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RADICALS RELATED TO THE BROWN–McCOY RADICAL IN SOME VARIETIES OF ALGEBRAS

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Abstract

The Brown–McCoy radical is the upper radical defined by the class of simple rings with identities. For associative or alternative rings the Brown–McCoy radical is hereditary, and its semi-simple class consists of all subdirect products of simple rings with identities. In this paper we present some classes of simple non-associative algebras whose upper radicals behave similarly. Classifications are then obtained of 'most' semi-simple radical classes of (γ, δ) and right alternative rings.

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It is well known that in the universal class of all associative rings, the Brown–McCoy radical class, the upper radical class $\mathcal{G}$ defined by the class $\mathcal{U}$ of simple rings with identities, has the following properties:

(i) $\mathcal{G}(A) = \bigcap\{I \lhd A \mid A/I \in \mathcal{U}\}$ for all A.

(ii) $\mathcal{G}(A) = 0$ if and only if A is a subdirect product of rings in $\mathcal{U}$.

(iii) $\mathcal{G}(I) = I \cap \mathcal{G}(A)$ for all $I \lhd A$, for all A.

(Here $\mathcal{G}(A)$ is the largest ideal of A which belongs to $\mathcal{G}$.)

Suliński (1958) showed that the only classes of simple associative rings for which the analogues of (i), (ii) and (iii) are true are the subclasses of $\mathcal{U}$. (We note in passing that recently Leavitt (preprint) has shown that these are, in fact, the only classes for which the analogues of (i) and (ii) are true.) Later, Suliński (1966) proved that in the universal class of all alternative rings, any class of simple rings with identities satisfies the three conditions.

No other examples of classes of simple rings or algebras exhibiting this behaviour appear to have been reported. Suliński (1966) proposed as a candidate a certain class of Lie algebras, but Andrunakievich and Ryabukhin (1968) subsequently proved that in fact *no* class of simple Lie algebras behaves appropriately.

Our main purpose in this paper is to present some further examples of classes of simple rings and algebras which *do* satisfy the conditions. We also point out that the class of simple rings with identities need not satisfy the conditions, even in universal classes where all simple rings are associative.

Our results should be of some interest as sources of hereditary semi-simple classes in non-associative universal classes. Among the hereditary semi-simple classes are the semi-simple radical classes. After discarding rings of characteristic 2 (resp. 2 and 3) we are able to characterize the latter for the universal classes of right alternative rings (resp. (γ,δ) rings, where $(\gamma,\delta)\neq(\pm 1,0)$, $(1,1)$).

Throughout, we shall use as universal classes varieties of (not necessarily associative) algebras over a commutative, associative ring with identity, sometimes specializing to rings (Z-algebras) or algebras over a field.

1. Generalities

In an algebra A, associative or otherwise, an element a is said to be *F_1-regular* if a is in the ideal generated by

$$\{ax-x+ya-y \mid x,\, y\in A\}.$$

The algebra A is said to be F_1-regular if it consists of F_1-regular elements.

The *Brown-McCoy radical class* $\mathscr{G}$ (in any variety) is the class of all F_1-regular algebras. It is also the upper radical class defined (in the given variety) by the class of all simple algebras with identities.

For associative rings it is customary to define the Brown–McCoy radical by a property different from F_1-regularity (see, for example, Wiegandt (1974), p. 116), but for rings in general, this latter property defines a different radical. For some comments on this, see Smiley (1950), pp. 96–97.

In their original paper, where they considered only associative rings, Brown and McCoy (1947) obtained further information about $\mathscr{G}$. For any algebra A we define

$$\Gamma(A)=\bigcap\{I\lhd A \mid A/I \text{ is a simple algebra with identity}\}.$$

(This of course makes sense in any variety, if appropriately interpreted.) For associative algebras we have

(*) $$\Gamma(A)=\mathscr{G}(A) \quad \text{for all } A$$

(**) $\mathscr{G}(A)=0\neq A \Leftrightarrow A$ is a subdirect product of simple rings with identities.

