1
Partial Differential Equations

1.1 Introduction

Most physical as well as engineering systems one encounters in real life can be mathematically modeled using a system of partial differential equations subject to appropriate boundary conditions. These partial differential equations are coupled as well as nonlinear in nature. Owing to their nonlinearity, systems of partial differential equations that represent physical and engineering phenomena do not have closed-form or analytical solutions. Thus, the only alternative available to a scientist or a engineer is to seek a numerical solution for the aforementioned systems of partial differential equations.

There are countless examples of the manifestation of partial differential equations with appropriate boundary conditions in various fields of physics, including magnetism, optics, statistical physics, general relativity, superconductivity, liquid crystals, turbulent flow in plasma and solitons. Furthermore, diverse fields such as fluid mechanics, atmospheric physics, and ocean physics have rich and exhaustive examples of partial differential equations. In this book an effort has been made to familiarize the readers to a general introduction of partial differential equations as well as equations of fluid motion before acquainting them with the various numerical methods. The well-known method of finite differences is introduced and important aspects such as consistency and stability are discussed while applying the above method to standard partial differential equations of the parabolic, hyperbolic, and elliptic types. The method of finite differences is then applied to equations of motion of the atmosphere and oceans. The book also introduces the readers to advanced numerical methods such as semi-Lagrangian methods, spectral method, finite volume, and finite element methods and provides for the application of the above methods to the equations of motion of the atmosphere and oceans.

Towards this end, it is important to introduce partial differential equations (PDE) and the various numerical methods that can be employed to solve PDEs numerically. A PDE is an equation that represents a relationship between an unknown function of two or more independent variables and the partial derivatives of this unknown function with respect to the independent variables. Although the independent variables are either space (x, y, z) or space and time (x, y, z, t) related, the nature of the unknown function depends on the physical/engineering problem being modeled.

The function $f(x)$ is defined as a linear function of x if $f(x)$ can be expressed as $f(x) = mx + b$, where m and b are constants. The order of a PDE is determined by the highest-order derivative that appears in the PDE.

If $u(x, y)$ is a dependent variable, which is a function of two independent variables x and y, then the general second-order PDE can be written as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) + G(x, y) = 0, \tag{1.1}$$

where A, B, C are functions of $x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, F may be a nonlinear function, and G may be a function of x and y. In such cases, Equation (1.1) is known as a second-order quasilinear PDE. A quasilinear PDE is a PDE that is linear in the highest derivative. A partial differential equation is called a quasilinear PDE if all the terms with the highest-order derivatives of dependent variables are linear. The coefficients of the highest-order derivative terms in the PDE are functions of only the lower order derivatives of the dependent variables. However, for the quasilinear PDE, the terms in the PDE with lower order derivatives can occur in any manner.

A partial differential equation is called a semilinear PDE if all the terms with the highest-order derivatives of dependent variables are functions of independent variables only. In such cases, the coefficients of the highest-order derivative terms in the PDE are functions of only the independent variables. Equation (1.1) is known as a second-order semilinear PDE if A, B, and C are functions of x and y only.

If the dependent variable and all its partial derivatives appear linearly in any PDE, i.e., there are no terms in the PDE that involve the product of the dependent variables with itself or with its derivatives, then such an equation is called a linear PDE. If F is a linear function, and A, B, and C are functions of only x and y, then Equation (1.1) is called a linear PDE.

If all the terms of a PDE contain the dependent variable or its partial derivatives, then such a PDE is called a homogeneous partial differential equation.

If function F involves the dependent variable u and its derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and also $G = 0$, the Equation (1.1) is called a homogeneous PDE; if $G \neq 0$, then Equation (1.1) is called a nonhomogeneous PDE.

Equation (1.1) can also be written in the following form known as the implicit form

$$f(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1.2}$$

If f is a linear function of u and its derivatives, then the PDE is said to be linear. It is necessary to classify PDE, as different types of PDE arise naturally in very different physical problems; dissimilar types of PDE have different nature of conditions (boundary/initial) to be satisfied and hence, dissimilar types of PDE need to employ different numerical methods for their solution.

It is known that the general solution of ordinary differential equations (ODEs) involve arbitrary constants of integration; in contrast, the general solution of PDEs involves arbitrary functions. Consider, for example, the equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0. \tag{1.3}$$

Integrating Equation (1.3) with respect to y, one gets $\partial u/\partial x = F(x)$, where $F(x)$ is an arbitrary function of x. Integrating the equation once again with respect to x, one gets

$$u(x, y) = f(x) + g(y), \tag{1.4}$$

where $f(x) = \int F(x)dx$ and $g(y)$ are arbitrary functions of x and y respectively. To obtain $f(x)$ and $g(y)$, one needs to have additional information, for example, the initial conditions (if time is one of the independent variables) and/or boundary conditions.

To be specific, suppose that one were to find $u(x, y)$ satisfying Equation (1.4) in the region $x \geq 0, y \geq 0$ and that one is given the following boundary conditions $u = x$, when $y = 0$ and $u = y$, when $x = 0$. Then, the surface $u(x, y)$ must intersect the plane $x = 0$ in the line $u = y$ and the plane $y = 0$ in the line $u = x$. The functions $f(x)$ and $g(y)$ in Equation (1.4) are determined in the following manner. As $u(x, 0) = f(x) + g(0) = x$ and $u(0, y) = f(0) + g(y) = y$, it follows that $u(x, y) = f(x) + g(y) = x - g(0) + y - f(0) = x + y - g(0) - f(0)$. The only way this can satisfy the PDE and the boundary conditions are if $f(0)$ and $g(0)$ are both zero, which implies $u(x, y) = x + y$.

It can be easily verified that the equation satisfies the PDE and the two boundary conditions. The aforementioned example clearly illustrates the importance of the boundary conditions in obtaining the solution of the PDE. For an ordinary differential equation of the second-order, it is known that two conditions are required to obtain an unique solution. It is clear that depending on the nature of the PDE, the sufficient set of boundary conditions that are required for a meaningful solution may vary.

