Canad. Math. Bull. Vol. 19 (1), 1976

A METRIZATION THEOREM FOR 2-MANIFOLDS[1]

BY
PAUL A. VINCENT

1. **Introduction**. There are few known metrization theorems for manifolds (locally Euclidean, connected, Hausdorff space). It is well known that for manifolds metrizability, second countability, Lindelöf's condition, σ-compactness and paracompactness are equivalent. Although these conditions imply separability, the latter does not imply any of the former (see Example 2.2), as is often believed. A common source of metrization for a covering manifold is that lifted from the base manifold [8; p. 181].

For 2-manifolds, the presence of a complex analytic structure gives us a metrization theorem; it has been shown [1] that such manifolds are topologically characterized as those which are orientable and second countable. More recently, Cannon [3] has generalized this to 2-manifolds which carry a K-quasiconformal structure by showing it admits a complex analytic one. This situation is particular to complex manifolds of complex dimension one. Calibi and Rosenlicht [2] have produced examples of non second countable complex manifolds of complex dimension $n>1$.

A topological condition resulting in the existence of a conformal structure (and hence a metric) on a 2-manifold is due to the work of Stoilow. He [9] showed that a light open map (a map preserving continua and open sets) between 2-manifolds is locally topologically equivalent to the map z^n. Thus the existence of a light open map from a 2-manifold M into the Riemann sphere S^2 describes M as a covering of a domain of S^2 so that [1; p. 119] a conformal structure may be lifted back to M.

This paper is concerned with a generalization of this last metrization theorem. The level sets of the real and imaginary parts of a light open map are generalized in the following concept. Two families of locally connected generalized continua on a 2-manifold form a *conjugate net* (see Definition §2) if locally up to homeomorphism they are given by the level sets of Re z^n and Im z^n. In general, there need not exist a global light open map to the complex plane with a prescribed conjugate net contouring it [7]. Thus it is natural to ask under what conditions can a

Received by the editors August 28, 1974 and, in revised form, March 12, 1975.

[1] This paper represents a chapter of the author's dissertation, submitted to the University of Rochester in partial fulfillment of the requirements for the degree of Doctor of Philosophy. The author wishes to express his gratitude to Professor Gail S. Young who directed his dissertation.

conjugate net produce metrizability or orientability. There are simple examples to show that a conjugate net cannot force orientability. The purpose of this paper is to show that if the generalized continua of both families are separable then the manifold is metrizable.

On an open Riemann surface the concept of a conjugate net here is a conjugate net in the sense of Jenkins and Morse ([5], [6], [7]), though not identical with it, for local connectivity does not permit self limiting (non recurrence).

It would have been just as natural to use the level sets of $z^{n/2}$, $n \geqq 2$, for the local structure of a conjugate net since we are interested in the topology of the structure and not, for example, in coherent sensing of the curves. Furthermore such local structures occur in the trajectories of quadratic differentials at its zeros [4]. On the other hand it is possible [10] to construct a double covering of the manifold ramified at the singular points ($n>2$) for which n is odd such that the conjugate net lifts to one locally structured by z^n.

Another point of view of the presence of a conjugate net on a manifold is the following. There is a submanifold, whose complement is the discrete set of singular points, on which there is a topological coordinate covering for which the transition maps send verticals to verticals and horizontals to horizontals.

Notation. Capital letters will be used to denote spaces and their subsets and small letters will denote points of the space. Script letters will denote families of subsets of a space. If $\mathscr{A}$ is a family of connected subsets of a space X and S is a subset of X then $\mathscr{A}_S$ will denote the family $\{\text{components of } A \cap S \mid A \in \mathscr{A}\}$ If the family $\mathscr{A}$ consists of mutually disjoints subsets of X and $p \in \bigcup \mathscr{A}$ then the element of $\mathscr{A}$ containing p will be denoted by A_p.

An arc (open arc) is the continuous 1–1 image of a closed (open) interval. They will be denoted by lower case Greek letters. An arc joining two points p and q will also be denoted by pq.

2. **Preliminary definitions.** Let M be a 2-manifold.

