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RESEARCH ARTICLE

An extension of the van Hemmen–Ando norm inequality

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Abstract

Let $C_{|||.|||}$ be an ideal of compact operators with symmetric norm $|||.|||$. In this paper, we extend the van Hemmen–Ando norm inequality for arbitrary bounded operators as follows: if f is an operator monotone function on $[0, \infty)$ and S and T are bounded operators in $\mathbb{B}(\mathscr{H})$ such that $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \Gamma_a = \{z \in \mathbb{C} \mid \mathrm{re}(z) \geq a\}$, then

$$|||f(S)X - Xf(T)||| \leq f'(a)\ |||SX - XT|||,$$

for each $X \in C_{|||.|||}$. In particular, if $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \Gamma_a$, then

$$|||S^rX - XT^r||| \leq ra^{r-1}\ |||SX - XT|||,$$

for each $X \in C_{|||.|||}$ and for each $0 \leq r \leq 1$.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded operators on a complex separable Hilbert space $\mathscr{H}$. Let $\mathcal{C}(\mathscr{H})$ be the algebra of all compact operators on $\mathscr{H}$, and $C_{\mathrm{fin}}(\mathscr{H})$ denotes the set of all finite rank operators on $\mathscr{H}$. A norm $|||.|||$ on $C_{\mathrm{fin}}(\mathscr{H})$ is called to be unitarily invariant or a symmetric norm if

$$|||UTV||| = |||T|||,$$

for every $T \in C_{\mathrm{fin}}(\mathscr{H})$ and any unitaries U, V, on $\mathscr{H}$. By the relation between the symmetric gauge functions and the unitarily invariant norms, we can define $|||T|||$ for all $T \in \mathbb{B}(\mathscr{H})$, see [6, Section 2]. Let

$$I_{|||.|||} = \{T \in \mathbb{B}(\mathscr{H}) :\ |||T||| < \infty\},$$

and $C_{|||.|||}$ be the norm closure of $C_{\mathrm{fin}}(\mathscr{H})$ in $I_{|||.|||}$. It is known that $C_{|||.|||}$ is a Banach space with respect to the norm $|||.|||$ and $C_{|||.|||} \subseteq \mathcal{C}(\mathscr{H})$. Also,

$$|||SXT||| \leq \|S\|\ |||X|||\ \|T\|,$$

for all $S, T \in \mathbb{B}(\mathscr{H})$ and all $X \in C_{|||.|||}$; see [6, Corollary 3.1]. For example, the Schatten p norms are unitarily invariant. Let S_p denote the Schatten ideal of compact operators with norms $\|.\|_p$ for each $1 \leq p < \infty$. For more details about unitarily invariant norms, we refer the reader to [4, 6, 13].

Let J be a subset of $\mathbb{R}$. We say that a continuous function f on an interval J is operator monotone, if $A \leq B$ implies that $f(A) \leq f(B)$ for all self-adjoint operators A and B, whose spectrums are contained in J. Ando and van Hemmen [15] showed that if f is an operator monotone function on $[0, \infty)$ and A and B are positive operators and $\mathrm{sp}(A + B) \subseteq [2a, \infty)$ for some positive scalar a, then

$$|||f(A) - f(B)||| \leq \left(\frac{f(a) - f(0)}{a}\right)\ |||A - B|||,$$

for every symmetric norm $|||.|||$. In continuation, Kittaneh and Kosaki [10] improved this inequality and showed that if f is an operator monotone function on $[0, \infty)$ and A and B are two positive operators that

$\operatorname{sp}(A) \subseteq [a, \infty)$ and $\operatorname{sp}(B) \subseteq [b, \infty)$, then

$$|||f(A)X - Xf(B)||| \leq d_{a,b}(f)\ |||AX - XB|||, \tag{1.1}$$

where $|||.|||$ is a symmetric norm, $X \in C_{|||.|||}$, and

$$d_{a,b}(f) = \begin{cases} \dfrac{f(b) - f(a)}{b - a} & \text{if } a \neq b \\ f'(a) & \text{if } a = b \end{cases}$$

