2, \ldots$, are open, the sets

$$X_{\alpha,k} = \{x \in X : |\mathrm{st}(x, \mathscr{V}_\alpha)| = k\}$$

are c$\aleph$-Borel. Clearly,

$$\bigcup\{X_{\alpha,k} : \alpha \in T, k = 1, 2, \ldots\} = X.$$

Let $\mathscr{W}_{\alpha,k}$ consist of all sets $A \cap X_{\alpha,k} \cap V_1 \cap \ldots \cap V_k$ where $V_1, \ldots, V_k$ are distinct elements of $\mathscr{V}_\alpha$. Then $\mathscr{W}_{\alpha,k}$ is separated in $X_{\alpha,k}$ and $\bigcup\{W : W \in \mathscr{W}_{\alpha,k}\} = A \cap X_{\alpha,k}$. Since $\mathscr{W}_{\alpha,k} \subset \mathscr{B}_\aleph$, we have $A \cap X_{\alpha,k} \in \mathscr{B}_\aleph$ for $\alpha \in T$ and $k = 1, 2, \ldots$. The theorem follows.

Throughout, let κ be an *uncountable* ordinal. By W we shall denote the set of all ordinals less than κ equipped with the order topology, and we let $\aleph = |W|$. The family of all closed cofinal subsets of W is denoted by $\mathscr{H}$. Thus if κ is a *regular* ordinal, then $\mathscr{H}$ consists of all closed sets $F \subset W$ for which $|F| = \aleph$.

LEMMA *Let κ be a regular ordinal, $\{F_\alpha : \alpha \in T\} \subset \mathscr{H}$, and let $F = \bigcap\{F_\alpha : \alpha \in T\}$. If $|T| < \aleph$ then $F \in \mathscr{H}$.*

PROOF. Using the interlacing lemma (see Kelley, 1955, Chap. 4, Prob. E, (a)) in W, it is easy to see that the lemma is correct if $|T| = 2$. By induction it is correct whenever $|T| < \aleph_0$. Let $\aleph_0 \leq m < \aleph$ and suppose that the lemma is correct if $|T| < m$. Let ξ be the initial ordinal for m and let $T = \{\alpha : \alpha < \xi\}$. Replacing F_α by $\bigcap\{F_\beta : \beta \leq \alpha\}$, we may assume that $F_\alpha \subset F_\beta$ for each $\beta < \alpha < \xi$. Given $\gamma < \kappa$, there are $\gamma_\alpha \in F_\alpha$ such that $\gamma < \gamma_\alpha < \gamma_\beta$ for each $\alpha < \beta < \xi$. Let $\delta = \sup\{\gamma_\alpha : \alpha < \xi\}$. Since κ is a regular ordinal, $\delta < \kappa$. It follows that $\delta \in F$ and so $F \in \mathscr{H}$.

Let $\mathscr{A}$ consist of all sets $A \subset W$ such that either A or $W - A$ contain a set $F \in \mathscr{H}$. For $A \in \mathscr{A}$ let $\mu(A) = 1$ if A contains a set $F \in \mathscr{H}$ and $\mu(A) = 0$ otherwise. The next proposition follows immediately from the lemma.

PROPOSITION 3. *Let κ be a regular ordinal. Then the family $\mathscr{A}$ is an $\aleph$-algebra in W containing all open subsets of W and μ is a complete $\aleph$-measure on $\mathscr{A}$.*

COROLLARY. *Let κ be a regular ordinal such that the cardinal $\aleph$ is not two-valued measurable. Then W is not $\aleph$-saturated and hence not $\aleph$-weakly θ-refinable.*

PROOF. The space W is Hausdorff and each $x\in W$ has a neighborhood U with $|U|<\aleph$. Thus $\mathscr{L}_\aleph=\exp W$. Because the cardinal $\aleph$ is not two-valued measurable, $\mathscr{A}\neq\exp W$. Being finite, the $\aleph$-measure μ is saturated. By Proposition 1 and Remark 1, $\mathscr{B}_\aleph\subset\mathscr{A}$. The corollary follows from the theorem.

Bennett and Lutzer (1972) proved that W is weakly θ-refinable if and only if it is paracompact (Theorem 11). A simple modification of this proof will show that W is not $\aleph$-weakly θ-refinable for *any* uncountable regular ordinal κ.

REMARK 2. If the cardinal $\aleph$ is two-valued measurable, we cannot use Proposition 1 to show that $\mathscr{B}_\aleph\subset\mathscr{A}$. However, K. Prikry kindly pointed out to the author that $\mathscr{A}\neq\exp W$ for *any* uncountable regular ordinal κ. Indeed, this is clear if $\kappa=\omega_1$, for $\aleph_1$ is not measurable (see Ulam, 1930, Theorem (A)). If $\kappa>\omega_1$ then each closed cofinal subset of W contains an ordinal α cofinal with ω_0 and also an ordinal β cofinal with ω_1. Hence if B is the set of all ordinals $\alpha\in W$ cofinal with ω_0, then $B\notin\mathscr{A}$.

We shall close this paper with an example indicating the necessity of the cardinality assumption in Proposition 1.

EXAMPLE 2. Let $\aleph$ be a two-valued measurable cardinal and let Z be a discrete space of cardinality $\aleph$. Denote by ν a two-valued $\aleph$-measure on $\exp Z$ such that $\nu(Z)=1$ and $\nu(\{z\})=0$ for each $z\in Z$. If κ is the initial ordinal for $\aleph$, then κ is regular (see Ulam (1930)). Thus we can define the $\aleph$-measure μ in W as in Proposition 3. Let $X=W\times Z$. For $C\subset X$ and $\alpha\in W$ set $C^\alpha=\{z\in Z\colon (\alpha,z)\in C\}$ and $C'=\{\alpha\in W\colon \nu(C^\alpha)=1\}$. Denote by $\mathscr{C}$ the family of those $C\subset X$ for which $C'\in\mathscr{A}$ and let $\lambda(C)=\mu(C')$ for each $C\in\mathscr{C}$. It is easy to see that $\mathscr{C}$ is an $\aleph$-algebra in X and that λ is a complete two-valued $\aleph$-measure on $\mathscr{C}$. Let $G\subset X$ be open and let $\alpha\in G'$ be a limit ordinal. For each $\beta<\alpha$ let

$$A_\beta=\{z\in G^\alpha\colon (\beta,\alpha]\times\{z\}\subset G\}.$$

Since G is open, $G^\alpha=\bigcup\{A_\beta\colon \beta<\alpha\}$. It follows that $\nu(A_\beta)=1$ for some $\beta<\alpha$. Consequently, $(\beta,\alpha]\subset G'$ and G' is open. Therefore, $\mathscr{G}\subset\mathscr{C}$. Choose $A\subset W$ for which $A\notin\mathscr{A}$ (see Remark 2). Clearly, we can consider Z as W with the discrete topology. Let $B=\{(\alpha,z)\in X\colon \alpha\in A \text{ and } z>\alpha\}$. Then $B'=A$ and thus $B\notin\mathscr{C}$. However,

$$B=\bigcup\{(A\cap[0,z))\times\{z\}\colon z\in Z\}$$

from which it follows that $B\in\mathscr{B}_\aleph$.

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