DEFINITIONS. A pair $[\mathscr{A}, \mathscr{B}]$ of families of locally connected generalized continua on M forms a *conjugate net* if for each point $p \in M$ there is a neighborhood N about p and a homeomorphism h of N onto the unit disc $B(0, 1)$ in the z-plane such that $h(p)=0$ and each element or $\mathscr{A}_N$ or $\mathscr{B}_N$ is carried onto a component of a level curve of $\operatorname{Re} z^n$ or $\operatorname{Im} z^n$, $n>0$, respectively. The elements of $\mathscr{A}$ and $\mathscr{B}$ are called the *fibers* of the conjugate net.

In keeping with Jenkins and Morse [6] the neighborhood N and homeomorphism h will be termed *canonical.* If A_p and B_p are the fibers of $\mathscr{A}_N$ and $\mathscr{B}_N$ respectively containing p then the $2n$ components of $N-A_p$ are called *$\mathscr{A}$-sectors* of N incident with p and the $2n$ components of $N-B_p$ *$\mathscr{B}$-sectors* of N incident with p. Each fiber of $\mathscr{A}_N-A_p$ or $\mathscr{B}_N-B_p$ is an open arc, as well as each component of A_p-p

https://doi.org/10.4153/CMB-1976-014-x Published online by Cambridge University Press

or B_p-p. Also it is clear from the structure of the level curves of z^n in $B(0, 1)$ that A_p meets each fiber of $\mathscr{B}_N$ exactly once and B_p meets each fiber of $\mathscr{A}_N$ exactly once. Any other fiber of $\mathscr{A}_N$ meets a fiber of $\mathscr{B}_N$ at most once and vice-versa. The n is called the *order of p in* $[\mathscr{A}, \mathscr{B}]$ and is denoted by $O(p)$. The set $S=\{p \in M \mid O(p)>1\}$ is the set of *singular points* of $[\mathscr{A}, \mathscr{B}]$.

One way in which this definition differs from that of Jenkins and Morse is that the fibers here need not be separable. The object of this paper is to show the following.

THEOREM 2.1. *If the fibers of a conjugate net* $[\mathscr{A}, \mathscr{B}]$ *on a 2-manifold M are separable then M is metrizable.*

This must be true of the fibers of both families as the next example shows.

Example 2.1. Let L be the long line without its initial point. Take $M=\mathbb{R}\times L$ and $\mathscr{A}$ and $\mathscr{B}$ as slices parallel to the first and second factor respectively. Then $[\mathscr{A}, \mathscr{B}]$ is clearly a conjugate net but M is not metrizable.

This example shows also that using one family only won't yield metrizability. In fact this remains true even if we assume that M is separable and each fiber is homeomorphic to $\mathbb{R}$. The next example of R. L. Moore, as modified by G. S. Young, realizes such a family as the level sets of the imaginary part of a real analytic monotone map.

EXAMPLE 2.2. The complex plane $\mathbb{C}$ is modified by "stuffing" a continuum of points onto each point of the real axis as follows. Let x be a real number and consider the addition of points $P_{x,\theta}$, $0<\theta<\pi$, topologized by neighborhoods

$$N_\varepsilon(P_{x,\theta}) = \{r\exp(i\phi)+x, P_{x,\phi} \mid 0<|r|<\varepsilon, 0<(\theta-\varepsilon)<\phi<(\theta+\varepsilon)<\pi)\}$$
$$\cong (\theta-\varepsilon, \theta+\varepsilon)\times(-\varepsilon, \varepsilon) \subset \mathbb{R}^2.$$

This last homeomorphism under which $P_{x,\phi}$ is carried to $(\phi, 0)$ shows that $M=(\mathbb{C}\text{-}\{\text{real axis}\}) \cup \{P_{x,\theta} \mid x \in \mathbb{R}, \theta \in (0, \pi)\}$ is locally Euclidean at the new points. M is separable (use a countable dense set of $\mathbb{C}$-{real axis}) but is not metrizable ($\{P_{x,\pi/2} \mid x \in \mathbb{R}\}$ has no condensation point). Define $f{:}M\to\mathbb{R}^2$ by $f(x+iy)=x+iy$, $y\neq 0$ and $f(P_{x,\theta})=x$ for all $x \in \mathbb{R}$, $\theta \in (0, \pi)$ and let $\mathscr{A}=\{\text{components of } (\operatorname{Im} f)^{-1}(c) \mid c \in \mathbb{R}\}$.

The first step, however, in proving Theorem 2.1 is to show that M is itself separable (see §3). Then, using the conjugate net, onto each point of a countable dense set in M is built (§4, §5) a metrizable neighborhood. These neighborhoods are shown to cover M and thus give the metrization of M.