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The question that is posed is as follows: what is a sufficient set of boundary conditions for a given PDE? The answer to this question depends on the type of PDE, the latter in turn, depending on the nature of the associated physical problem. Two different types of boundary conditions applied to the same PDE, will invariably lead to two different types of solution. Hence, methods of solution of PDEs will depend on the nature and type of the boundary conditions.

One expects that a given PDE subject to suitable boundary conditions will possess an unique solution. Any physical or engineering problem defined by Equation (1.1) in a given two-dimensional domain is said to be "well-posed" if

1. there exists at least one solution (existence)
2. there exists atmost one solution (uniqueness)
3. the solution is stable.

Three types of PDEs arise when one classifies PDE and these are (i) parabolic type, (ii) elliptic type, and (iii) hyperbolic type. Examples of the parabolic type of PDEs are the diffusion equation whereas examples of elliptic and hyperbolic PDEs are the Laplace equation and wave equation, respectively.

1.2 Diffusion Equation

The most common form of diffusion equation is as follows:

$$\frac{\partial c}{\partial t} = D\nabla^2 c, \tag{1.5}$$

where c is the concentration, which is in general a function of space and time, D is the diffusion coefficient, and ∇^2 is the Laplacian operator, which in Cartesian coordinates is

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

By definition, a flux $\bar{J}$ is a movement of particles (or other quantities) through a unit measure (point, length, area) per unit time. From Ficks' law of diffusion, it follows that flux $\bar{J}$ is related to concentration c through the following equation

$$\bar{J} = -D\nabla c, \tag{1.6}$$

where ∇ is the gradient operator. The negative sign signifies that the flux is always in the direction opposite to the gradient operator. The direction of the gradient operator,

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also known as the "ascendant," is in the maximum rate of change of 'c' and is always directed from low values to high values of c. Hence Equation (1.6) clearly shows that the direction of flux is always directed from high values to low values of c.

For a normal diffusion process, particles cannot be created or destroyed. This implies that the flux of particles into one region must be the sum of the particle flux flowing out of the surrounding regions. The aforementioned statement can be easily expressed mathematically by the continuity equation given by

$$\frac{\partial c}{\partial t} + \nabla \cdot \bar{J} = 0. \tag{1.7}$$

Using Equation (1.7) in Equation (1.6), one gets

$$\frac{\partial c}{\partial t} - \nabla \cdot (D \nabla c) = 0. \tag{1.8}$$

If the diffusion coefficient D is a constant, then Equation (1.8) becomes the diffusion equation (1.5). The diffusion equation can be applied to solving problems in mass diffusion, momentum diffusion, and heat diffusion. It is clear that under different situations, the diffusion equation assumes different forms. For example, in the case of heat diffusion, c will be the temperature T whereas D will become the coefficient of thermal diffusivity α. Equation (1.5) for the case of heat diffusion is also known as the heat conduction equation, whose one-dimensional form is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \tag{1.9}$$

where T is the temperature of a heated rod, α is the coefficient of thermal conductivity, x is the distance along the rod, and t is the time. In Figure 1.1, the heated rod extends from $x = 0$ to $x = a$ with $T(x,t)$ being the temperature of the rod at location x, the distance from the end $x = 0$ and time t.

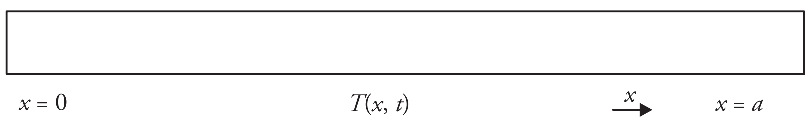

Figure 1.1 Temperature distribution of a heated rod of length a.

It is extremely helpful to picturize the solution of a PDE. In the case of Equation (1.9), the solution can be expressed as a surface, $z = T(x,t)$ in a three-dimensional space (x,t,z), as shown in Fig 1.2. The domain of the solution, Ω, is the region $0 \le t < \infty$ and

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$0 \leq x \leq a$. The temperature distribution at some time $t_o > 0$ is the curve $z = T(x, t_o)$, where the plane $t = t_o$ intersects the solution curve. The curve $z = T(x, 0)$ is the initial temperature distribution that is assumed to be given. Equation (1.9) states that at any point (i.e., at any point x, t) in the solution surface, the slope of the surface in the t-direction is related locally to the rate of change of the slope in the x-direction. It is abundantly clear that in order to obtain a unique solution, there is a need to prescribe the nature of the solution (the behaviour of the surface) at the edges of the solution domain: at $t = 0$ and at $x = 0$, and $x = a$. It makes sense to expect, on the basis of physical reasoning, that in order to predict the future evolution of the temperature, one needs to have knowledge of the initial state, i.e., the initial temperature distribution in the rod, $T(x, 0)$. In a similar manner, it makes sense to expect that the temperature values at the ends of the rod at any particular time would affect the temperature distribution in the rod.

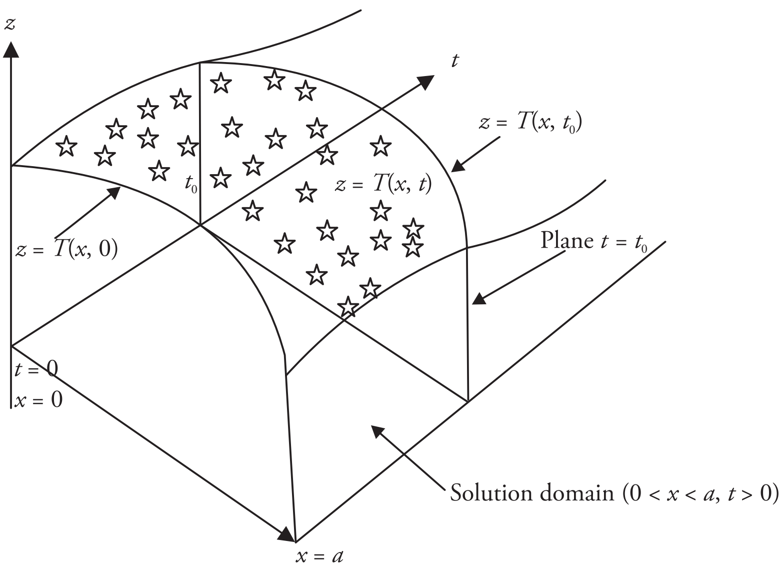

Figure 1.2 Solution surface and solution domain, Ω, for Equation (1.9).