Smiley (1950) transferred the radical theory of Brown and McCoy to rings in an arbitrary variety and called $\Gamma(A)$ the Brown–McCoy radical of A. In general, $\Gamma(A)\neq\mathscr{G}(A)$ (see Section 2 below) though of course $\mathscr{G}(A)\subseteq\Gamma(A)$ in every case.

PROPOSITION 1.1. *In any variety of algebras,* (*) *and* (**) *are equivalent.*

PROOF. (*)$\Rightarrow$(**): $\mathscr{G}(A)=0=\Gamma(A)$ if and only if $A \cong A/\Gamma(A)$, while the latter is a subdirect product of simple algebras with identities.

(**)$\Rightarrow$(*): For any A, $A/\mathscr{G}(A)$ is a subdirect product of simple algebras with identities, so A has a set $\{I_\lambda \mid \lambda \in \Lambda\}$ of ideals such that $A/I_\lambda \cong (A/\mathscr{G}(A))/(I_\lambda/\mathscr{G}(A))$ is a simple algebra with identity for each λ and $\bigcap_{\lambda\in\Lambda} I_\lambda = \mathscr{G}(A)$. Hence $\mathscr{G}(A) \supseteq \Gamma(A)$, and so $\mathscr{G}(A) = \Gamma(A)$.

The apparent assumption that $\mathscr{G}(A) = \Gamma(A)$ in the class of all rings leads to some inaccuracies in Friedman (1965).

Suliński (1966) imposed a set of conditions on a class $\mathscr{M}$ of simple objects in a category (subject to constraints we need not list here, but which make the category 'at least as good as' a variety of rings) which ensure that the upper radical class defined by $\mathscr{M}$ satisfies (*) and (**).

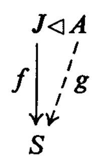
Adapting Suliński's terminology, we shall call a class $\mathscr{M}$ of simple algebras *modular* if it satisfies the following conditions.

(M1) If $S \in \mathscr{M}$ and $S \lhd A$, then $A = S \oplus X$ for some $X \lhd A$.

(M2) If $I \lhd J \lhd A$ and $J/I \in \mathscr{M}$, then $I \lhd A$.

(Suliński included the requirement, in (M1), that there be a unique maximal normal subobject M of A such that $S \cap M = 0$, but we shall not need this.)

PROPOSITION 1.2. *A class $\mathscr{M}$ of simple algebras is modular if and only if for every algebra A and every surjective homomorphism $f: J \to S$, with $J \lhd A$ and $S \in \mathscr{M}$, there is a homomorphism $g: A \to S$ such that $g \mid J = f$, i.e. the diagram*



can be completed to a commutative diagram.

PROOF. Suppose the stated condition is satisfied. If $S \lhd A$ and $S \in \mathscr{M}$, then there is a homomorphism $g: A \to S$ such that $g(s) = s$ for each $s \in S$. Then $A = S \oplus \operatorname{Ker}(g)$ and (M1) is satisfied. If $I \lhd J \lhd A$ and $J/I \in \mathscr{M}$, the following diagram can be completed.

We then have $I = J \cap \operatorname{Ker}(g) \lhd A$, so (M2) is satisfied. Hence $\mathscr{M}$ is modular.

Conversely, if $\mathcal{M}$ is modular, consider a situation $J \lhd A$, $f\colon J \to S \in \mathcal{M}$. We have $\operatorname{Ker}(f) \lhd J \lhd A$ and $J/\operatorname{Ker}(f) \in \mathcal{M}$, so by (M2), $\operatorname{Ker}(f) \lhd A$. But then

$$S \cong J/\operatorname{Ker}(f) \lhd A/\operatorname{Ker}(f),$$

so by (M1),

$$A/\operatorname{Ker}(f) = J/\operatorname{Ker}(f) \oplus X/\operatorname{Ker}(f)$$

for some $X \lhd A$. If $a \in A$, then

$$a + \operatorname{Ker}(f) = (j + \operatorname{Ker}(f)) + (x + \operatorname{Ker}(f)), \quad j \in J, \quad x \in X.$$

Define $g\colon A \to S$ by $g(a) = f(j)$. If $j + \operatorname{Ker}(f) = j' + \operatorname{Ker}(f)$, then $f(j) = f(j')$, so g is well defined. Also, for $a \in J$, we have $g(a) = f(a)$. This completes the proof.