A few remarks are needed before proceeding.

REMARK 2.1. The fibers of $\mathscr{A}$ (or $\mathscr{B}$) are disjoint since if, for fibers $A_1, A_2 \in \mathscr{A}$, we have $p \in A_1 \cap A_2$ then by local connectedness there is a canonical neighborhood N of p such that $N \cap A_1$ and $N \cap A_2$ are connected. But since p belongs to just one fiber of $\mathscr{A}_N$, $N \cap A_1 = N \cap A_2$. Thus $A_1 \cap A_2$ is open in A_1. By the same reasoning $A_1 - A_2$ is open in A_1 and so by connectedness must be empty. This gives $A_1 \subset A_2$. By symmetry then the fibers are identical.

REMARK 2.2. For any $A \in \mathscr{A}$ and $B \in \mathscr{B}$, $A \cap B$ is discrete for if $p \in A \cap B$ is a cluster point of $A \cap B$ then in a canonical neighborhood N of p there is a sequence $\{p_n\}$ in $A \cap B - p$ coverging to p. Since each fiber of $\mathscr{A}_N$ meets each fiber of $\mathscr{B}_N$ in at most one point there are infinitely many distinct components of $A \cap N$ or $B \cap N$ converging to a continuum in A_p or B_p, as $p_n \to p$, showing A or B is not locally connected at p.

REMARK 2.3. When M is a plane, a conjugate net here is a conjugate net in the sense of Jenkins and Morse. Thus [7] there is a single valued light open map f from M to the complex plane with $\mathscr{A}$ and $\mathscr{B}$ the family of level curves of Re f and Im f respectively. In particular for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$, $A \cup B$ cannot contain a simple closed curve. For such a simple closed curve is mapped by f into $\{x = c_1\} \cup \{y = c_2\}$ where $f(A) = c_1$ and $f(B) = c_2$, which is not possible for a light open map.

REMARK 2.4. The sphere S^2 cannot support a conjugate net since on $S^2 - p$ there is [7] a light open map f to the plane contoured by the conjugate net which can clearly be extended to p. This is impossible since $f(S^2)$ would be a compact open set in the plane. In fact this is Liouville's theorem via Stoïlow's work.

REMARK 2.5. Let M be a 2-manifold and M^* a covering manifold with projection $\pi: M^* \to M$. If $\mathscr{A}$ is a family of generalized continua on M the *lift* of $\mathscr{A}$ will be $\mathscr{A}^* = \{\text{components of } \pi^{-1}(A) \mid A \in \mathscr{A}\}$. If $[\mathscr{A}, \mathscr{B}]$ is a conjugate net on M then the *lift* of $[\mathscr{A}, \mathscr{B}]$ will be $[\mathscr{A}^*, \mathscr{B}^*]$. It is easy to see that the lift $[\mathscr{A}^*, \mathscr{B}^*]$ is also a conjugate net in M^*. In particular if M is second countable M^* is homeomorphic to $\mathbb{R}^2$ or S^2 but supporting a conjugate net $[\mathscr{A}^*, \mathscr{B}^*]$ it must be homeomorphic to $\mathbb{R}^2$ by Remark 2.4.

3. *M* is separable.

Here I show the following theorem

THEOREM 3.1 *If the fibers of the net $[\mathscr{A}, \mathscr{B}]$ on a 2-manifold M are separable, then M is separable.*

Proof. For the point q in M let D_q denote the countable dense set of $A_q \cup B_q$. Let p_0 be a fixed point in M. Let $D_0 = D_{p_0}$. Then set $D_1 = \bigcup_{q \in D_0} D_q$. By induction when D_n has been defined, set $D_{n+1} = \bigcup_{q \in D_n} D_q$. Finally set $D = \bigcup_n D_n$ which is clearly countable. Now it suffices to show that D is dense in M. Assume not and let $p \in \beta C\ell D$. Let N be a canonical neighborhood of p. Thus N meets the open set $E = M - C\ell D$. Let $\mathrm{O} = N \cap E$. I show now that O contains each $\mathscr{A}$-sector it meets.