1.3 First-order Equations

One of the most important first-order PDE is the one-dimensional advection equation,

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0 \tag{1.10}$$

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where ρ is the air density, u is a constant velocity, t is the time, and x is space coordinate. Equation (1.10) expresses the statement that the rate of change of air density of an air parcel with respect to time is zero following the motion, i.e., the air density of an air parcel is constant following the motion. Alternatively, Equation (1.10) states that the total or substantive derivative of air density is zero, following the motion. A fluid flow is said to be incompressible if the fractional change in density of an air parcel, associated with a change in pressure, following the motion is very small. In effect, incompressible fluid flow is one for which the rate of change of air density of an air parcel with respect to time is zero, following the motion. Hence, Equation (1.10) is valid for an incompressible fluid flow.

Consider a one-dimensional flow of an incompressible fluid. The continuity equation that expresses the principle of conservation of mass for a one-dimensional flow of density $\rho(x,t)$ for an incompressible fluid is expressed as

$$\frac{d\rho}{dt} = 0, \tag{1.11}$$

where $d\rho/dt$ signifies the rate of change of density $\rho(x,t)$ following the motion as expressed in the Lagrangian description of fluid motion. Its equivalent expression in the Eulerian description of motion is given by (1.10), where u is the nonzero constant velocity component in the x direction. The aforementioned equation called the advection equation can also be easily derived from the following consideration

Consider a one-dimensional flow of an incompressible fluid. Assuming that the fluid density $\rho(x,t)$ changes only due to convective/advective processes, one can write the following

$$\rho(x,t+\Delta t) = \rho(x-u\Delta t,t).$$

If Δt is sufficiently small, one can expand both sides of the equation by Taylor series expansion and retain only up to the linear term

$$\rho(x,t) + \Delta t \frac{\partial \rho(x,t)}{\partial t} = \rho(x,t) - u\Delta t \frac{\partial \rho(x,t)}{\partial x}$$

or canceling of Δt on both sides, one gets the one-dimensional advection equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0.$$

From this discussion it is clear that the exact solution of Equation (1.10) is given as

$$\rho(x,t) = F(x-ut), \tag{1.12}$$

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where the initial condition $\rho(x,0) = F(x)$. Equation (1.12) defines a right-traveling wave that propagates (i.e., convects or advects) the initial property (density) distribution to the right at the convection/advection velocity u. The aforementioned analytical solution indicates that the initial property (density) profile $\rho(x,0) = F(x)$ simply propagates (i.e., convects/advects) to the right with the constant velocity u, its shape and magnitude is unchanged.

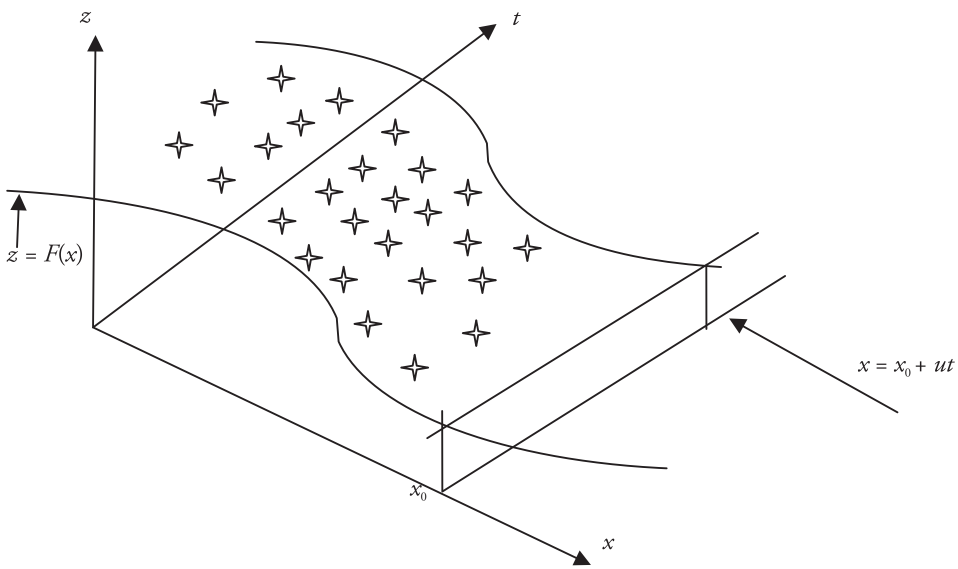

Figure 1.3 Solution surface for Equation (1.10).

If one moves with the solution point $x(t) = x_o + ut$, Equation (1.10) tells us that the rate of change of ρ is zero, i.e., in other words, ρ is a constant along a line $x = x_o + ut$. Figure 1.3 shows the solution surface in the (x,t,z)-space. It is clear from Figure 1.3 that the lines $x = x_o + ut$ are a family of parallel lines in the plane $z = 0$ that intersect the plane $t = 0$ at $x = 0$. The equation says that the height of the solution surface is always the same along such a line, i.e., the intersection of this solution surface with the plane $t = constant$ is a curve that is identical with the curve $z = F(x)$ at $t = 0$, but displaced in the x-direction by a distance ut. Thus, the solution represents a disturbance with arbitrary shape $F(x)$ translating uniformly with speed u in the positive x-direction if $u > 0$, or in the negative x-direction if $u < 0$.

