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ANGULAR DERIVATIVE AND COMPACTNESS OF COMPOSITION OPERATORS ON LARGE WEIGHTED HARDY SPACES

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ABSTRACT. We show that a restriction on the angular derivative of the inducing map does not determine compact composition operators on large weighted Hardy spaces, thus answering in the negative a question posed by T. Kriete.

1. **Introduction.** In [3] T. Kriete and B. MacCluer study composition operators on weighted Bergman spaces with fast regular weights. They determine that compact composition operators are precisely those for which the inducing function has angular derivatives always strictly greater than one. In the case of a more general weighted Hardy space (to be defined later) they show that the same restriction on the angular derivatives is necessary for compactness of composition operators. Is it a sufficient condition, and if not, is there a smaller class of weighted Hardy spaces (including weighted Bergman spaces with fast regular weights) where it is a sufficient condition?

T. Kriete has raised the question of sufficiency of the above restriction of angular derivatives for compactness of one such class of spaces, namely the class of large weighted Hardy spaces. The example we construct below shows that there exists a noncompact composition operator on large weighted Hardy spaces for which the inducing function has angular derivatives equal to two (thus strictly greater then one). The example provides the negative answer to the raised question and reveals that compactness on large weighted Hardy spaces differs essentially from the compactness on weighted Bergman spaces with fast regular weights.

2. **Preliminaries.** Let ϕ be an analytic map that maps the unit disc D into itself. The composition operator C_ϕ is defined by $C_\phi f = f \circ \phi$ for any function f analytic on D. A good reference paper on composition operators is C. Cowen status report (see [1]).

Composition operators have been studied on many different Hilbert spaces of analytic functions. In this paper we deal with composition operators on large weighted Hardy spaces. The reference that concerns us the most is the article by T. Kriete and B. MacCluer (see [3]).

We proceed with the definition of weighted Hardy space. Let the sequence $\beta = \{\beta_n\}_{n=0}^{\infty}$

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of positive numbers be such that the power series

$$\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} t^n$$

has radius of convergence equal to one. The set $H^2(\beta)$ of all complex formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which

$$\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$$

is a Hilbert space of functions analytic on the unit disc D with a norm defined by

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2$$

(see [6]). We say that $H^2(\beta)$ is a weighted Hardy space. If $\beta_n = 1$ for every n, then we get the classical Hardy space H^2.

We shall be interested in the *large weighted Hardy spaces*, *i.e.* the spaces $H^2(\beta)$ for which $n^\alpha \beta_n \longrightarrow 0$ as $n \longrightarrow \infty$ for every $\alpha > 0$. For example, if $\beta_n = \exp(-n^{1/2})$, then the corresponding space $H^2(\beta)$ is a large weighted Hardy space. Weighted Bergman spaces with fast weights are all large weighted Hardy spaces (for details see [3]).

An analytic self map ϕ of the unit disc D has a *finite angular derivative at a point* ξ in ∂D if the nontangental limit

$$\lim_{z \to \xi} \frac{\phi(z) - \omega}{z - \xi}$$

exists for some point ω in ∂D. In that case we write

$$\phi'(\xi) = \lim_{z \to \xi} \frac{\phi(z) - \omega}{z - \xi} \quad (z \longrightarrow \xi \text{ nontangentially}).$$

By Julia-Caratheodory theorem ([5]) the map ϕ has an angular derivative at ξ if and only if ϕ has a nontangential limit of modulus 1 at ξ, and ϕ' has a nontangential limit at ξ. In case $\phi'(\xi)$ does not exist, we shall assume that $|\phi'(\xi)|$ is infinity.

Angular derivatives play an important role in the problem of determination of compact composition operators.

The condition that $|\phi'(\xi)| = \infty$ for all ξ in ∂D is necessary (but not sufficient) for compactness of C_ϕ on H^2, whereas the same condition on ϕ completely determines compact composition operators on weighted Bergman spaces (for details see [4].)