Let $\Gamma_a = \{z \in \mathbb{C} \mid \operatorname{re}(z) \geq a\}$ for each $a \in \mathbb{R}$. In this paper, by a different argument than those of [10, 15], we extend Inequality (1.1) for arbitrary bounded operators. Indeed, we show that if f is an operator monotone function on $[0, \infty)$ and S and T are bounded operators such that $\operatorname{sp}(S), \operatorname{sp}(T) \subseteq \Gamma_a$, then

$$|||f(S)X - Xf(T)||| \leq f'(a)\ |||SX - XT|||,$$

for each symmetric norm $|||.|||$ and each $X \in C_{|||.|||}$. In particular, for any bounded operators S, T with $\operatorname{sp}(S), \operatorname{sp}(T) \subseteq \Gamma_a$, we have

$$|||S^r X - XT^r||| \leq ra^{r-1}\ |||SX - XT|||,$$

for each $X \in C_{|||.|||}$ and for each $0 \leq r \leq 1$.

2. Operator Lipschitz functions

Let $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ be a linear map. Let

$$\|\Phi\| = \sup\{\|\Phi(T)\| : \|T\| \leq 1\},$$

$$\|\Phi\|_1 = \sup\{\|\Phi(T)\|_1 : \|T\|_1 \leq 1\},$$

It is well known that if $\|\Phi\| = \|\Phi\|_1 = d$, then

$$|||\Phi(X)||| \leq d|||X|||, \tag{2.1}$$

for all $X \in C_{|||.|||}$. For details, see the first part of proof of [7, Proposition 2.7.].

Let $A(\mathbb{D})$ be the disk algebra of all continuous complex-valued functions on the unit disk $\mathbb{D}$, which are holomorphic in the interior of $\mathbb{D}$. It is well known that any function in $A(\mathbb{D})$ acts on the set of all contraction operators in $\mathbb{B}(\mathscr{H})$.

A continuous function f on the unit disk $\mathbb{D}$ is called operator Lipschitz with constant d, if

$$||f(S) - f(T)|| \leq d\ ||S - T||, \tag{2.2}$$

for all normal contraction operators T and S on any Hilbert space $\mathscr{H}$.

Kissin and Shulman in [9] proved that if $f \in A(\mathbb{D})$ is an operator Lipschitz function with constant d, then

$$||f(S) - f(T)|| \leq d\ ||S - T||,$$

for all arbitrary contraction operators S and T. Moreover, by using the interpolation theory, they proved that if $f \in A(\mathbb{D})$ is an operator Lipschitz function with constant d, then

$$||f(T) - f(S)||_p \leq d\ ||T - S||_p,$$

for any $1 \leq p < \infty$ and any contraction operators S and T with $S - T \in S_p$; see also [8, Theorem 6.4].

We can extend the results of [9] for a unitarily invariant norm ideals by using the majorization property that state in the first part of this section. Although, the proof of the following theorem is similar to

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[9, Theorem 4.2.], for the convenience of the reader we prove the following theorem. For more results on Lipschitz-type estimates for general symmetrically normed ideals, we refer the reader to [14].

Theorem 2.1. *Let $f \in A(\mathbb{D})$ be operator Lipschitz with constant d. Then, for arbitrary contraction operators S and T and an arbitrary operator $X \in C_{|||.|||}$, we have*

$$|||f(S)X - Xf(T)||| \leq d\ |||SX - XT|||.$$

Proof. First, assume that $\sigma(S) \cap \sigma(T) = \varnothing$. As the operator $\Delta = L_S - R_T$ on $\mathbb{B}(\mathscr{H})$ is invertible, we can consider the operator $F = (L_{f(S)} - R_{f(T)})\Delta^{-1}$. The proof of [9, Theorem 4.2.] shows that $\|F\| \leq d$ on $\mathbb{B}(\mathscr{H})$ and $\|F|_{S_1}\|_1 \leq d$. Now, by interpolation theory (equation (2.1)), for each unitarily invariant norm $|||.|||$ and for each $X \in C_{|||.|||}$, we have