It is clear that "information" about the initial distribution of ρ "propagates" or is "carried along" the lines $x = x_o + ut$ in the plane $z = 0$. These lines are called the

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characteristic curves, or simply the characteristics of the equation. The characteristic equation is then given as

$$\frac{dx}{dt} = u. \tag{1.13}$$

Integrating Equation (1.13) provides us the characteristic curves. The solution of the ODE Equation (1.13) involves one integration constant that determines where the characteristic curves intersect the x axis. Subsequently one needs to construct the solution surface that has the same value $F(x)$ along each characteristic in the x–t plane as that in the initial plane, $t = 0$.

Consider a general first-order partial differential equation as follows:

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c. \tag{1.14}$$

If a, b, and c are functions of x, y, and u, then Equation (1.14) is called a quasilinear PDE. If a and b are functions of x and y while c is a function of x, y and u, then Equation (1.14) is called a semilinear PDE. If a, b, and c are functions of x and y only, then Equation (1.14) is called a linear PDE. Quasilinear PDE is one in which the highest-order terms are linear.

1.4 First-order Equations: Method of Characteristics

Consider the simplest case of the following first-order linear partial differential equation

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y). \tag{1.15}$$

Assume that one can find a solution $u(x,y)$. Consider the function $S = \{x, y, u(x,y)\}$. If $u(x,y)$ is a solution of Equation (1.15), at each point (x,y), it is possible to express Equation (1.15) as the dot product

$$[a(x,y), b(x,y), c(x,y)] \cdot [u_x(x,y), u_y(x,y), -1] = 0. \tag{1.16}$$

From calculus, the normal to the surface $S = \{x, y, u(x,y)\}$ at the point $[x, y, u(x,y)]$ is given by

$$N(x,y) = [u_x(x,y), u_y(x,y), -1].$$

It is thus clear that if the vector $[a, b, c]$ is perpendicular to $[u_x, u_y, -1]$, then the vector $[a, b, c]$ lies in the tangent plane to S. Hence, to obtain a solution to Equation (1.15),

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one needs to find a surface S such that at each point (x, y, u) on S, the vector $[a, b, c]$ lies in the tangent plane. To construct such a surface, one first obtains a curve that lies in S. It is clear that the vector $[a, b, c]$ need to lie in the tangent plane to the surface S at each point (x, y, u) on the surface. Let there be a curve C parameterized by s such that at each point on the curve C, the vector $[a, b, c]$ will be tangent to the curve. That is, for a curve C parameterized as $C = \{(x(s), y(s), u(s)\}$, the following three conditions need to be satisfied

$$\frac{dx}{ds} = a(x(s), y(s)) \tag{1.17}$$

$$\frac{dy}{ds} = b(x(s), y(s)) \tag{1.18}$$

$$\frac{du}{ds} = c(x(s), y(s)) \tag{1.19}$$

Such a curve when it exists is called an integral curve for the vector field $[a, b, c]$. For solving a PDE of the form given in Equation (1.15), we need to find the integral curves for the vector field $V = [a(x, y), b(x, y), c(x, y)]$ associated with the PDE. These integral curves are known as *characteristic curves*. The aforementioned characteristic curves are obtained by solving the system of ordinary differential equations (1.17)-(1.19)(ODE).

Once the characteristic curves for Equation (1.15) are obtained, one needs to construct a solution of Equation (1.15) by forming a surface S as a union of these characteristic curves. Such a surface $S = x, y, u$ for which the vector field $V = [a(x, y), b(x, y), c(x, y)]$ lies in the tangent plane to S at each point (x, y, u) on S is known as the integral surface. Through the introduction of characteristic equations, the original PDE (Equation (1.15)) can be reduced to a system of ODEs. The concept is to solve the characteristic equations, obtain an union of the so-called characteristic curves to form a surface that would provide for the solution of the PDE (Equation (1.15)).

1.5 Second-order Quasilinear PDEs: Classification Using Method of Characteristics

The general quasilinear second-order nonhomogeneous PDE in two independent variables x and y are given as

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g. \tag{1.20}$$

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The classification of PDEs to parabolic, elliptic, and hyperbolic PDEs are analogous to the classification of conic section. For example, conics are generally described by the second-order algebraic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1.21}$$

The conics as described by Equation (1.21) are classified as parabolic, elliptic, and hyperbola based on the sign of the discriminant, $B^2 - 4AC$, ($B^2 - 4AC = 0$ is defined as a parabola, $B^2 - 4AC < 0$ is defined as an elliptic and $B^2 - 4AC > 0$ is defined as a hyperbola). In exactly the same way, the second-order quasilinear PDE (Equation (1.20)) is classified based on the sign of the discriminant $b^2 - 4ac$, where a, b, and c refer to the coefficients of the highest (second-order) derivative; $b^2 - 4ac = 0$ is referred to as a parabolic partial differential equation, $b^2 - 4ac < 0$ is referred to as an elliptic partial differential equation, and $b^2 - 4ac > 0$ is referred to as hyperbolic partial differential equation. In this section, the classification of Equation (1.20) using the characteristics is examined. Earlier it was shown that the characteristic curves for the one-dimensional advection equation [Equation (1.10)] are the lines $x = x_o + ut$ in the plane $z = 0$. The solution for Equation (1.10) represents a disturbance with arbitrary shape $F(x)$ translating uniformly with speed u in the positive x-direction if $u > 0$, or in the negative x-direction if $u < 0$, i.e., "information" about the initial distribution of ρ "propagates" or is "carried along" the characteristic curves.