Let S be an $\mathscr{A}$-sector and $q \in \mathrm{O} \cap S$. Let A_q be the fiber of $\mathscr{A}_N$ containing q. If S contains a point of $C\ell D$ it must contain a point r_0 of D since it is open. In

$\mathscr{A}_N$, either $B_{r_0} \cap A_q \neq \varnothing$ or $A_{r_0} \cap B_q \neq \varnothing$, say $r = B_{r_0} \cap A_q$ (see Figure 1). Since $r_0 \in D$ then there is a first n such that $r_0 \in D_n$. Therefore there is a countable dense set D' in the fiber B of $\mathscr{B}$ containing B_{r_0} such that $D' \subset D_{n+1} \subset D$. Since B_{r_0} is an open subset of B, there is a sequence $\{r_i\} \subset D' \cap B_{r_0}$ converging to r Similarily for each i, the fiber A_{r_i} of $\mathscr{A}_N$ containing r_i has a countable dense subset $D_{r_i} \cap A_{r_i} \subset D_{n+2} \subset D$, so that $A_{r_i} \subset C\ell D$. Since r_i converges to r then by the structure of an $\mathscr{A}$-sector lim sup $A_{r_i} = A_q$. But this means $A_q \subset C\ell D$ contradicting the choice of q. Thus $S \subset$ O.

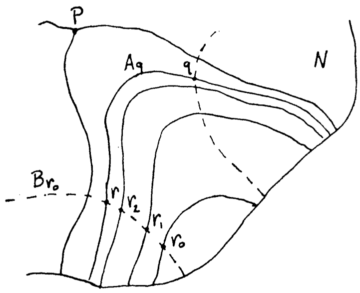

Figure 1

Similarily it is shown that O contains any $\mathscr{B}$-sector it meets. Now since any $\mathscr{A}$-sector straddles two consecutive $\mathscr{B}$-sectors and vice-versa, and since O must meet some $\mathscr{A}$-sector or $\mathscr{B}$-sector, O must contain all $\mathscr{A}$-sectors and all $\mathscr{B}$-sectors Therefore $\mathrm{O} = N - p$. But then p cannot be a boundary point of $C\ell D$. Thus $\beta\, C\ell D$ is empty and by connectedness of M, $C\ell D = M$. Hence M contains the countable dense set D establishing Theorem 3.1.

REMARK 3.1. As Example 2.2 shows, separability is not enough to guarantee metrizability in a 2-manifold. However, for 1-manifold it is enough. In fact separability of a fiber of a conjugate net $[\mathscr{A}, \mathscr{B}]$ gives metrizability of the fiber. If A is a fiber of $\mathscr{A}$ and if S is the singular set of $[\mathscr{A}, \mathscr{B}]$ then by local connectivity each component of $A - S$ is a 1-manifold whose boundary in A consists of at most two points of S. Thus the components form a disjoint collection of open subsets of A. If A is separable there are at most countably many such components each of which is separable and so a metrizable 1-manifold. Thus A is regular Lindelöf and so paracompact and hence metrizable.

4. **The blocks $A(\beta)$ and $B(\alpha)$.** For the rest of this paper M is a 2-manifold and $[\mathscr{A}, \mathscr{B}]$ is a conjugate net on M, whose fibers are separable and whose singular set is S. Before describing the building blocks for M, I give the following useful structural lemma.

LEMMA 4.1. (*a*) *Let A be a fiber of $\mathscr{A}$ and α an arc in $A - S$. There exists a neighborhood $N(\alpha)$ of a such that there is a homeomorphism $h: C\ell N(\alpha) \to R$, where R is the* *rectangle* $[-1, 1]\times[-1, 1]$ *in* $\mathbb{R}^2$ *under which* $[\mathscr{A}_{ClN(\alpha)}, \mathscr{B}_{ClN(\alpha)}]$ *is mapped to the lines* $[\{y=\text{constant}\}, \{x=\text{constant}\}]$ *with a going onto* $[-\frac{1}{2}, \frac{1}{2}]\times\{0\}$.

(*b*) *Let* α *be an arc on a fiber* A *of* $\mathscr{A}$ *containing one point* $p\in S$ *and such that* $a-p$ *is formed of two arcs bounding the same* $\mathscr{A}$*-sector of some canonical neighborhood of* p. *There exists a "rectangle"* $R(\alpha)$ *such that there is a homeomorphism* $h:R(\alpha)\to[-1, 1]\times[0, 1]$ *in* $\mathbb{R}^2$ *under which* p *is mapped to* $(0, 0)$ *and* $[\mathscr{A}_{R(\alpha)}, \mathscr{B}_{R(\alpha)}]$ *is mapped to* $[\{y=\text{constant}\}, \{x=\text{constant}\}]$ *with a going onto* $[-1, 1]\times\{0\}$.