$$|||F(X)||| \leq d\ |||X|||.$$

The definition of F implies that for each $S, T \in \mathbb{B}(\mathscr{H})$ with $\sigma(S) \cap \sigma(T) = \varnothing$ and for each $X \in C_{|||.|||}$, we have

$$|||f(S)X - Xf(T)||| \leq d\ |||SX - XT|||. \tag{2.3}$$

Now, if $\dim(\mathscr{H}) < \infty$ and $S, T \in \mathbb{B}(\mathscr{H})$, we can see that there exist contractions S_n such that $\sigma(S_n) \cap \sigma(T) = \varnothing$ and $\|S_n - S\| \to 0$. We have

$$\begin{aligned} |||f(S)X - Xf(T)||| &\leq |||f(S_n)X - Xf(T)||| + |||f(S_n)X - f(S)X||| \\ &\leq |||f(S_n)X - Xf(T)||| + \|f(S_n) - f(S)\|\ |||X||| \\ &\leq d\ |||S_nX - XT||| + \|f(S_n) - f(S)\|\ |||X|||. \end{aligned}$$

By the previous observation, we can prove (2.3) for finite rank operators S, T.

In the general case, let P_n be an increasing sequence of finite-dimensional projections such that $P_n \to I$ in the strong operator topology. We have

$$\begin{aligned} |||f(P_nS)XP_n - P_nXf(TP_n)||| &\leq d\ |||P_nSXP_n - P_nXTP_n||| \\ &= d\ |||P_n(SX - XT)P_n||| \\ &\leq d\ ||P_n||\ |||SX - XT|||\ ||P_n|| \\ &\leq d\ |||SX - XT|||. \end{aligned}$$

Since $C_{|||.|||}$ is an ideal of compact operators, $f(S)X - Xf(T)$ is compact. Now $f(P_nS)XP_n - P_nXf(TP_n) \to f(S)X - Xf(T)$ in the strong operator topology and $f(S)X - Xf(T)$ is compact, so by the noncommutative Fatou's lemma [13], we have

$$|||f(S)X - Xf(T)||| \leq \sup_{n \in \mathbb{N}} |||f(P_nS)XP_n - P_nXf(TP_n)||| \leq d\ |||SX - XT|||.$$

□

Let $\mathcal{O}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ be a closed disk in $\mathbb{C}$. We can see that $f \in A(\mathbb{D})$ is an operator Lipschitz function with constant d, if and only if $g(z) = f\left(\frac{1}{r}(z - z_0)\right)$ is an operator Lipschitz function with constant d on $\mathcal{O}_r(z_0)$. Hence, we have the following corollary.

Corollary 2.2. *Let f be an analytic function on the disk $\mathcal{O}_r(z_0)$ such that*

$$||f(S) - f(T)|| \leq d\ ||S - T||, \tag{2.4}$$

for all normal operators T,S on any Hilbert space $\mathscr{H}$ with $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \mathcal{O}_r(z_0)$. Then, for arbitrary operators S and T with $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \mathcal{O}_r(z_0)$ and an arbitrary operator $X \in C_{|||.|||}$, we have

$$|||f(S)X - Xf(T)||| \leq d\ |||SX - XT|||.$$

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3. Operator monotone functions

Let Π_+ be the upper half-plane and Π_- be the lower half-plane. Let $\Omega = \Pi_+ \cup \Pi_- \cup [0, \infty)$. Let f be an operator monotone function on $[0, \infty)$. The Löwner theorem [11] states that f is analytic on $(0, \infty)$ and has an analytic continuation to Ω, which again we denote by f, such that $f(\Pi_+) \subseteq \Pi_+$. Let $S \in \mathbb{B}(\mathscr{H})$ with $\mathrm{sp}(S) \subseteq \Omega \setminus \{0\}$ and f be an operator monotone function on $[0, \infty)$. Since f is analytic on Ω, we can define the operator $f(S)$ by the integral representation:

$$f(S) = \frac{1}{2\pi i} \int_\gamma f(z)(z - S)^{-1}\, dz, \tag{3.1}$$

where γ is a closed rectifiable curve in Ω such that $\mathrm{sp}(S) \subset \mathrm{ins}(\gamma)$.