The following two results from [3] motivate the problem on compactness and angular derivatives that we deal with in this paper.

(PART OF) THEOREM 1.3, (SEE [3], P. 758). *Let $H^2(\beta)$ be a weighted Hardy space with fast regular weight and let ϕ be an analytic self map of D. Then C_ϕ is compact on $H^2(\beta)$ if and only if $|\phi'(\xi)| > 1$ for all ξ in ∂D.*

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PROPOSITION 4.2, (SEE [3], P. 774). *Let $H^2(\beta)$ be a space for which the series $\sum \frac{1}{\beta_n^2}$ diverges, and let ϕ be an analytic self map of D. If C_ϕ is bounded on $H^2(\beta)$ and if there exists ξ in ∂D with $|\phi'(\xi)| \leq 1$, then C_ϕ is not compact.*

THE QUESTION (T. KRIETE). If ϕ is an analytic self map of D for which $|\phi'(\xi)| > 1$ for all ξ in ∂D, must C_ϕ be compact on the large weighted Hardy spaces?

3. **The example.** We construct a large weighted Hardy space on which the function $\phi(z) = z^2$ induces a non-compact bounded composition operator C_ϕ.

Let us write a positive sequence $\beta = \{\beta_n\}$ in the form

$$\beta_n = \exp\big(-h(n)\big)$$

for some function $h: N \longrightarrow R$. Suppose that $\frac{\beta_{n+1}}{\beta_n} \longrightarrow 1$ as $n \longrightarrow \infty$, *i.e.*, suppose that $h(n) - h(n+1) \longrightarrow 0$ as $n \longrightarrow \infty$. This restriction guarantees that the functions in $H^2(\beta)$ are analytic in the unit disc and that the operator multiplication by z is bounded.

The condition $n^\alpha \beta_n \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\alpha > 0$, transforms into the condition $\frac{h(n)}{\log n} \longrightarrow \infty$ as $n \longrightarrow \infty$.

Define $e_n(z) = \frac{z^n}{\beta_n}$ for $n = 1, 2, \ldots$. Then $\{e_n\}$ forms an orthonormal basis for the space $H^2(\beta)$ and so $\{e_n\}$ converges weakly to 0 as $n \longrightarrow \infty$. In order for C_ϕ to be compact on $H^2(\beta)$, $\{C_\phi e_n\}$ must converge strongly to 0. Since $\phi(z) = z^2$, we have that

$$(C_\phi e_n)(z) = (e_n \circ \phi)(z) = e_n(z^2) = \frac{z^{2n}}{\beta_n},$$

and so

$$\|C_\phi e_n\| = \frac{\beta_{2n}}{\beta_n} = \exp\big(h(n) - h(2n)\big).$$

We look for a sequence $\beta = \{\beta_n\}$ for which $\frac{\beta_{2n}}{\beta_n}$ does not converge to 0, *i.e.* for which $h(2n) - h(n)$ does not converge to infinity.

To summarize, we need a function $h: N \longrightarrow R$ such that

1. $h(n+1) - h(n) \longrightarrow 0$ as $n \longrightarrow \infty$, (*i.e.* $\frac{\beta_{n+1}}{\beta_n} \longrightarrow 1$ and the sequence $\{\beta_n\}$ defines an $H^2(\beta)$ space)
2. $\frac{h(n)}{\log n} \longrightarrow \infty$ as $n \longrightarrow \infty$, (*i.e.* $n^\alpha \beta_n \longrightarrow 0$ for all $\alpha > 0$, and $H^2(\beta)$ is a large weighted Hardy space)
3. $h(2n) - h(n) \not\longrightarrow \infty$ as $n \longrightarrow \infty$ (*i.e.* $\frac{\beta_{2n}}{\beta_n} \not\longrightarrow 0$ and C_ϕ is not compact on $H^2(\beta)$).