Similar statements hold for arcs β *in fibers* B *of* $\mathscr{B}$.

Proof. Covering α by a finite chain of canonical neighborhoods whose centers belong to α and, for a given order on α, two consecutive centers belong to each other's canonical neighborhoods, both (a) and (b) follow by routine arguments.

The description of the blocks $B(\alpha)$ only will be given since $A(\beta)$ is obtained by interchanging the roles of $\mathscr{A}$ and $\mathscr{B}$. Let A be a fiber of $\mathscr{A}$ and α an open arc in $A-S$. Define $B(\alpha)=\bigcup\{B\in\mathscr{B}_{M-S}\mid B\cap\alpha\neq\varnothing\}$. $B(\alpha)$ is connected since any two points can be connected to α via the fibers of $\mathscr{B}_{M-S}$ in which they lie. It is open as well for let $p\in B(\alpha)$ and pr be an arc in the fiber B_p of $\mathscr{B}_{M-S}$ joining p to $r\in\alpha$. If $N(pr)$ is the neighborhood of pr given in Lemma 4.1 (a) then the component α' of $N(pr)\cap\alpha$ containing r is mapped under the homeomorphism of $N(pr)$ to $(-1, 1)\times(-1, 1)$ onto $\{-\frac{1}{2}\}\times(t, t')$ where $(t, t')\subset(-1, 1)$. Thus the inverse image of $(-1, 1)\times(t, t')$ is a neighborhood of p in $B(\alpha)$. Hence $B(\alpha)$ is a submanifold. To show it is metrizable we show it to be the appropriate union of smaller blocks $K(\alpha)$.

Now let γ be an arc such that $\gamma\subset A-S$ for some fiber $A\in\mathscr{A}$. Let p and q be its end points. By Remark 3.1 the fiber B_p of $\mathscr{B}_{M-S}$ is homeomorphic to $\mathbb{R}$ or S^1. Let $N(\gamma)$ be a neighborhood of γ given by Lemma 4.1 (a) such that p is mapped to $(-\frac{1}{2}, 0)$. This homeomorphism carries the component of $N(\gamma)\cap B_p$ containing p to $\{-\frac{1}{2}\}\times[-1, 1]$ giving it a sensing corresponding to increasing parameter. This induces a sense in B_p. Let $p(t)$, $t\in\mathbb{R}$ be a sense preserving parametrization of B_p such that $p(0)=p$ and such that $p(t)$ has period 2π if B_p is a topological circle. Let T_+ be the set of real numbers $t>0$ such that $A_{p(t)}\cap B_q\neq\varnothing$, where $A_{p(t)}$ is the fiber of $\mathscr{A}_{M-S}$ containing $p(t)$, and there is a point $q(t)\in A_{p(t)}\cap B_q$ such that the arcs $pp(t)$ of B_p, $p(t)\,q(t)$ of $A_{p(t)}$, $q(t)q$ of B_q and γ bound a domain $K(t)$ whose closure is homeomorphic to $[-1, 1]\times[0, t]$ in which $[\mathscr{A}_{ClK(t)}, \mathscr{B}_{ClK(t)}]$ corresponds to $[\{y=\text{constant}\}, \{x=\text{constant}\}]$, γ to $[-1, 1]\times\{0\}$ and p to $(-1, 0)$ (see Figure 2). From $N(\gamma)$ it is clear that $T_+\neq\varnothing$. Similarily for $t<0$ define T_- (here $K(t)\cong(-1, 1)\times(t, 0)$). Set $T=T_+\cup\{0\}\cup T_-$. It is clear using $N(p(t)\,q(t))$ of Lemma 4.1 (a) that T is an open interval about 0. Let $\{(s_n, t_n)\}_n$ be a sequence of intervals containing 0 such that $T=\bigcup_n(s_n, t_n)$. Set $\gamma^\circ=\gamma-p-q$ and $K(\gamma)=\gamma^\circ\cup(\bigcup_n(K(t_n)\cup K(s_n)))$. Since $(-1, 1)\times(s_n, t_n)$ covers $K(t_n)\cup\gamma^\circ\cup K(s_n)$, then the latter, and hence $K(\gamma)$, is seen to be a metrizable submanifold of M.

