A note on the correlation of classes.

By J. M. Whittaker.

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1. Let R be a $(1-1)$ relation between the members of two similar classes A, B_1. It correlates the members of a subclass X of A to the members of a certain subclass Y of B_1 and thus defines a relation ρ connecting X and Y. It is clear that ρ is a $(1-1)$ relation and that it has the property (M). *If* $X_1 \rho Y_1$, $X_2 \rho Y_2$, *then* $X_1 \subset X_2$ *implies* $Y_1 \subset Y_2$.

It will be shown that

If $A \rho B_1 \subset B$, $B \sigma A_1 \subset A$, *there are subclasses* A_0, B_0 *of* A, B *such that* $A_0 \rho B_0$, $B - B_0 \sigma A - A_0$.

The proof consists in making a kind of Dedekind section of the subclasses X, and may be explained as follows.

If X' is defined by $X \rho Y$, $B - Y \sigma X'$ we say that X is a U if X, X' overlap, and that X is an L if they do not. The subclass A_0 whose existence we wish to demonstrate is to be such that $A_0' = A - A_0$. *i.e.* it is to be an L but as nearly as possible a U. Thus we might expect that there will be a largest L and that this will be A_0. It is not difficult to prove that this is the case.

2. An immediate consequence of (M) is the following lemma.

If $X_1 \subset X_2$, *then* $X_2' \subset X_1'$.

Let $A_0 =$ sum of all L's.[1] Then in the first place

(1) $\quad A_0 \subset A - A_0'$.

For by the lemma $A_0' \subset L'$ for every L and so

$$L \subset A - L' \subset A - A_0'.$$

[1] There may be no L's, but this does not matter since the null class is counted as a subclass of A. It will be noticed that the proof depends only on the fact that ρ, σ are $(1-1)$ relations with the property (M), so that the theorem is true for any relations with these properties. Thus it is not necessary that the members of X should be in $(1-1)$ relation with those of Y, nor that those subclasses of B to which the subclasses of A are correlated by ρ should be all the subclasses of a certain part B_1 of B.

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Thus $A - A_0'$ contains every L and so it contains A_0. By (1) and the lemma

$$(A - A_0')(A - A_0')' \subset (A - A_0')A_0' = 0$$

i.e. $A - A_0'$ is an L and so is contained in A_0. But by (1) A_0 is contained in $A - A_0'$. Thus

$$A_0 = A - A_0'$$

or

$$A_0' = A - A_0$$

which is the result stated.

3. An immediate corollary is the Schröder-Bernstein theorem.

If A is similar to a part of B and B is similar to a part of A, then A is similar to B.[1]

Again let A, B be simply ordered classes. We deduce that

If A is ordinally similar to a part of B and B is ordinally similar to a part of A, then there is a part A_0 of A which is ordinally similar to a part B_0 of B and such that $A - A_0$ is ordinally similar to $B - B_0$.

That the premisses of this proposition do not necessarily imply that A is ordinally similar to B is illustrated by the following trivial example. A consists of the real numbers in $(0 \leqslant x \leqslant 1)$ together with the rational numbers in $(1 \leqslant x \leqslant 2)$, B of the real numbers in $(0 \leqslant y \leqslant 2)$. Then A is not ordinally similar to B, but A is ordinally similar to a part of B by the relation $y = x$, and B is ordinally similar to a part of A by the relation $y = 2x$. A_0, B_0 are in this case the sets of rational numbers in $(0 \leqslant x \leqslant 2)$, $(0 \leqslant y \leqslant 2)$. These sets are ordinally similar by the first relation, while the set of irrational numbers in A is ordinally similar to the set of irrational numbers in B by the second relation.

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[1] *i.e.* if a, b are cardinal numbers, $a \leqq b$ and $b \leqq a$ together imply $a = b$.