THEOREM 1.3. *Let $\mathcal{M}$ be a modular family of simple algebras, and let $\mathcal{R}$ be the upper radical class defined by $\mathcal{M}$. Then the following conditions hold.*

(i) *$\mathcal{R}$ is hereditary.*

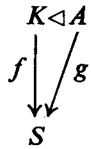
(ii) *$\mathcal{R}(A) = \bigcap\{J \lhd A \mid A/J \in \mathcal{M}\}$ for all A.*

(iii) *$\mathcal{R}(A) = 0$ if and only if A is a subdirect product of algebras from $\mathcal{M}$.*

(iv) *$\mathcal{R}$ has a hereditary semi-simple class.*

PROOF. (i) If $J \lhd A \in \mathcal{R}$ and $J \notin \mathcal{R}$, then J has a homomorphic image in $\mathcal{M}$. By Proposition 1.2, so does A—contradiction.

(ii) Let $K = \bigcap\{J \lhd A \mid A/J \in \mathcal{M}\}$. If K has a homomorphic image S in $\mathcal{M}$, then by Proposition 1.2 we have a commutative diagram



But then $K \subseteq \operatorname{Ker}(g)$, by definition of K. This is impossible, so $K \in \mathcal{R}$ and thus $K \subseteq \mathcal{R}(A)$. The reverse inclusion holds, as noted previously.

(iii) now follows as in Proposition 1.1.

(iv) The class of subdirect products of any family of simple objects is hereditary.

COROLLARY 1.4. *In a variety of algebras, if the class of simple algebras with identities is modular, then the following conditions are satisfied.*

(i) *$\mathcal{G}$ is hereditary.*

(ii) *$\mathcal{G}(A) = \Gamma(A)$ for all A.*

(iii) *$\mathcal{G}(A) = 0$ if and only if A is a subdirect product of simple algebras with identities.*

(iv) *$\mathcal{G}$ has a hereditary semi-simple class.*

(v) *$\mathcal{G}(J) = J \cap \mathcal{G}(A)$ whenever $J \lhd A$.*

Note that in a variety wherein every algebra can be embedded, as an ideal, in an algebra with identity, every modular family of simple algebras must consist of simple algebras with identities (by (M1)).

As noted previously, in the variety of all associative algebras, the class of all simple algebras with identities is modular. Suliński (1966) showed that the analogous statement for alternative algebras is also true. Further examples do not seem to have been pointed out. Suliński (1966) observed that the class of *complete* simple Lie algebras satisfies (M1) and asked whether this class is in fact modular. This question was answered in the negative by Andrunakievich and Ryabukhin (1968).

In the next section we shall present some further examples. For later use, we mention the following obvious result.

PROPOSITION 1.5. *If $\mathcal{M}$ is a modular class of simple algebras, so is any non-empty subclass.*

2. Examples

EXAMPLE 2.1. Corollary 1.4 is false for the variety of all rings. Let A be the algebra over the two-element field K_2 with basis $\{u, v, w\}$ and multiplication table

	u	v	w
u	u	w	0
v	0	v	0
w	v	v	w .

A is subdirectly irreducible, with heart $H = \{0, v, w, v+w\}$. Since

$$A/H = \{0, u+H\} \cong K_2,$$

we have $\Gamma(A) \subseteq H$, and thus $\Gamma(A) = H$. However, we have

$$K_2 \cong \{0, v\} \lhd \{0, v, w, v+w\}; \quad \{0, v, w, v+w\}/\{0, v\} \cong K_2,$$

so that $\mathcal{G}(A) = 0$.

A (γ, δ)-*algebra*, where γ, δ are scalars, is an algebra satisfying the identity

$$(x, y, z) + \gamma(y, x, z) + \delta(z, x, y) = 0,$$

where $(x, y, z) = (xy)z - x(yz)$.

THEOREM 2.2. *In the variety of (γ,δ)-algebras over a field K, where*

$$(\gamma,\delta)\neq(\pm 1,0),\ (1,1)$$

and K does not have characteristic 2, 3, *the class of simple algebras with identities is modular, and thus the Brown–McCoy radical satisfies all the conditions of Corollary* 1.4.