Let $P[0, \infty)$ denote the set of all positive operator monotone functions defined in the positive half-line and consider the convex set:

$$\mathcal{P} = \{f \in P[0, \infty) | f(1) = 1\}.$$

Hansen in [5] showed that $\mathcal{P}$ is compact in the topology of point-wise convergence and extreme points in $\mathcal{P}$ are necessarily of the form:

$$f_\alpha(t) = \frac{t}{\alpha + (1-\alpha)t},$$

where $0 \le \alpha \le 1$. The next theorem shows that the family $\mathcal{P}$ is generated in the uniformly compact topology by the convex hull of its extreme points.

Theorem 3.1. [12, Theorem 3.1] *Let f be a nonnegative operator monotone function on* $[0, \infty)$ *such that* $f(1) = 1$. *Then, there exists a sequence f_n which is uniformly convergent to f on every compact subset of* Ω. *Moreover, for each n the following property hold:*

$$f_n = \sum_{i=1}^{k_n} \gamma_i f_{\alpha_i}, \tag{3.2}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{k_n}$ and $\gamma_1, \gamma_2, \ldots, \gamma_{k_n}$ are positive scalars such that $\sum_{i=1}^{k_n} \gamma_i = 1$.

In the last theorem, since f_n converges uniformly on compact sets to f, we can conclude that $f_n^{'}$ is also uniformly convergent to f' on compact sets. The following lemma will be useful.

Lemma 3.2. *Let $0 \le \alpha \le 1$, and let S,T be bounded invertible operators such that* $(\mathrm{sp}(S) \cup \mathrm{sp}(T)) \cap (-\infty, 0) = \emptyset$. *Then,*

$$f_\alpha(S) - f_\alpha(T) = \alpha f_{1-\alpha}(S^{-1})(S - T) f_{1-\alpha}(T^{-1}).$$

Proof. We can see that $f_\alpha(t) = (\alpha t^{-1} + (1-\alpha))^{-1}$. Since $\alpha S^{-1} + (1-\alpha)$ and $\alpha T^{-1} + (1-\alpha)$ are invertible, so

$$\begin{aligned}
f_\alpha(S) - f_\alpha(T) &= (\alpha S^{-1} + (1-\alpha))^{-1} - (\alpha T^{-1} + (1-\alpha))^{-1} \\
&= \alpha(\alpha S^{-1} + (1-\alpha))^{-1}(T^{-1} - S^{-1})(\alpha T^{-1} + (1-\alpha))^{-1} \\
&= \alpha(\alpha S^{-1} + (1-\alpha))^{-1} S^{-1}(S - T) T^{-1}(\alpha T^{-1} + (1-\alpha))^{-1} \\
&= \alpha(\alpha + (1-\alpha)S)^{-1}(S - T)(\alpha + (1-\alpha)T)^{-1} \\
&= \alpha f_{1-\alpha}(S^{-1})(S - T) f_{1-\alpha}(T^{-1}).
\end{aligned}$$

□

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Proposition 3.3. *Let f be an operator monotone function on* $[0,\infty)$. *Let S and T be bounded normal operators in* $\mathbb{B}(\mathscr{H})$ *such that* $\mathrm{sp}(S)\subseteq\Gamma_a$ *and* $\mathrm{sp}(T)\subseteq\Gamma_b$ *for some* $a,b>0$. *Then,*

$$\|f(S)-f(T)\|\le d_{a,b}(f)\ \|S-T\|,$$

for each $X\in C_{|||.|||}$.