Define $h: N \longrightarrow R$ as follows. For every $m = 0, 1, 2, \ldots$

(i) $h(2^{2m}) = (2m)^2$

(ii) $h(2^{2m} + k) = (2m)^2 + k\frac{m}{2^{2m}}$ for $1 \leq k \leq 2^{2m}$

(iii) $h(2^{2m+1} + k) = (2m)^2 + m + k\frac{1}{2^{2m}}\Big[\big(2(m+1)\big)^2 + \frac{m+1}{2} - (2m)^2 - m\Big]$ for $1 \leq k \leq 2^{2m}$

(iv) $h(2^{2m+1} + 2^{2m} + k) = \big(2(m+1)\big)^2 + \frac{m+1}{2} - k\frac{1}{2^{2m}}\frac{m+1}{2}$ for $1 \leq k \leq 2^{2m}$.

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This way we get a function that grows fast enough but still repeats some values from time to time,the latter being true since $h(3\cdot 2^{2m}) = h(3\cdot 2^{2m+1}) = \big(2(m+1)\big)^2 + \frac{m+1}{2}$ for $m = 0,1,2,\ldots$.

Note that $\phi(z) = z^2$ defines a bounded composition operator on $H^2(\beta)$ for $\beta_n = \exp\big(-h(n)\big)$. Calculation shows that $h(n) - h(2n) \leq 0$ for all n (n being the argument of h in (i) — —(iv)), *i.e.* that $\exp\big(h(n) - h(2n)\big) \leq 1$. Hence for $f(z) = \sum a_n z^n$ and f in $H^2(\beta)$ we have that

$$\|C_\phi f\|^2 = \|f\circ\phi\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \Big(\frac{\beta_{2n}}{\beta_n}\Big)^2 \leq \sum_{n=0}^{\infty} |a_n|^2\beta_n^2 = \|f\|^2$$

where the inequality follows from $\frac{\beta_{2n}}{\beta_n} = \exp\big(h(n) - h(2n)\big) \leq 1$.

Let us see now why the conditions 1, 2 and 3 are satisfied.

For every n there is an m such that the difference $h(n+1) - h(n)$ equals to one of the following:

$$\frac{m}{2^{2m}}$$

$$\frac{\big(2(m+1)\big)^2 + \frac{m+1}{2} - (2m)^2 - m}{2^{2m}}$$

$$\frac{m+1}{2^{2m+1}}.$$

All of them converge to 0 as $m \longrightarrow \infty$ and hence condition 1 follows.

For every n, there is an m such that the ratio $\frac{h(n)}{\log n}$ equals to (or is greater than) one of the following:

$$\frac{(2m)^2}{2m\cdot\log 2}$$

$$\frac{(2m)^2}{(2m+1)\log 2}$$

$$\frac{(2m)^2 + m}{2m\cdot\log 2 + \log 3}$$

$$\frac{\big(2(m+1)\big)^2 + \frac{m+1}{2}}{2(m+1)\log 2}.$$

All of these converge to infinity as $m \longrightarrow \infty$, so that condition 2 is satisfied.

The difference $h(2n) - h(n)$ does not converge to infinity because, as we have pointed out earlier,

$$h\big(2(3\cdot 2^{2m})\big) - h(3\cdot 2^{2m}) = 0, \quad \text{for } m = 0,1,2,\ldots.$$

Hence, the condition 3 is also fulfilled.

Observe that the angular derivative of $\phi(z) = z^2$ is the same as the derivative. So $|\phi'(\xi)| = 2$ for all $\xi \in \partial D$, and is thus always bigger than one, as required.

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We conclude that compact composition operators on large weighted Hardy spaces $H^2(\beta)$ can not be characterized by the considered condition on the angular derivatives. The main reason is that the nice integral structure of weighted Bergman spaces with fast regular weights has been replaced with not so restrictive series structure where the weight sequence β is allowed to oscillate (as in our example). Introducing some regularity restrictions on β (for example monotonicity) might yield interesting results regarding the compactness of composition operators on such $H^2(\beta)$ spaces.

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