PROOF. It has been shown by Hentzel and Cattaneo (1977) that all simple (γ,δ)-algebras are associative. Moreover, for every (γ,δ)-algebra A, the associator ideal A^a is locally nilpotent. Let S be simple, with identity, let J be an ideal of an algebra A and let $f\colon J\to S$ be a homomorphism. Then $J\cap A^a$ is locally nilpotent, so $f(J\cap A^a)=0$ and we have an induced map $\bar{f}\colon J/J\cap A^a\to S$, where

$$\bar{f}(j+J\cap A^a)=f(j)\quad\text{for each } j\in J.$$

But

$$J/J\cap A^a\cong(J+A^a)/A^a\lhd A/A^a,$$

and here everything is associative, so there is a homomorphism $g\colon A/A^a\to S$ such that $g(j+A^a)=\bar{f}(j+J\cap A^a)$ for each $j\in J$. Define $\bar{g}\colon A\to S$ by

$$\bar{g}(a)=g(a+A^a).$$

Then for each $j\in J$, we have $\bar{g}(j)=g(j+A^a)=\bar{f}(j+J\cap A^a)=f(j)$. The result now follows from Proposition 1.2.

Our next example concerns the *right alternative algebras* over a (commutative, associative) ring containing 1 and $\frac{1}{2}$. These are the algebras satisfying the identity

$$(x,y,y)=0.$$

We have not been able to show that the class of all simple algebras with identities is modular, but shall prove the modularity of the class of simple algebras with identities and with no nilpotent elements. These latter are alternative, by the following result of Mikheev (1969): *right alternative algebras over a ring containing $\frac{1}{2}$ satisfy the identity*

$$(x,x,y)^4=0.$$

In what follows, S is a simple right alternative algebra with identity and with no nilpotent elements, I is an ideal of A, $f\colon I\to S$ is a non-zero homomorphism, and a, b and c are elements of A.

We firstly note that linearization of the right alternative identity gives us

(1) $$(a,b,c)=-(a,c,b)$$

(compare Schafer (1966), p. 27). Lemma 1 of Kleinfeld (1953) says that

(2) $$a((cb)c)=((ac)b)c.$$

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Somewhat analogously to (1), we have

(3) $$f(a,b,u) = -f(b,a,u) \quad \text{for all } u \in I.$$

To see this, we first note that since $(ab)u$, $a(bu)$, and so on, are in I, the expressions in (3) are defined.

Now $(a+b, a+b, u)^4 = 0$, so $f(a+b, a+b, u)^4 = 0$, and thus

$$\begin{aligned} 0 = f(a+b,a+b,u) &= f(a,a,u)+f(a,b,u)+f(b,a,u)+f(b,b,u) \\ &= f(a,b,u)+f(b,a,u), \end{aligned}$$

since $f(a,a,u) = 0 = f(b,b,u)$, (a,a,u) and (b,b,u) being nilpotent.

We now choose an $i \in I$ such that $f(i) = 1$. We need to examine the effect of f on some elements involving i.

(4) $$f(i,a,i) = 0,$$

(5) $$f((ia)i) = f(i(ai)).$$

By (1), $f(i,a,i) = -f(i,i,a) = 0$, since $(i,i,a)^4 = 0$. This gives us (4), from which (5) immediately follows.

(6) $$f(ia - ai) = 0.$$

This is because

$$\begin{aligned} f(ia-ai) &= f(i)f(ia-ai) = f(i)f(ia) - f(i)f(ai) \\ &= f(ia) - f(i(ai)) = f(ia) - f((ia)i) \text{ (by (5))} \\ &= f(ia) - f(ia)f(i) = 0. \end{aligned}$$

(7) $$f(a, ib, i) = -f(a,i,b).$$

By (2), $f(((ai)b)i) = f(a((ib)i))$, so

$$\begin{aligned} 0 &= f(((ai)b)i - a((ib)i)) = f((a,i,b)i - (a,ib,i)) \\ &= f(a,i,b) - f(a,ib,i). \end{aligned}$$