Proof. Without loss of generality, we can assume that f is nonconstant. Let $T_\alpha=\alpha+(1-\alpha)T$ and $S_\alpha=\alpha+(1-\alpha)S$ for each $0\le\alpha\le 1$. As $\mathrm{sp}(S),\mathrm{sp}(T)\subseteq\Gamma_a$, we can conclude that T_α, S_α are invertible for each $0\le\alpha\le 1$. Moreover,

$$S_\alpha^*S_\alpha=\alpha^2+(1-\alpha)^2S^*S+\alpha(1-\alpha)(S+S^*).$$

Since S is normal, $S+S^*\ge 2a$ and $S^*S\ge a^2$. Therefore,

$$\begin{aligned}S_\alpha^*S_\alpha &\ge \alpha^2+(1-\alpha)^2S^*S+2a\alpha(1-\alpha)\\ &\ge \alpha^2+(1-\alpha)^2a^2+2a\alpha(1-\alpha)\\ &=(\alpha+(1-\alpha)a)^2.\end{aligned}$$

Hence, $(S_\alpha^*S_\alpha)^{-1}\le(\alpha+(1-\alpha)a)^{-2}$, and so

$$||S_\alpha^{-1}||=||S_\alpha^{*-1}S_\alpha^{-1}||^{\frac{1}{2}}=||(S_\alpha^*S_\alpha)^{-1}||^{\frac{1}{2}}\le(\alpha+(1-\alpha)a)^{-1}.$$

A similar argument implies that $||T_\alpha^{-1}||\le(\alpha+(1-\alpha)b)^{-1}$. By Lemma 3.2, we have

$$\begin{aligned}||f_\alpha(S)-f_\alpha(T)|| &= \alpha||S_\alpha^{-1}(S-T)T_\alpha^{-1}||\\ &\le \alpha||S_\alpha^{-1}||\ ||S-T||\ ||T_\alpha^{-1}||\\ &\le \frac{\alpha}{(\alpha+(1-\alpha)a)(\alpha+(1-\alpha)b)}||S-T||\\ &= d_{a,b}(f_\alpha)||S-T||.\end{aligned}$$

Now, assume that f is an arbitrary operator monotone function on $[0,\infty)$. By replacing $f(t)$ with $\frac{f(t)-f(0)}{f(1)-f(0)}$, we can assume that f is nonnegative and $f(1)=1$ (as f is non-constant, Lemma 3.2. in [2], implies that $f(1)\neq f(0)$). By Theorem 3.1, there exists a sequence $\{f_n\}$ in $\mathcal{P}$ that satisfies (3.2) and is uniformly convergent to f on compact sets. If

$$f_n=\sum_{i=1}^{k_n}\gamma_i f_{\alpha_i},$$

then $d_{a,b}(f_n)=\sum_{i=1}^{k_n}\gamma_i d_{a,b}(f_{\alpha_i})$ and we have

$$\begin{aligned}||f_n(S)-f_n(T)|| &= ||\sum_{i=1}^{k_n}\gamma_i f_{\alpha_i}(S)-\sum_{i=1}^{k_n}\gamma_i f_{\alpha_i}(T)||\\ &\le \sum_{i=1}^{k_n}\gamma_i\ ||f_{\alpha_i}(S)-f_{\alpha_i}(T)||\\ &\le \sum_{i=1}^{k_n}\gamma_i d_{a,b}(f_{\alpha_i})\ ||S-T||\\ &= d_{a,b}(f_n)||S-T||.\end{aligned}$$

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Letting $n \to \infty$ to get

$$||f(S) - f(T)|| \leq d_{a,b}(f)||S - T||. \tag{3.3}$$

□

We obtain the following theorem.

Theorem 3.4. *Let f be an operator monotone function on* $[0, \infty)$. *Let S and T be bounded operators in* $\mathbb{B}(\mathscr{H})$ *such that* $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \Gamma_a$ *for some* $a > 0$. *Then,*

$$|||f(S)X - Xf(T)||| \leq f'(a)\ |||SX - XT|||,$$

for each $X \in C_{|||.|||}$.