As discontinuities in the derivatives of the solution, if they exist, must propagate along the characteristics, it is possible to utilize the characteristics themselves to classify the second-order quasilinear PDEs. The following question is posed. Are there any curves in the solution domain passing through a general point P along which the highest-order derivatives [in the case of Equation (1.20)], the second-order derivatives of $u(x, y)$, i.e., u_{xx}, u_{xy}, and u_{yy}, are multi-valued or discontinuous? Such curves, if they exist, are the paths of information propagation. One equation that relates the three second-order derivatives of $u(x, y)$ is the PDE [Equation (1.20)] itself. One can obtain two more such equations as follows:

$$d\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}dx + \frac{\partial^2 u}{\partial y \partial x}dy; \tag{1.22}$$

$$d\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}dx + \frac{\partial^2 u}{\partial y^2}dy. \tag{1.23}$$

Equations (1.20), (1.22) and (1.23) can be written in matrix form with the second-order derivatives as unknown. If the determinant of the coefficient matrix vanishes, the second-order derivatives of $u(x, y)$ are indeterminate and thus, multi-valued or

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discontinuous. Setting the determinant of the coefficient matrix to zero, yields $a(dy)^2 - b(dy)(dx) + c(dx)^2 = 0$, whose solution is obtained from the quadratic formula

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.24}$$

Equation (1.24) is the ordinary differential equation for the two families of characteristic curves in the x, y plane, corresponding to the $\pm$ signs. The two families of characteristic curves, if they exist, may either be real and repeated, complex, or real and distinct. This requirement is equivalent to the discriminant $b^2 - 4ac = 0$, $b^2 - 4ac < 0$, and $b^2 - 4ac > 0$, i.e., the original PDE being either parabolic PDE, elliptic PDE, or hyperbolic PDE. Hence, while elliptic PDEs do not have any real characteristics, parabolic PDEs have one real and repeated characteristic, and hyperbolic PDEs have two real and distinct characteristic curves.

The existence of characteristic curves in the solution domain provides for the introduction of concepts such as domain of dependence and range of influence. The domain of dependence of a point $P(x, y)$ in the solution domain is defined as the region of the solution domain upon which the solution at point $P(x, y)$ depends. In other words, the solution at any point P depends on the solution over the domain of dependence. The range of influence of a point $P(x, y)$ in the solution domain is defined as the region of the solution domain in which the solution is influenced by the solution at point $P(x, y)$. That is, the solution at a point P influences the solution over the range of influence. As parabolic and hyperbolic PDEs have real characteristic curves, they will have a definite domain of dependence and range of influence in the real domain. However, elliptic PDEs do not have real characteristic curves. Hence, elliptic PDEs do not have a definite domain of dependence and range of influence in the real domain; thus, the entire solution domain of an elliptic PDE is both its domain of dependence and range of influence of every point in the solution domain. In order to further understand the concept of domain of dependence and range of influence, let us consider specific examples of (i) parabolic, (ii) hyperbolic, and (iii) elliptic PDEs.

The one-dimensional linear heat conduction equation is an example of a parabolic PDE,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}. \tag{1.25}$$

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Two other equations that can relate the second-order derivatives are as follows:

$$d\left(\frac{\partial T}{\partial x}\right) = \frac{\partial^2 T}{\partial x^2}dx + \frac{\partial^2 T}{\partial x \partial t}dt \tag{1.26}$$

$$d\left(\frac{\partial T}{\partial t}\right) = \frac{\partial^2 T}{\partial t \partial x}dx + \frac{\partial^2 T}{\partial t^2}dt \tag{1.27}$$

The characteristic differential equation is found by setting the determinant of the coefficient matrix to zero. This yields $\alpha[dt]^2 = 0$, which when integrated provides for time being equal to a constant for the characteristic paths. An example of a hyperbolic PDE is the second-order linear wave equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1.28}$$

Two other equations that can relate the second-order derivatives are as follows:

$$d\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}dx + \frac{\partial^2 u}{\partial x \partial t}dt \tag{1.29}$$

$$d\left(\frac{\partial u}{\partial t}\right) = \frac{\partial^2 u}{\partial t \partial x}dx + \frac{\partial^2 u}{\partial t^2}dt. \tag{1.30}$$

The characteristic differential equation is found by setting the determinant of the coefficient matrix to zero. This yields $c^2[dt]^2 = [dx]^2$, which consequently yields

$$\frac{dt}{dx} = \pm\frac{1}{c},$$

indicating that two distinct and real families of characteristic paths exist for a hyperbolic PDE.

An example of an elliptic PDE is the second-order linear Laplace equation given by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{1.31}$$

Two other equations that can relate the second-order derivatives are as follows:

$$d\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x^2}dx + \frac{\partial^2 \phi}{\partial x \partial y}dy \tag{1.32}$$

$$d\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial y \partial x}dx + \frac{\partial^2 \phi}{\partial y^2}dy \tag{1.33}$$

The characteristic differential equation is found by setting the determinant of the coefficient matrix to zero. This yields $[dy]^2 = -[dx]^2$, which subsequently yields

$$\frac{dy}{dx} = \pm i$$

indicating that the characteristic curves for an elliptic PDE do not lie in the real domain. The aforementioned concepts of domain of dependence and range of influence are illustrated for each of the aforementioned second-order linear PDE in Figure 1.4 (a) for a parabolic PDE, Figure 1.4(b) for a hyperbolic PDE and Figure 1.4 (c) for an elliptic PDE.