(8) $$f(i,a,b) = f(i,a,bi).$$

We have

$$\begin{aligned} f(i,a,b) - f(i,a,bi) &= f((ia)b) - f((ab)i) - f((ia)(bi)) + f(i)f(a(bi)) && \text{(by (6))} \\ &= f((ia)b) - f(a,b,i) - f((ia)(bi)) \\ &= f(((ia)b)i - (a,b,i) - (ia)(bi)) \\ &= f(ia,b,i) - f(a,b,i) \\ &= f(-(b,ia,i)) + f(b,a,i) && \text{(by (3))} \\ &= f(b,i,a) + f(b,a,i) && \text{(by (7))} \\ &= 0. \end{aligned}$$

(9) $$f(a, ib, i) = f(a,b,i).$$

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We have

$$\begin{aligned} f(a, ib, i) - f(a, b, i) &= f(a, ib, i) + f(a, i, b) \\ &= f((a(ib))\,i - a((ib)\,i) + (ai)\,b - a(ib)) \\ &= f(-a((ib)\,i)) + f(a(ib)) + f((ai)\,b) - f(a(ib)) \\ &= f(-a((ib)\,i) + ((ai)\,b)\,i) = 0 \quad \text{by (2).} \end{aligned}$$

(10) $$f(b, a, i) = -f(i, a, b).$$

Here we have

$$\begin{aligned} f(b, a, i) &= f(b, ia, i) = -f(b, i, ia) = f(i, b, ia) = -f(i, ia, b) = -f(i, ia, bi) \\ &= f(ia, i, bi) = -f(ia, bi, i) = f(bi, ia, i) = -f(bi, i, a) = f(bi, a, i) \\ &= -f(a, bi, i) = f(a, i, bi) = -f(i, a, bi) = -f(i, a, b). \end{aligned}$$

(11) $$f((i(ab))\,i) = f(i((ab)\,i)) = f((ia)\,(bi)).$$

The first equality comes from (4). Furthermore,

$$\begin{aligned} f((ia)\,(bi)) - f((i(ab))\,i) &= f(-(ia, b, i) + ((ia)\,b)\,i - (i(ab))\,i) \\ &= f((b, ia, i) + (i, a, b)) = f((b(ia))\,i - b((ia)\,i) + (i, a, b)) \\ &= f((b(ia))\,i - ((bi)\,a)\,i + (i, a, b)) = f(-(b, i, a) + (i, a, b)) \\ &= f((b, a, i) + (i, a, b)) = f(b, a, i) + f(i, a, b) = 0. \end{aligned}$$

THEOREM 2.3. *In the variety of right alternative algebras over a commutative associative ring containing* 1 *and* $\frac{1}{2}$, *the class of simple algebras with identities and without nilpotent elements is modular.*

PROOF. Let $I \lhd A$, let S be simple, with identity and without nilpotent elements. Let $f\colon I \to S$ be a surjective homomorphism. Let $i \in I$ be such that $f(i) = 1 \in S$. Define $g\colon A \to S$ by

$$g(a) = f(iai).$$

(This is unambiguous by (4).)

It is clear that g preserves addition, while for $a, b \in A$ we have

$$\begin{aligned} g(ab) &= f(i(ab)\,i) = f((ia)\,(bi)) \qquad \text{(by (11))} \\ &= f(ia)f(bi) = [f(ia)f(i)]\,[f(i)f(bi)] = f(iai)f(ibi) \\ &= g(a)\,g(b). \end{aligned}$$

Thus g is a ring homomorphism, and for $j \in I$ we have $g(j) = f(iji) = f(i)f(j)f(i)$ (unambiguously, since S is alternative) $= f(j)$. The result now follows from Proposition 1.2.

Each modular class of simple algebras thus far exhibited has consisted of alternative algebras. Alternativity is, however, neither necessary nor sufficient for modularity.

An algebra over the two-element field K_2 is *autodistributive* (Fiedorowicz (1974); see also Kepka (1977), Gardner (1979c) if it satisfies the identities

$$x(yz) = (xy)(xz); \quad (xy)z = (xz)(yz).$$

Simple autodistributive algebras with identities need not be alternative; see Fiedorowicz (1974), Theorems 14, 16 or Kepka (1977), Theorem 4.4.