Proof. Let S, T be arbitrary and $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \{z \in \mathbb{C} \mid \mathrm{re}(z) > a\}$. Since $\mathrm{sp}(S)$ and $\mathrm{sp}(T)$ are compact, there exists a closed disk $\mathcal{O} \subset \Gamma_a$ such that $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \mathcal{O}$. Proposition 3.3 shows that f is operator Lipschitz with constant $f'(a)$ on the closed disk $\mathcal{O}$. Hence, Corollary 2.2 implies that

$$|||f(S)X - Xf(T)||| \leq f'(a)|||SX - XT|||,$$

for any symmetric norm $|||.|||$ and any $X \in C_{|||.|||}$.

In the general case, the assumptions $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \Gamma_a$ imply that $\mathrm{sp}(S + 1/n), \mathrm{sp}(T + 1/n) \subseteq \{z \in \mathbb{C} \mid \mathrm{re}(z) > a\}$ for each $n \in \mathbb{N}$. We use the noncommutative Fatou's lemma to get

$$\begin{aligned}
|||f(S)X - Xf(T)||| &\leq \sup_{n\in\mathbb{N}}\ |||f(S + 1/n)X - Xf(T + 1/n)||| \\
&\leq f'(a)\ \sup_{n\in\mathbb{N}} |||(S + 1/n)X - X(T + 1/n)||| \\
&= f'(a)\ \limsup_{n} |||(S + 1/n)X - X(T + 1/n)||| \\
&= f'(a)|||SX - XT|||.
\end{aligned}$$

□

Corollary 3.5. *Let f be an operator monotone function on* $[0, \infty)$. *Let S and T be bounded operators in* $\mathbb{B}(\mathscr{H})$ *such that* $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \Gamma_a$ *for some* $a > 0$ *and* $T - S \in C_{|||.|||}$. *Then,*

$$|||f(S) - f(T)||| \leq f'(a)\ |||S - T|||.$$

Proof. Let P_n be an increasing sequence of finite-dimensional projections such that $P_n \to I$ in the strong operator topology. We have

$$\begin{aligned}
|||f(P_nS)P_n - P_nf(TP_n)||| &\leq f'(a)|||P_nSP_n - P_nTP_n||| \\
&= f'(a)|||P_n(S - T)P_n||| \\
&\leq f'(a)||P_n||\ |||S - T|||\ ||P_n||| \\
&\leq f'(a)|||S - T|||.
\end{aligned}$$

Since f is an analytic function and $S - T$ is a compact operator, $f(S) - f(T)$ is compact. Now $f(P_nS)P_n - P_nf(TP_n) \to f(S) - f(T)$ in the strong operator topology and $f(S) - f(T)$ is compact, so by the noncommutative Fatou's lemma, we have

$$|||f(S) - f(T)||| \leq \sup_{n\in\mathbb{N}}\ |||f(P_nS)P_n - P_nf(TP_n)||| \leq f'(a)|||S - T|||.$$

□

As $t \mapsto t^r$ and $t \mapsto \log(t+1)$ are operator monotone functions on $[0, \infty)$ for each $0 \le r \le 1$, we obtain the following corollaries.

Corollary 3.6. *Let* $0 \le r \le 1$*, and let* S,T *be bounded operators such that* $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \Gamma_a$*. Then*

$$|||S^rX - XT^r||| \le ra^{r-1}\ |||SX - XT|||,$$

for each $X \in C_{|||.|||}$*. In particular, if* $T - S \in C_{|||.|||}$*, then*

$$|||S^r - T^r||| \le ra^{r-1}\ |||S - T|||.$$

Corollary 3.7. *If* S *and* T *are bounded operators such that* $\mathrm{sp}(S), \mathrm{sp}(T) \subseteq \Gamma_a$*, then*

$$|||\log(S+1)X - X\log(T+1)||| \le \frac{1}{a+1}\ |||SX - XT|||,$$

for each $X \in C_{|||.|||}$.

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