https://doi.org/10.4153/CMB-1976-014-x Published online by Cambridge University Press

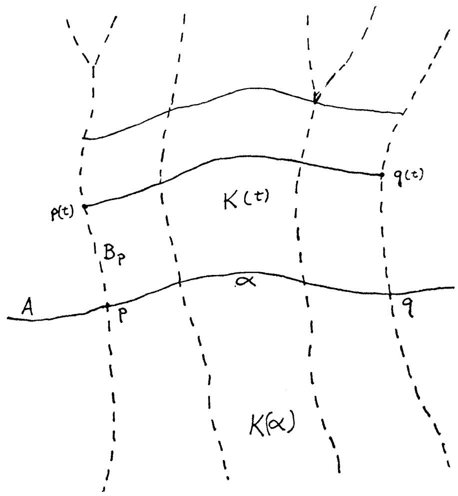

Figure 2

THEOREM 4.1. *$B(\alpha)$ is a metrizable submanifold of M.*

Proof. Since α is homeoporphic to $\mathbb{R}$ (Remark 3.1), let $\{\gamma_n^0\}$ be a countable basis for the topology of α obtained as the images of a basis of relatively compact open intervals in $\mathbb{R}$ under the above homeomorphism and $\gamma_n = \mathrm{Cl}\gamma_n^0$.

I show now that $B(\alpha) = \bigcup_n K(\gamma_n)$ from which the theorem follows. Since $K(\gamma_n)$ is composed of subsets of fibers of $\mathscr{B}_{M-S}$, meeting $\gamma_n \subset \alpha$ then $\bigcup_n K(\gamma_n) \subset B(\alpha)$. On the other hand for any point $p \in B(\alpha)$, let pq be an arc in $B_p \in \mathscr{B}_{M-S}$ joining p to $q \in B_p \cap \alpha$, and let $N(pq)$ be given by Lemma 4.1 (a). The open arc of $\alpha \cap N(pq)$ containing q contains an open basic set γ_n^0 containing q. Then it is clear from $N(pq)$ that $p \in K(\gamma_n)$ and so $B(\alpha) \subset \bigcup_n K(\gamma_n)$.

A fiber of $\mathscr{B}_{M-S}$ is either a fiber of $\mathscr{B}$ or else a component of a fiber of $\mathscr{B}$ with at least one point of S in its closure. Let $S_{B(\alpha)}$ be the subset of S such that for each $s \in S_{B(\alpha)}$ there is a fiber of $\mathscr{B}_{M-S}$ in $B(\alpha)$ which limits at s. It will result from Theorem 2.1 that S is at most countable. But for the proof of this theorem we need the following information about the subsets $S_{B(\alpha)}$ of S.

THEOREM 4.2. *$S_{B(\alpha)}$ is at most countable.*

Proof. Let D be a simply connected domain, containing α, with a given orientation and let α be given a sense. Let $p \in S_{B(\alpha)}$ and B_p be the fiber of $\mathscr{B}$ containing p. Since $B_p \cap \alpha$ is non empty and discrete, there is an arc qp in B_p joining $q \in \alpha$ to p such that $\alpha \cap qp = q$. Let γ be an arc in α with q interior to it. Let $N(\gamma) \subset D$ be a neighborhood of α' given by Lemma 4.1 (a) such that the homeomorphism $h: N(\gamma) \to (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ is sense preserving and the sense of γ induced by the increasing parameter of $h(\gamma)$ coincides with that induced by α. The image of $N(\gamma) \cap qp$ is in the upper half plane or else in the lower half plane. This clearly

https://doi.org/10.4153/CMB-1976-014-x Published online by Cambridge University Press

depends only on the sensing of α and the orientation of D and $\mathbb{R}^2$. Let $S^+_{B(\alpha)}$ ($S^-_{B(\alpha)}$) be the points p of $S_{B(\alpha)}$ such that there are arcs pq and γ as above with $N(\gamma) \cap qp$ mapped into the upper (lower) half plane of $\mathbb{R}^2$ with the counter clockwise orientation. Then $S_{B(\alpha)}=S^+_{B(\alpha)} \cup S^-_{B(\alpha)}$ (not necessarily a disjoint union). Thus it suffices to show that $S^+_{B(\alpha)}$ is at most countable.