If, however, S is a simple autodistributive algebra with identity, A is autodistributive, $I \lhd A$, $f\colon I \to S$ is a surjective homomorphism and $f(i) = 1$, then if as in the previous proof we define $g\colon A \to S$ by

$$g(a) = f((ia)i),$$

we have

$$g(ab) = f((i(ab))i) = f(((ia)(ib))i) = f(((ia)i)((ib)i)) = f((ia)i)f((ib)i) = g(a)g(b).$$

THEOREM 3.4. *In the variety of autodistributive algebras, the class of simple rings with identities is modular and so the Brown–McCoy radical satisfies the conclusions of Corollary* 1.4.

Now consider the variety of all *associative-by-associative rings*:

$$\{A \mid \text{there exists } I \lhd A \text{ with } I \text{ and } A/I \text{ associative}\}.$$

It is clear that all simple rings in this variety are associative.

THEOREM 2.5. *In the variety of associative-by-associative rings, the class of simple rings with identities is not modular.*

PROOF. If the class were modular, then by Corollary 1.4, $\mathscr{G}$ would be a radical class containing all zerorings and having a hereditary semi-simple class. But by Theorem 2.10 of Gardner (1979a) the only such radical class is the whole variety.

Suliński (1966) dealt with a generalization of the Brown–McCoy radical to categories, and sought examples of modular classes of simple objects in this context. We have given here some further examples of modular classes of simple algebras. It would be nice to know about modular classes of simple objects of other types. We mention in passing a marginal case, for the category of modules over a ring (not necessarily commutative). The set of injective simple modules (if there are any; see, for example, Zaks (1968) for some relevant information) is clearly modular.

3. Semi-simple radical classes

Most known examples of semi-simple radical classes are varieties generated by finite sets of simple algebras with identities (see, for example, Stewart (1970), Gardner (1975), Gardner and Stewart (1975)). In this section we shall characterize 'most' semi-simple radical classes of (γ,δ)-rings (for $(\gamma,\delta)\neq((\pm 1,0),(1,1))$ and right alternative rings.

A semi-simple radical class is, *inter alia,* a variety (that is, a subvariety of the universal variety under consideration) which is closed under extensions (in the given universal variety) (see Gardner (1975), Theorems 1.4, 1.5 and Corollary 1.6). A variety $\mathscr{V}$ is the *product* of subvarieties

$$\mathscr{V}_1,\mathscr{V}_2,\ldots,\mathscr{V}_n;\ \mathscr{V}=\mathscr{V}_1\times\ldots\times\mathscr{V}_n,$$

if every $A\in\mathscr{V}$ is uniquely expressible in the form

$$A=A_1\times\ldots\times A_n;\quad A_i\in\mathscr{V}_i.$$

Our result is obtained from Theorem 1.1 of Gardner (1979b) in which the hypothesis can be weakened to require only the rings without nilpotent elements to be power-associative. This condition is clearly met by right alternative rings. In the case of (γ,δ)-rings, it follows from Theorem 11 of Hentzel and Cattaneo (1977) that the condition is satisfied.

THEOREM 3.1. *Let $\mathscr{V}$ be a proper extension-closed subvariety of the variety of (γ,δ)-rings $((\gamma,\delta)\neq(\pm 1,0),(1,1))$ or the variety of right alternative rings. Then there exist finitely many primes $p_1,p_2,\ldots,p_n$ and finitely many subvarieties $\mathscr{V}_1,\mathscr{V}_2,\ldots,\mathscr{V}_n$ such that*

(i) $\mathscr{V}=\mathscr{V}_1\times\mathscr{V}_2\times\ldots\times\mathscr{V}_n$ *and*

(ii) *each ring in $\mathscr{V}_i$ has characteristic p_i.*

Furthermore, $\mathscr{V}$ contains no ring A for which $A^2=0\neq A$.