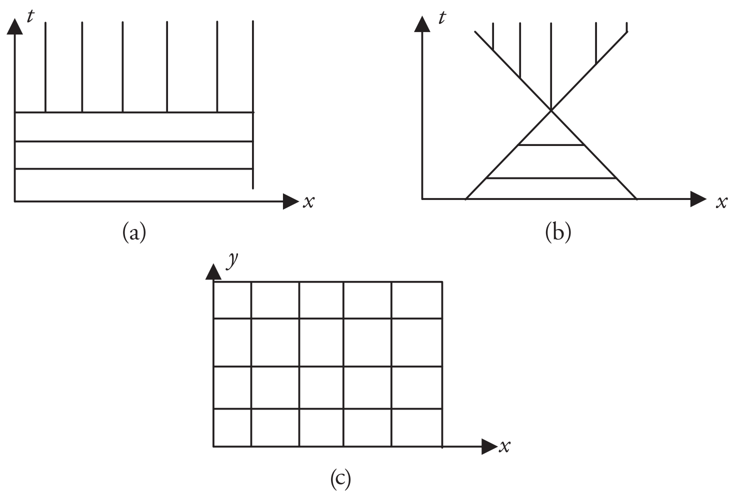

Figure 1.4 Domain of dependence (horizontal hatching) and range of influence (vertical hatching) for (a) parabolic, (b) hyperbolic, and (c) elliptic PDEs.

One can employ a similar strategy to classify the first-order quasilinear PDE. Consider classification of the first-order quasilinear PDE

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = c. \tag{1.34}$$

One additional equation that relates the first-order derivative is as follows:

$$d(u) = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx \tag{1.35}$$

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The characteristic differential equation is found by setting the determinant of the coefficient matrix to zero. This yields $adx - bdt = 0$. Solving for dx/dt gives

$$\frac{dx}{dt} = \frac{b}{a}. \tag{1.36}$$

Equation (1.36) is the differential equation for a family of characteristic curves in the solution domain along which the first derivatives of u may be discontinuous or multi-valued. As a and b are real functions, the characteristic curves always exist and they are real characteristic curves. Hence, a single quasilinear first-order PDE is always a hyperbolic PDE. The one-dimensional first-order advection equation is an example of a hyperbolic PDE.

1.6 Wave Equation

In this section, we derive the one-dimensional wave equation, which is the simplest form of the wave equation for an idealized string. The following assumptions on the physical string are presumed to hold. Assume that a flexible string of length L is tightly stretched along the x-axis with one of its end point at $x = 0$ and the other end point at $x = L$. It is further assumed that the tension force on the string is the only dominant force, whereas all other forces acting on the string are negligible. Moreover, it is assumed that no external forces are applied to the string. Furthermore, it is assumed that the weight of the string is negligible and that the damping forces can also be neglected. Considering the string to be flexible, it follows that at each point, the tension force has constant magnitude; moreover, it has the direction of the tangent line to the string. It is also assumed that each point of the string moves only vertically. Let $u(x,t)$ denote the vertical displacement at time t of the point x on the string. At a fixed initial time, $t = t_o$, the shape of the string is given by the known function $u(x,t_o)$. The objective is to find the shape of the string at all points x and at time t, i.e., $u(x,t)$. To find the shape of the string at all points at a later time, one needs to solve the one-dimensional wave equation with associated initial and boundary conditions.

Consider a small element of the string between the points x and $x + \Delta x$ ($\Delta x > 0$ is assumed small; moreover, it is assumed that this element moves vertically). The total force to which this element is subject to is the tension force exerted at the left end $T(x,t)$ and the tension force exerted at the right end $T(x+\Delta x,t)$ by the rest of the string. These forces have the same constant magnitude T. Let $\theta(x,t)$ be the angle between $T(x,t)$ and the x-axis and $\theta(x+\Delta x,t)$ be the angle between $T(x+\Delta x,t)$ and the x-axis. It is assumed that these angles are between 0 and π. As we are assuming that we are dealing with

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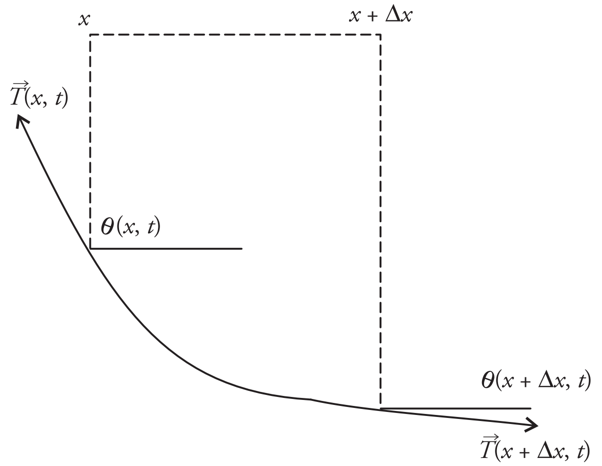

Figure 1.5 String element at time t subject to tension forces.

small vibrations, then either θ is close to 0 (at location $(x+\Delta x, t)$) or close to π (at location (x,t)).

The total vertical force acting on the element is given by $F =$ vertical component of tension force at (x,t) plus vertical component of tension force at $(x+\Delta x, t)$, i.e.,

$$F = T(x+\Delta x,t)\sin[\theta(x+\Delta x,t)] + T(x,t)\sin[\theta(x,t)] = T\{\sin[\theta(x+\Delta x,t)] + \sin\theta(x,t)\}. \tag{1.37}$$

For θ close to zero, $\sin\theta \sim \tan\theta \sim \theta$, whereas for θ close to π, $\sin\theta \sim -\tan\theta \sim \pi - \theta$. Moreover, the shape of the string at a fixed time t is given as the graph of the function $u(x,t)$ (t fixed and x varies); slope of the tangent line at location x_o is given by $\tan\theta_o$, where θ_o is the inclination angle. It follows then that

$$\tan\theta(x,t) = \frac{\partial u(x,t)}{\partial x} \quad \text{and} \quad \tan\theta(x+\Delta x,t) = \frac{\partial u(x+\Delta x,t)}{\partial x}. \tag{1.38}$$

Substituting Equation (1.38) in Equation (1.37), one obtains the total vertical force acting on the element as

$$F = T(x,t)\left(\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right). \tag{1.39}$$

Using Newton's second law of motion, $F = \Delta m\, a$, where Δm is the mass of the element and a is the acceleration of the element at time t. The mass of the element $\Delta m = \rho\Delta x$,

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where ρ is the density of the string material. The acceleration of the element at time t can be written as

$$a = \frac{\partial^2 u}{\partial t^2}. \tag{1.40}$$

Using the expressions for Δm and a, Newton's second law of motion becomes

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{T}{\Delta x}\left(\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right) \tag{1.41}$$

In the limit when $\Delta x \to 0$, Equation (1.41) becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1.42}$$

where $c^2 = T/\rho$ is the square of the speed of the wave. Equation (1.42) is known as the one-dimensional wave equation.