Let $p \in S_{B(\alpha)}$ and arc qp in B_p chosen as above to verify that $p \in S^+_{B(\alpha)}$. Let N be a canonical neighborhood of p such that $C\ell N \cap \alpha=\varnothing$. Now let β be the component $C\ell N \cap (B_p-qp)$ such that $C\ell N \cap qp$ and β bound a $\mathscr{B}$-sector and such that if $\beta_p=qp \cup \beta$ and $R(\beta_p)$ the rectangle given by Lemma 4.1 (b) then $R(\beta_p) \cap \alpha$ is an (half open) arc with q as the initial point in the sensing induced by α. If h is the homeomorphism of $[-1, 1]\times[0, 1]$ to $R(\beta_p)$, h can be chosen such that $h((1, 0))=q$. Let $U_p=h((-1, 0))\times(0, 1)) \cap B(\alpha)$ which is clearly non empty since $R(\beta_p) \cap \alpha\neq\varnothing$ (see Figure 3).

Since the work from here on is done in $B(\alpha)$ it can be assumed that $B(\alpha)$ is the plane. For if not then consider the universal covering surface $B(\alpha)^*$ of $B(\alpha)$. It is by Remark 2.5 the plane. Let $[\mathscr{A}^*, \mathscr{B}^*]$ be the lift of $[\mathscr{A}_{B(\alpha)}, \mathscr{B}_{B(\alpha)}]$. Then working on a lift α^* of α, $h:(h^{-1}(R(\beta_p) \cap B(\alpha)), (1, 0))\to(B(\alpha), q)$ can be lifted [8; V, 5–11] to a map $h^*:(h^{-1}(R(\beta_p) \cap B(\alpha)), (1, 0))\to(B(\alpha^*), q^*)$ where $q^* \in \alpha^*$ covers q and gives $U^*_p=h^*(h^{-1} U_p)$. It is clear that U^*_p has the same structure relative to $[\mathscr{A}^*, \mathscr{B}^*]$ as U_p does to $[\mathscr{A}_{B(\alpha)}, \mathscr{B}_{B(\alpha)}]$.

With this assumption it is shown that if p and p' are two distinct points of $S^+_{B(\alpha)}$, $U_p \cap U_{p'}=\varnothing$. Let q, $R(\beta_p)$ and U_p be as above for p and q', $R(\beta_{p'})$ and $U_{p'}$ for p'. Assume there is a point $s \in U_p \cap U_{p'}$. Let α_q and $\alpha_{q'}$ be the open arcs of $R(\beta_p) \cap \alpha$ and $R(\beta_{p'}) \cap \alpha$ respectively. If $\alpha_q \cap \alpha_{q'}=\varnothing$ then since the fiber B_s of $\mathscr{B}_{B(\alpha)}$ containing s must meet α in α_q and in $\alpha_{q'}$ (since $U_p\subset(\operatorname{Int} R(\beta_p) \cap B(\alpha))\subset B(\alpha_q)$ and $U_p'\subset(\operatorname{Int} R(\beta_{p'}) \cap B(\alpha))\subset B(\alpha_{q'})$ (see Figure 3)) this would contradict Remark 2.3. So assume now that $\alpha_q \cap \alpha_{q'}\neq\varnothing$. In the order of α induced by the sensing of α described earlier assume $q<q'$. Then $q' \in \alpha_q$ and

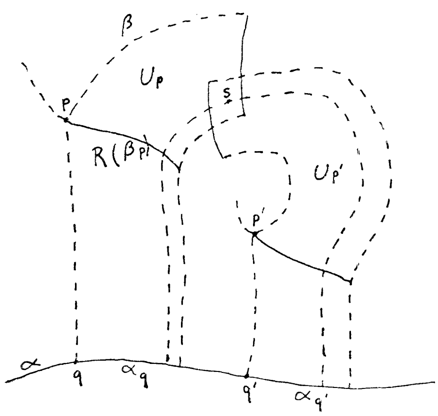

Figure 3

since the arc $q'p'$ limits to the singular point p' and since p is the only singular point of $R(\beta_p)$ then $q'p'$ must separate $R(\beta_p)$ and hence U_p (see Figure 4). But the fiber A_s of $\mathscr{B}_{B(\alpha)}$ containing s separates U_p and so must meet $q'p'$. This means that A_s must separate $(\text{Int } R(\beta_{p'}) \cap B(\alpha))-U_{p'}$. Hence for any $r \in \alpha_q \cap \alpha_{q'}$ since the fiber B_r of $\mathscr{B}_{B(\alpha)}$ separates Int $R(\beta_{p'}) \cap B(\alpha)$, it must cross A_s twice there, contradicting once again Remark 2.3. This establishes the claim that $U_p \cap U_{p'}=\varnothing$.