Let $\mathscr{V}$ be a subvariety of some universal variety $\mathscr{W}$. For $A\in\mathscr{W}$, let

$$A(\mathscr{V})=\bigcap\{I\lhd A\mid A/I\in\mathscr{V}\}.$$

Then $\mathscr{V}$ is said to have *attainable identities* if $A(\mathscr{V})(\mathscr{V})=A(\mathscr{V})$ for every A. (This terminology is due to Tamura (1966).) A semi-simple radical class is the same thing as a variety with attainable identities (Gardner (1975), Theorem 1.5).

THEOREM 3.2. *Let $\mathscr{V}$ be a variety of (γ,δ)-rings where $(\gamma,\delta)\neq(\pm 1,0),(1,1)$, such that*

$$\{0\}\subset\mathscr{V}\subseteq\{A\mid pA=0\},$$

where p is a prime >3. The following conditions are equivalent.

(i) *$\mathscr{V}$ is closed under extensions.*

(ii) *$\mathscr{V}$ is a semi-simple radical class.*

(iii) *$\mathscr{V}$ has attainable identities.*

(iv) *$\mathscr{V}$ is generated, as a variety, by a finite set of finite (associative) fields.*

PROOF. (i) $\Rightarrow$ (ii): By Theorem 3.1, $\mathscr{V}$ contains no nilpotent rings, so by Theorem 11 of Hentzel and Cattaneo (1977) $\mathscr{V}$ consists of associative rings. As an extension-closed variety of associative rings, $\mathscr{V}$ is therefore generated by a finite strongly hereditary set $\mathscr{F}$ of finite fields (see Stewart (1970) or Gardner and Stewart (1975)). This means that $\mathscr{V}$ consists of all subdirect products of fields in $\mathscr{F}$. Now every field in $\mathscr{F}$ has characteristic p, so by Theorem 2.2, Theorem 1.3 and Proposition 1.5, $\mathscr{V}$ is a semi-simple class of (γ,δ)-algebras over the field $GF(p)$ of p elements. Arguing as in the final paragraph of the proof of Theorem 3.3 of Gardner (1975), we can now show that $\mathscr{V}$ is a semi-simple class of (γ,δ)-rings.

(ii) $\Leftrightarrow$ (iii), as noted above.

(iii) $\Rightarrow$ (i): See Corollary 1.6 of Gardner (1975) (or Mal'tsev 1967)).

(i) $\Rightarrow$ (iv): This has already been shown.

(iv) $\Rightarrow$ (i): $\mathscr{V}$ consists of all subdirect products of finite fields from a finite set and the latter is modular, so $\mathscr{V}$ is a semi-simple class, and hence $\mathscr{V}$ is closed under extensions.

THEOREM 3.3. *Let $\mathscr{V}$ be a variety of right alternative rings such that*

$$\{0\} \subset \mathscr{V} \subseteq \{A \mid pA = 0\},$$

where p is an odd prime. The following conditions are equivalent.

(i) *$\mathscr{V}$ is closed under extensions.*

(ii) *$\mathscr{V}$ is a semi-simple radical class.*

(iii) *$\mathscr{V}$ has attainable identities.*

(iv) *$\mathscr{V}$ is generated, as a variety, by a finite set of finite fields.*

PROOF. (i) $\Rightarrow$ (ii). By Theorem 3.1, $\mathscr{V}$ contains no nilpotent rings, so by the result of Mikheev cited in Section 2, each ring in $\mathscr{V}$ satisfies the identity $(x,x,y)=0$, that is, $\mathscr{V}$ consists of alternative rings. The rest of the proof of (i) $\Rightarrow$ (ii) is now like that of (i) $\Rightarrow$ (ii) in Theorem 3.2, provided Theorem 4.3 of Gardner (1979b) is used instead of the results of Gardner and Stewart (1975).

(ii) $\Leftrightarrow$ (iii); (iii) $\Rightarrow$ (i); (i) $\Leftrightarrow$ (iv): As for Theorem 3.2.

Theorem 3.1, 3.2 and 3.3 combine to yield complete descriptions of those radical semi-simple classes of (γ,δ)-rings (respectively right alternative rings) which do not contain rings of characteristic 2 or 3 (respectively 2).

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