1.7 Linear Advection Equation

The wave equation is closely related to the so-called *advection equation*, which in one dimension takes the form

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0. \tag{1.43}$$

The aforementioned equation describes the passive advection of some scalar field u carried along by a flow of constant speed v. Let the initial condition be $u(x,0) = u_o$. Based on the method of characteristics discussed in Section 1.4, it follows that the characteristic equations are

$$\frac{dt}{ds} = 1; \quad \text{with } t(0) = 0;$$

this implies $t = s$;

$$\frac{dx}{ds} = v; \quad \text{with } x(0) = x_o;$$

this implies that $x = x_o + vs$ and $x = x_o + vt \to x_o = x - vt$. Furthermore,

$$\frac{du}{ds} = 0; \quad \text{with } s(0) = u_o(x_o).$$

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The unique solution of the advection equation is

$$u(x,t) = u_o(x - vt). \tag{1.44}$$

The solution (Equation (1.44)) is just an initial condition u_o shifted by vt to the right (for $v > 0$) or to the left ($v < 0$), which remains constant along the characteristic curves, $du/ds = 0$.

1.8 Laplace Equation

Consider a thin plate having some width w and some length l; it also has a very small thickness t. The faces of this plate are insulated to ensure that no heat flows in the direction of the thickness t. Assume that the top edge of the plate is maintained at a higher temperature while the other three edges are maintained at a same lower temperature. In this situation, heat flows into the plate through the top edge and out of the plate through the other three edges. Assume that there are no mechanism/processes for generation of internal energy within the plate. In the aforementioned circumstance, one is interested in obtaining the temperature (T) distribution within the plate. The temperature within the plate will vary within the horizontal plane in terms of x (width-wise coordinate) and y (length-wise coordinate) and the temperature distribution within the plate $T(x,y)$ will be governed by the following two-dimensional Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1.45}$$

The non-homogeneous (right-hand side is a known function of space) form of the Laplace equation is called Poisson equation. Solving Equation (1.45) subject to the boundary conditions (specified temperature on all the four edges) will determine the temperature distribution $T(x,y)$ within the plate. The Laplace equation arises in several problems in ideal fluid flow, heat diffusion, mass diffusion, and in electrostatics. As time does not appear in the Laplace equation, and the prescribed temperature on all the four edges is also independent of time, the solution of the Laplace equation (temperature distribution within the plate) will also not depend on time. Such problems in which time does not appear in the governing equations are known as equilibrium problems.

Equation (1.45) can be solved by the *method of separation of variables* in which it is assumed that the equation has a solution of the form

$$T(x,y) = X(x)Y(y). \tag{1.46}$$

Substituting Equation (1.46) in Equation (1.45), one obtains after dividing by XY, the following

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2. \tag{1.47}$$

As the first term and second terms of Equation (1.47) depend only on x and y, respectively, each of them should depend on a constant, say $-k^2$. The solution is then a product of $X(x) = c_1 \sin(kx) + c_2 \cos(kx)$ and $Y(y) = c_3 \sinh(ky) + c_4 \cosh(ky)$. It is possible to reduce the number of constants from 5 to 4. For example, if $c_1 c_3 \neq 0$, it is possible to redefine $c_1 c_3 = A$; $c_2/c_1 = B$; $c_4/c_3 = C$; and write the solution as

$$T(x,y) = A\left[\sin(kx) + B\cos(kx)\right]\left[\sinh(ky) + C\cosh(ky)\right]. \tag{1.48}$$

The constants A, B, C, and k are to be determined from the given initial and boundary conditions.

1.9 Method of Separation of Variables for the One-dimensional Heat Equation

In this section, the method of separation of variables is used to solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}. \tag{1.49}$$

In this method, the solution is assumed to be of the form

$$u(x,t) = X(x)\,T(t). \tag{1.50}$$

Substituting Equation (1.50) in Equation (1.49) and dividing by XT, one gets

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha T}\frac{dT}{dt} = -k^2 \tag{1.51}$$

As the first term and second terms of Equation (1.51) depend only on x and t, respectively, each of them should depend on a constant, say $-k^2$. The solution is then a product of

$$X(x) = c_1 \cos(kx) + c_2 \sin(kx) \quad \text{and} \quad T(t) = c_3 e^{-\alpha k^2 t} \quad \text{given by}$$

$$u(x,t) = e^{-\alpha k^2 t}\left[c_1 \cos(kx) + c_2 \sin(kx)\right], \tag{1.52}$$

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where c_3 is assumed to be unity without any loss of generality. With boundary conditions $u(x=0,t)=0$ and $u(x=L,t)=0$, one gets $c_1=0$ and $k=n(\pi/L)$; the solution is as follows:

$$u_n(x,t) = b_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L} \qquad \text{for} \quad n = 1, 2, \cdots \tag{1.53}$$

Satisfying the initial condition $u(x,t=0)=f(x)$, one gets the solution of the heat equation as

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L} \tag{1.54}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

1.10 Method of Separation of Variables for the One-dimensional Wave Equation

In this section, the method of separation of variables is used to solve the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{1.55}$$