Thus to the points of $S^+_{B(\alpha)}$ there correspond disjoint open sets in $B(\alpha)$ so that $S^+_{B(\alpha)}$ must be at most countable.

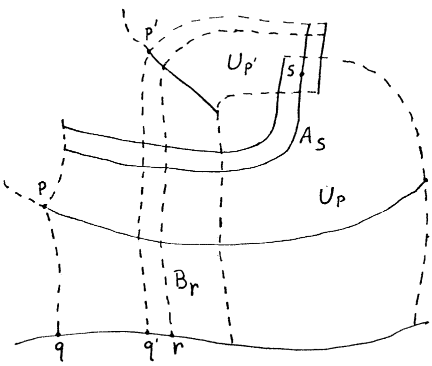

Figure 4

5. **Proof of theorem 2.1.** Let D be a countable dense set in M (Theorem 3.1). Using the blocks of §4, I build a metrizable neighborhood about each point of D and show they cover M. Thus let $p \in M$ and $O(p)=n$. Let α_i, $i=1,\ldots,n_p$ be the $n_p(n_p \leq 2n)$ components of $A_p-(S \cup \{p\})$ which have p in each of their closures and let β_j, $j=1,\ldots,m_p$ $(m_p \leq 2n)$ components of $B_p-(S \cup \{p\})$ which have p in each of their closures. Define $V(p)=\{p\} \cup \bigcup_i B(\alpha_i) \cup \bigcup_j A(\beta_j)$ and $S(p)= \bigcup_i S_{B(\alpha_i)} \cup \bigcup_j S_{A(\beta_j)}$. Then $V(p)$ is a metrizable submanifold of M (Theorem 4.1) and $S(p)$ is at most countable (Theorem 4.2). Finally we consider an augmentation of $V(p)$ defined by $W(p)=V(p) \cup \bigcup_{q \in S(p)} V(q)$, which again is a metrizable submanifold of M.

To prove Theorem 2.1 it suffices to show $\{W(p) \mid p \in D\}$ is a cover of M. Let $r \in M$ and $N(r)$ be a canonical neighborhood of r. Then there is a point p of D in $N(r)$. Now it is clear from $N(r)$ that if $O(r)=1$ then $r \in V(p) \subset W(p)$ and if $O(r) \geq 2$ then $r \in W(p)$. This concludes the proof of Theorem 2.1.

Corollary. *The singular set S is at most countable.*

References

1. L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton Univ. Press, Princeton, N.J., 1960.

2. E. Calabi and M. Rosenlicht, *Complex analytic manifolds without countable base*, Proc. Amer. Math. Soc. **4** (1953), 335–340.

3. R. J. Cannon, *Quasiconformal structures and the metrization of 2-manifolds*, Trans. Amer. Math. Soc., Vol. **135** (1969), 95–103.

4. J. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, Berlin, 1958.

5. J. Jenkins and M. Morse, *The existence of pseudoconjugates on Riemann surfaces*, Fund. Math. **39** (1952) 269–287.

6. ——, ——, *Topological methods on Riemann surfaces*, Annals of Math. Studies, No. **30,** 111–139. Princeton, 1953.

7. ——, ——, *Conjugate nets on an open Riemann surface*, *Lectures on Functions of a Complex Variable*, ed. W. Kaplan et al., 123–185. The University of Michigan Press, 1955.

8. W. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace and World, Inc., New York, 1967.

9. S. Stoilow, *Principles Topologiques de la Théorie des Fonctions Analytiques*, Paris, Gauthiers-Villars, 2nd ed., 1956.

10. P. Vincent, *Families of Generalized Continua on 2-manifolds*, Ph.D. Thesis, University of Rochester, 1973.

UNIVERSITY OF MONCTON
MONCTON, N.B.
CANADA E1A 3E9.