As in the case of the heat equation, the solution is assumed to be of the form

$$u(x,t) = X(x)T(t). \tag{1.56}$$

Substituting Equation (1.56) in Equation (1.55) and dividing by XT, one obtains

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -k^2. \tag{1.57}$$

As the first term and second terms of Equation (1.57) depend only on x and t, respectively, each of them should depend on a constant, say $-k^2$. The solution is then a product of

$$X(x) = c_1\cos(kx)+c_2\sin(kx) \quad \text{and} \quad T(t) = c_3\cos(kct)+c_4\sin(kct) \quad \text{given by}$$

$$u(x,t) = [c_1\cos(kx)+c_2\sin(kx)]\,[c_3\cos(kct)+c_4\sin(kct)]\,. \tag{1.58}$$

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With boundary conditions $u(x=0,t)=0$ and $u(x=L,t)=0$, one gets $c_1=0$ and $k=n(\pi/L)$; solution is as follows:

$$u_n(x,t) = \left(\alpha_n \cos\frac{n\pi ct}{L} + \beta_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L} \qquad \text{for} \quad n = 1, 2, \cdots \tag{1.59}$$

Satisfying the initial conditions $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$; one gets the solution of the wave equation

$$u(x,t) = \sum_{n=1}^{\infty}(\alpha_n \cos\omega_n t + \beta_n \sin\omega_n t)\sin k_n x, \tag{1.60}$$

where

$$\omega_n = \frac{n\pi c}{L}, \qquad k_n = \frac{n\pi}{L},$$

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\,\sin k_n x\,dx, \quad \text{and}$$

$$\beta_n = \frac{2}{n\pi c}\int_0^L g(x)\,\sin k_n x\,dx.$$

Exercises 1a (Question and answer)

1. Find the type (linear, semilinear, quasilinear, or nonlinear) of the following partial differential equations:

   (a) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$

   (b) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z^3$

   (c) $(x+y)\dfrac{\partial z}{\partial x} + (x-y)\dfrac{\partial z}{\partial y} = xy$

   (d) $z\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$

   (e) $x\left(\dfrac{\partial z}{\partial x}\right)^2 + y\left(\dfrac{\partial z}{\partial y}\right)^2 = z$

   Answer: (a) Linear (b) Semilinear (c) Linear (d) Quasilinear (e) Nonlinear

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Exercises 1b (Questions only)

1. Find the general integral of the first-order partial differential equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

Answer: $F(x/y, z/y) = 0$

2. Given the first-order partial differential equation

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0,$$

find the nature of the characteristic curves.

Answer: The characteristic curves are a family of circles passing through the origin.

3. Given the first-order partial differential equation

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 0,$$

where a and b are constants. Find the general solution.

Answer: $z = f(ay - bx)$

4. Given the first-order partial differential equation

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c,$$

where a, b, and c are constants, find the general solution.

Answer: $z = f(ay - bx) + (c/a)x$

5. Solve the first-order partial differential equation

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = z,$$

with initial conditions $z = x^2$ on $y = x$; $1 \le y \le 2$.

Answer: $z(x,y) = x\sqrt{xy}$

6. Solve the first-order partial differential equation

$$x(y-z)\frac{\partial z}{\partial x} + y(x+z)\frac{\partial z}{\partial y} = (x+y)z,$$

with initial conditions $z = x^2 + 1$ on $y = x$.

Answer: $\dfrac{xy}{u} = \dfrac{x+u-y-1}{x+u-y}$

7. Show that the following second-order partial differential equation of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0, \tag{E1.1}$$

subject to the following transformation of independent variables from x and y to ξ and η, where A, B, C, D, E, F, and G are functions of x and y only, which can be put in the canonical or normal form. Show that the transformed equations are of the following form

$$\overline{A}(\xi_x, \xi_y)\frac{\partial^2 u}{\partial \xi^2} + \overline{B}(\xi_x, \xi_y, \eta_x, \eta_y)\frac{\partial^2 u}{\partial \xi \partial \eta} + \overline{C}(\eta_x, \eta_y)\frac{\partial^2 u}{\partial \eta^2} = F[\xi, \eta, u(\xi, \eta), u_\xi(\xi, \eta), u_\eta(\xi, \eta)] \tag{E1.2}$$

where

$$\begin{aligned} \overline{A}(\xi_x, \xi_y) &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \\ \overline{B}(\xi_x, \xi_y, \eta_x, \eta_y) &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y, \\ \overline{C}(\eta_x, \eta_y) &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \end{aligned}$$

In the aforementioned set of equations, the subscripts indicate partial derivatives.

8. Moreover, show that for the aforementioned second-order partial differential equation (E1.1), the following relation can be obtained

$$\overline{B}^2 - 4\overline{AC} = (\xi_x\eta_y - \xi_y\eta_x)^2(B^2 - 4AC).$$

9. For the hyperbolic case, where $B^2 - 4AC > 0$, show that Equation (E1.2) will be transformed and result in the following simple, canonical form given by

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$$

10. For the parabolic case, where $B^2 - 4AC = 0$, show that Equation (E1.2) will be transformed and result in the following simple, canonical form given by

$$\frac{\partial^2 u}{\partial \eta^2} = \phi(\xi, \eta, u, u_\xi, u_\eta)$$

11. For the elliptical case, where $B^2 - 4AC < 0$, show that Equation (E1.2) will be transformed and result in the following simple, real canonical form given by

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = \psi[\alpha, \beta, u, u_\alpha(\alpha, \beta), u_\beta(\alpha, \beta)],$$

where

$$\alpha = \frac{1}{2}(\xi + \eta) \qquad \text{and} \qquad \beta = \frac{1}{2}(\xi - \eta).$$

12. Reduce the following second-order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = x^2 \frac{\partial^2 u}{\partial y^2}$$

to its canonical form.

Answer: $u_{\xi\eta} = \frac{1}{4(\xi - \eta)}(u_\xi - u_\eta)$

13. Reduce the following second-order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

to its canonical form.

Answer: $u_{\eta\eta} = 0$

14. Reduce the following second-order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} = 0$$

to its canonical form.

Answer: $u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\alpha} u_\alpha$